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Integrable SYK models

Hosho Katsura (UTokyo)

東京大学
THE UNIVERSITY OF TOKYO

Institute for Physics of Intelligence

Trans-Scale Quantum Science Institute

With
Soshun Ozaki (UTokyo)
Eiki Iyoda (Tokai Univ.)
Takahiro Sagawa (UTokyo)
Kareljan Schoutens (Univ. Amsterdam)
Yuan Miao (IPMU, UTokyo)

When it all started…

- Sagawa-san's talk @ Gakushuin in 2017
  - Hamiltonian of the clean SYK

$$H_{\mathrm{cSYK}} = \sum_{1\le j<i\le N} \sum_{1\le k<l\le N} c_i^\dagger c_j^\dagger c_k c_l$$

NOTE) *c*'s are complex fermions

  - Classified as a *non-integrable* model
  - My question: Really true?
- Exact diagonalization for small systems

The number of E=0 states (N: the number of sites)

N	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Z_N	6	10	20	35	70	126	252	462	924	1716	3432

At half-filling (the num. of particles N_P=N/2)

N	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
Z_N	-	5	-	14	-	42	-	132	-	429	-

Exponentially many ground states! What are these numbers?

Online encyclopedia at work!

https://oeis.org/

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

6, 10, 20, 35, 70, 126, 252 Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:6,10,20,35,70,126,252**

Displaying 1-6 of 6 results found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A001405 a(n) = binomial(n, floor(n/2)). +30 384

(Formerly M0769 N0294)

1, 1, 2, 3, **6, 10, 20, 35, 70, 126, 252**, 462, 924, 1716, 3432, 6435, 12870, 24310, 48620, 92378, 184756, 352716, 705432, 1352078, 2704156, 5200300, 10400600, 20058300, 40116600, 77558760, 155117520, 300540195, 601080390, 1166803110 (list; graph; refs; listen; history; text; internal format)

$$Z_N = \binom{N+1}{\lfloor \frac{N+1}{2} \rfloor}$$

- The number of ground states for fixed N_P
- There must be something deep behind them…

$$Z_{N,N_P} = \binom{N}{N_P} - \binom{N}{N_P - 2}$$

Catalan numbers at half filling!

Outline

1. Majorana fermion models
   - Introduction & Motivation
   - Clean Majorana SYK
   - Clean supersymmetric Majorana SYK
2. Complex fermion models
   - Clean complex SYK
   - Clean supersymmetric complex SYK
3. Dynamics
   - Out-of-time order correlator (OTOC)
   - Spectral form factor
4. Summary

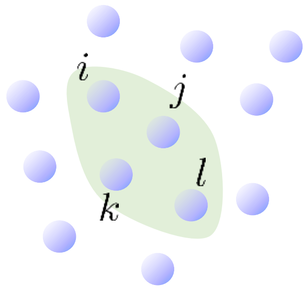
$$H_4 = -\sum_{i<j<k<l} \gamma_i \gamma_j \gamma_k \gamma_l$$

$$Q_3 = \mathrm{i} \sum_{i<j<k} \gamma_i \gamma_j \gamma_k$$

$$H_{c4} = \sum_{j<i} \sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$

$$Q_{c3} = \sum_{i<j<k} c_i c_j c_k$$

Original Majorana SYK model



- Majorana (real) fermions $\gamma_i \quad (i, j = 1, 2, \ldots, N)$
  - Obey $\gamma_i^\dagger = \gamma_i, \quad \{\gamma_i, \gamma_j\} = 2\delta_{i,j}$
- Sachdev-Ye-Kitaev model
  - Hamiltonian $$H_{\mathrm{SYK}} = \sum_{1 \le i < j < k < l \le N} J_{ijkl}\, \gamma_i \gamma_j \gamma_k \gamma_l$$
  - $J_{ijkl} \in \mathbb{R}$: Gaussian with zero mean and variance $J/N^{3/2}$
  - Consists of all-to-all coupling terms
  - Tractable in the large-*N* limit
  - Toy model for holographic duality
  - Maximally chaotic (OTOC saturates the chaos bound)
- What if all the couplings are equal?
  - Turns out to be *integrable!*
  - Can we still find a signature/remnant of chaos?

Sachdev & Ye, PRL **70** (1993), arXiv:cond-mat/9212030; Kitaev, Talks at KITP (2015); Maldacena, Stanford, PRD **94** (2016)

Proposals for experiments:
Danshita *et al*., PTEP **2017** (2017); Pikulin & Franz, PRX **7** (2017)

Clean Majorana SYK model

Hamiltonian

$$H_4 = -\sum_{1\le i<j<k<l\le N} \gamma_i\gamma_j\gamma_k\gamma_l$$

Lau, Ma, Murugan & Tezuka, JPA **54**, 095401 (2021)

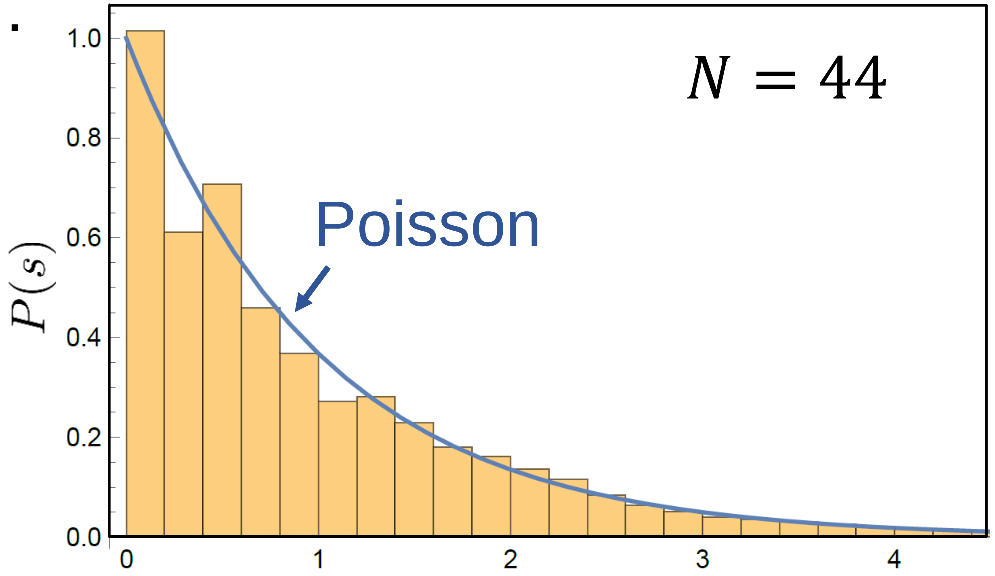
- Jordan-Wigner transformation

$$\gamma_1 = \sigma_1^x, \quad \gamma_2 = \sigma_1^y, \quad \gamma_3 = \sigma_1^z\sigma_2^x, \quad \gamma_4 = \sigma_1^z\sigma_2^y, \ldots$$

- *N*=4 $\quad H_4 = -\gamma_1\gamma_2\gamma_3\gamma_4 = \sigma_1^z\sigma_2^z$
- Just a 2-site Ising model! Trivially integrable.
- H_4 with *N* > 4 does not seem integrable…

Level-spacing distribution

- Use the energy levels in the middle of the spectrum $E_j - E_{\rm GS} \in [0.4N^2, 0.5N^2]$
- Level gaps $\delta_j = E_j - E_{j-1}$
- Poisson distribution!
  Is H_4 integrable??

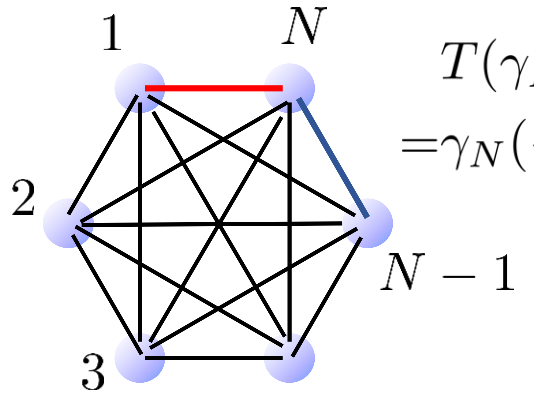


Integrability of quadratic case: warm-up

■Quadratic all-to-all Hamiltonian (N: even)

$$H_2 = \mathrm{i} \sum_{1\le i<j\le N} \gamma_i \gamma_j$$

Lau *et al*, JPA **54** (2021)



$$T(\gamma_{N-1}\gamma_N)T^{-1} = \gamma_N(-\gamma_1) = \gamma_1\gamma_N$$

- Subject to twisted boundary conditions:

$$T\gamma_j T^{-1} = \gamma_{j+1} \text{ if } 1 \le j < N; \quad T\gamma_N T^{-1} = -\gamma_1$$

- Fourier transform (k = 1, 2, …, N/2)

$$f_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{N} e^{\mathrm{i}(j-1)\theta_k} \gamma_j, \quad f_k^\dagger = \frac{1}{\sqrt{2N}} \sum_{j=1}^{N} e^{-\mathrm{i}(j-1)\theta_k} \gamma_j \quad \left(\theta_k = \frac{2k-1}{N}\pi\right)$$

- They are complex fermions obeying

$$\{f_k, f_\ell^\dagger\} = \delta_{k,\ell}, \quad \{f_k, f_\ell\} = \{f_k^\dagger, f_\ell^\dagger\} = 0 \quad \text{and} \quad T f_k^{(\dagger)} T^{-1} = e^{\pm \mathrm{i}\theta_k} f_k^{(\dagger)}$$

Diagonal form of H_2

$$H_2 = \sum_{k=1}^{N/2} \epsilon_k \left(f_k^\dagger f_k - \frac{1}{2} \right), \quad \epsilon_k = 2\cot\frac{\theta_k}{2}$$

Trivially solvable/integrable!

Dispersion is very weird, though…

Integrability of H_4

- Nontrivial identity

$$H_4 = \frac{1}{2}\left\{(H_2)^2 - \frac{N(N-1)}{2}\right\}$$

  - Proof by division into cases

$$(H_2)^2 = -\sum_{i<j}\sum_{k<l}\gamma_i\gamma_j\gamma_k\gamma_l = \cdots$$

  - Obviously, $[H_4, H_2] = 0$

- Eigenstates of H_4
  - Any eigenstate of H_2 is an eigenstate of H_4
  - Solvable structure is similar to that of the Hubbard + all-to-all interaction
    Hatsugai & Kohmoto, JPSJ **61**, 2056 (1992)

1	$i<j<k<l$	$\gamma_i\gamma_j\gamma_k\gamma_l$
2	$i<j=k<l$	$\gamma_i\gamma_l$
3	$i<k<j<l$	$-\gamma_i\gamma_k\gamma_j\gamma_l$
4	$i<k<j=l$	$-\gamma_i\gamma_k$
5	$i<k<l<j$	$\gamma_i\gamma_k\gamma_l\gamma_j$
6	$i=k<j<l$	$-\gamma_j\gamma_l$
7	$i=k<j=l$	-1
8	$i=k<l<j$	$\gamma_l\gamma_j$
9	$k<i<j<l$	$\gamma_k\gamma_i\gamma_j\gamma_l$
10	$k<i<j=l$	$\gamma_k\gamma_i$
11	$k<i<l<j$	$-\gamma_k\gamma_i\gamma_l\gamma_j$
12	$k<i=l<j$	$-\gamma_k\gamma_j$
13	$k<l<i<j$	$\gamma_k\gamma_l\gamma_i\gamma_j$

Integrability of H_4

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5	$i<k<l<j$	$\gamma_i\gamma_k\gamma_l\gamma_j$
6	$i=k<j<l$	~~$-\gamma_j\gamma_l$~~
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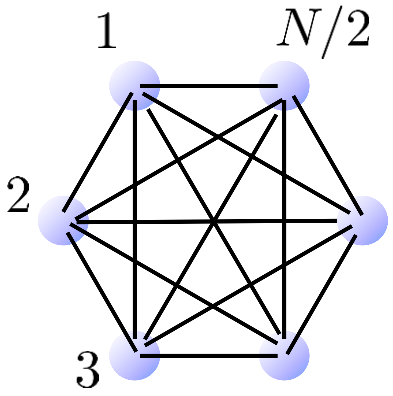
12	$k<i=l<j$	~~$-\gamma_k\gamma_j$~~
13	$k<l<i<j$	$\gamma_k\gamma_l\gamma_i\gamma_j$

$\binom{N}{2}$ cases

Equivalent Ising model



■ Spin Hamiltonian

- Occupation number $n_k = f_k^\dagger f_k \quad (k = 1, \ldots, N/2)$
- Ising variables $\sigma_k = 2n_k - 1$

$$H_4 = \sum_{k,\ell=1}^{N/2} J_{k\ell}\, \sigma_k \sigma_\ell + \text{const.} \qquad J_{k\ell} = \cot\left(\frac{2k-1}{2N}\pi\right)\cot\left(\frac{2\ell-1}{2N}\pi\right)$$

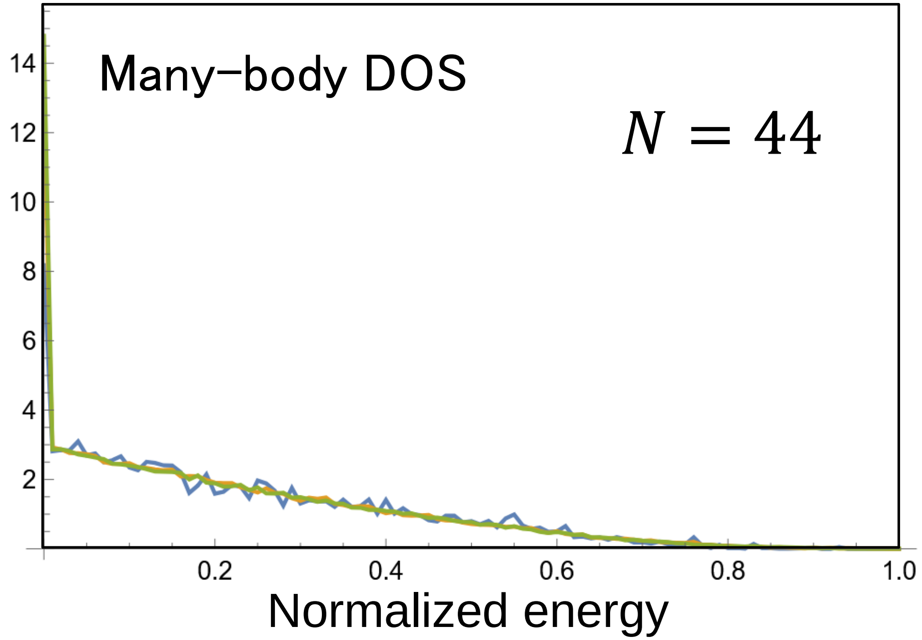
- Classical Ising model! Any Ising spin σ_k is a conserved quantity.

■ Ground states

- Exhibit complicated spin config. for $N > 24$
- Anything to do with number theory?
  Marinari, Parisi & Ritort, JPA **27**, 7615 (1994)

■ Energy spectrum

- Many low-energy states near the g.s.
- Residual entropy at T=0: $S/N \sim \frac{1}{2}\ln 2$



Infinite family of models

- Quadratic family (m=1,3,5,…)

$$H_2(\mathcal{A}^m) := \mathrm{i} \sum_{1\le i<j\le N} (\mathcal{A}^m)_{ij}\, \gamma_i \gamma_j,$$

Antisymmetric matrix

$$\mathcal{A}_{ij} = \begin{cases} 1 & i<j \\ 0 & i=j \\ -1 & i>j \end{cases}$$

  - They commute with $H_2 = H_2(\mathcal{A})$

- Most general form

$$H(\{C_{mn}\}) = \sum_{m\ge1}\sum_{n\ge1} C_{mn}\, H_2(\mathcal{A}^m)^n + \text{const.}, \qquad C_{mn}\in\mathbb{R} \qquad (*)$$

  - Conjecture

    The Hamiltonian of the generalized clean Majorana SYK

$$H_{2p} = \mathrm{i}^p \sum_{1\le i_1<i_2<\cdots<i_{2p-1}<i_{2p}\le N} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{2p-1}}\gamma_{i_{2p}}$$

    can be written in the form of (∗). $[H_{2p}, H_{2q}] = 0$ holds for all $p, q\in\mathbb{N}$.

  - Yet to be proved but substantial evidence for small *p* and *N*

Supersymmetric (SUSY) SYK models

- $\mathcal{N} = 1$ SUSY quantum mechanics

Witten, NPB **202**, 253 (1982)

  - Fermionic parity $(-1)^F$
  - Supercharge $Q \quad (Q^\dagger = Q)$ anti-commutes with $(-1)^F$
  - Hamiltonian $H = Q^2$
  - Symmetry $[H, (-1)^F] = [H, Q] = 0.$

- Spectrum of *H*
  - $E \geq 0$ for all states
  - $E > 0$ states come in pairs $\{|\psi\rangle, Q|\psi\rangle\}$
  - $E = 0$ state, if exists, must be annihilated by *Q*

- SUSY SYK
  - Supercharge
  - Hamiltonian

$$Q_{\rm SYK} = {\rm i} \sum_{1 \leq i < j < k \leq N} C_{ijk}\, \gamma_i \gamma_j \gamma_k$$

$$\langle C_{ijk} \rangle = 0, \quad \langle C_{ijk}^2 \rangle = \frac{2J}{N^2}$$

$$H_{\rm SYK}^{\rm SUSY} = (Q_{\rm SYK})^2$$

Fu, Gaiotto, Maldacena & Sachdev, *PRD* **95** (2017)

Clean SUSY Majorana SYK

■ Supercharge & Hamiltonian

$$Q_3 = \mathrm{i} \sum_{1 \le i<j<k \le N} \gamma_i \gamma_j \gamma_k \quad \Rightarrow \quad H_4^{\mathrm{SUSY}} = (Q_3)^2$$

- Commute with $\chi_0 := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \gamma_j$
- Nontrivial identity

$$Q_3 = \chi_0 H_{\mathrm{free}}, \quad H_{\mathrm{free}} = \frac{\mathrm{i}}{2} \sum_{i,j} (\tilde{A})_{ij}\, \gamma_j \gamma_k$$

Antisymmetric circulant matrix

$$\tilde{A}_{ij} = \begin{cases} \frac{1}{2} + \frac{i-j}{N} & i<j \\ 0 & i=j \\ -\frac{1}{2} + \frac{i-j}{N} & i>j \end{cases}$$

■ Integrability of H_4^{SUSY}

- $H_4^{\mathrm{SUSY}} = (H_{\mathrm{free}})^2$ follows from $[H_{\mathrm{free}}, \chi_0] = 0, \ (\chi_0)^2 = 1$
- Any eigenstate of H_{free} is an eigenstate of H_4^{SUSY}
- Diagonal form

$$H_{\mathrm{free}} = \sum_{k=1}^{\frac{N}{2}-1} \epsilon_k \left(g_k^\dagger g_k - \frac{1}{2} \right), \quad \epsilon_k = 2 \cot\left(\frac{k\pi}{N} \right), \quad g_k = \sqrt{\frac{2}{N}} \sum_{j=1}^{N} \exp\left(\mathrm{i} \frac{2(j-1)k}{N} \pi \right) \gamma_j$$

Infinite families of models

■ Non-SUSY yet solvable models

- Quadratic Hamiltonians of the form $H_{\mathrm{hop},r} = \mathrm{i}\sum_{j=1}^{N}\gamma_j\gamma_{j+r} \quad (r=1,2,\ldots)$ commute with Q_3
- E.g., $(1-s)(Q_3)^2 + sH_{\mathrm{hop},1}$, an interpolation between clean SUSY SYK and critical Kitaev chain is solvable

■ SUSY solvable models

- Supercharges $Q_{2p+1} = \mathrm{i}^p \sum_{1\le i_1<i_2<\cdots<i_{2p+1}\le N} \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_{2p+1}}, \quad (p=0,1,2,\ldots)$
- Properties

  1. They all commute: $[Q_{2p+1}, Q_{2q+1}] = 0$ (still a conjecture)

  2. $Q_{2p+1} = \frac{1}{2}\{Q_1, H_{2p}\} \qquad (Q_1 = \sqrt{N}\chi_0)$

- Multi-parameter family: $H^{\mathrm{SUSY}}(\{C_{2p+1}\}) = (C_1Q_1 + C_3Q_3 + \cdots)^2, \quad C_{2p+1}\in\mathbb{R}$
- All-to-all version of Sannomiya & Katsura, PRD **99** (2019)
  Is SUSY broken or unbroken?

Outline

1. Majorana fermion models
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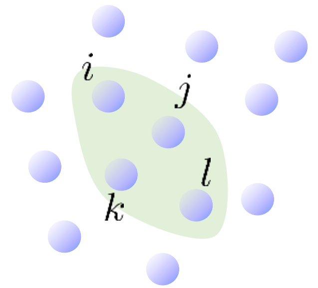
   - Clean supersymmetric complex SYK
3. Dynamics
   - Out-of-time order correlator (OTOC)
   - Spectral form factor
4. Summary

$$H_{\mathrm{c4}} = \sum_{j<i} \sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$

$$Q_{\mathrm{c3}} = \sum_{i<j<k} c_i c_j c_k$$

Clean complex SYK model



Complex (spinless) fermions

- Creation and annihilation operators: $c_i^\dagger, c_j \quad (i,j=1,2,...,N)$
- Canonical anti-commutation relations

$$\{c_i, c_j^\dagger\} = c_i c_j^\dagger + c_j^\dagger c_i = \delta_{i,j}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$$

Original model (disordered)

- Hamiltonian   Sachdev, PRX **5**, (2015); Fu & Sachdev, PRB **94** (2016)

$$H_{\mathrm{cSYK}} = \sum_{1\le j<i\le N} \sum_{1\le k<l\le N} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$

Complex Gaussian variables

$$J_{ij;kl} = -J_{ji;kl} = -J_{ij;lk} = J^*_{lk;ji}$$

$$\langle J_{ij;kl} \rangle = 0, \quad \langle |J_{ij;kl}|^2 \rangle = J^2/N^3$$

Clean complex SYK model

- Hamiltonian   Iyoda & Sagawa, PRA **97** (2018)

$$H_{c4} = \sum_{1\le j<i\le N} \sum_{1\le k<l\le N} c_i^\dagger c_j^\dagger c_k c_l$$

Is it Integrable?

YES! But not free-fermionic…

Integrability of H_{c4}

Iyoda, Katsura & Sagawa, PRD **98** (2018)

■ Factorization

$$H_{c4} = A^\dagger A \geq 0, \qquad A = \sum_{1\leq k<l\leq N} c_k c_l = \frac{1}{2}\boldsymbol{c}^{\mathrm{T}} \mathcal{A}\, \boldsymbol{c}$$

$\boldsymbol{c} = (c_1, ..., c_N)$

Antisymmetric matrix

$$\mathcal{A}_{ij} = \begin{cases} 1 & i<j \\ 0 & i=j \\ -1 & i>j \end{cases}$$

■ Canonical form

- $\mathcal{A}$ is non-diagonalizable, but $\mathcal{A}$ can be taken to $\mathcal{K} = \mathcal{O}\mathcal{A}\mathcal{O}^{\mathrm{T}} = \begin{pmatrix} 0 & \lambda_1 & & & O \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ O & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$, $\quad \lambda_k = \cot\left(\frac{2k-1}{2N}\pi\right)$
- A in the new basis:

$$A = \frac{1}{2}\boldsymbol{f}^{\mathrm{T}} \mathcal{K}\, \boldsymbol{f} = \sum_{k=1}^{N/2} \lambda_k\, f_{k\uparrow} f_{k\downarrow}, \qquad \boldsymbol{f} = \mathcal{O}\boldsymbol{c} = (f_{1\uparrow}, f_{1\downarrow}, f_{2\uparrow}, f_{2\downarrow}, ...)$$

■ Equivalent to a known model!

- Particular case of Richardson-Gaudin model

Richardson, JMP **6**, 1034 (1965); Gaudin's book

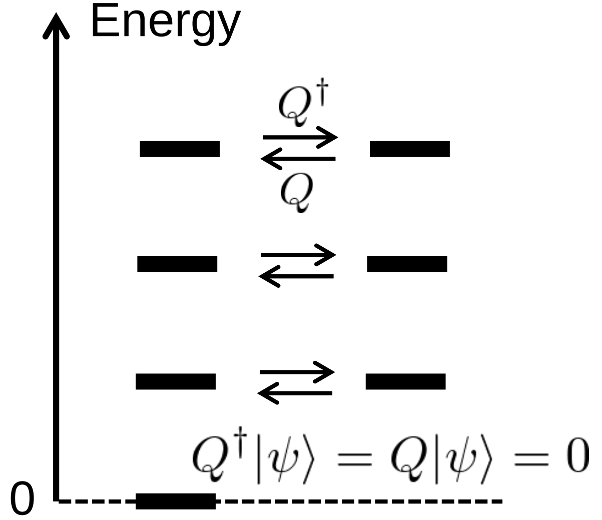
- Bethe-ansatz solvable
- E=0 states are in 1-to-1 correspondence with the lowest-weight states of ηSU(2) Yang, PRL **63** (1989)

$$\eta^- = \sum_{k=1}^{N/2} f_{k\uparrow} f_{k\downarrow}$$

Supersymmetric version

Nicolai, JPA **9**, 1497 (1976);
Witten, NPB **202**, 253 (1982)

- $\mathcal{N}=2$ SUSY quantum mechanics
  - Supercharges $Q,\ Q^\dagger,\quad Q^2=0,\ (Q^\dagger)^2=0$
  - Fermionic parity $\{Q,(-1)^F\}=\{Q^\dagger,(-1)^F\}=0$
  - Hamiltonian $H=\{Q,Q^\dagger\}=QQ^\dagger+Q^\dagger Q$
  - Symmetry $[H,Q]=[H,Q^\dagger]=[H,(-1)^F]=0$
- Spectrum of *H*
  - $E\geq 0$ for all states
  - $E>0$ states come in pairs $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$
  - $E=0$ iff a state is a SUSY singlet



- SUSY cSYK
  - Supercharge
  - Hamiltonian

$$Q_{\rm cSYK} = \mathrm{i}\sum_{1\leq i<j<k\leq N} D_{ijk}\, c_i c_j c_k$$

$$\langle D_{ijk}\rangle = 0,\quad \langle |D_{ijk}|^2\rangle = \frac{2J}{N^2}$$

$$H^{\rm SUSY}_{\rm cSYK} = \{Q_{\rm cSYK}, Q^\dagger_{\rm cSYK}\}$$

Fu, Gaiotto, Maldacena & Sachdev, PRD **95** (2017);
Sannomiya, Katsura & Nakayama, PRD **95** (2017)

Clean SUSY complex SYK

- Supercharge & Hamiltonian $Q_{c3} = \frac{1}{\sqrt{N}} \sum_{1 \le i < j < k \le N} c_i c_j c_k$ $\quad H_{c4}^{\mathrm{SUSY}} = \{Q_{c3}, Q_{c3}^\dagger\}$
- G.S. degeneracies

H_{c4}^{SUSY}

N	3	4	5	6	7	8	9	10	11	12	13
Z_N	6	12	20	40	70	140	252	504	924	1848	3432

…

H_{c4}

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OEIS: A063886

$$Z_N = 2\binom{N}{\lfloor \frac{N}{2} \rfloor}$$

- Integrability
  - $Q_{c3} = f_0 \tilde{A}$ with $f_0 := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c_j$ and $\tilde{A} = \sum_{j,k} (\tilde{A})_{jk} c_j c_k$ (Antisymmetric circulant matrix)
  - Hamiltonian

$$H_{c4}^{\mathrm{SUSY}} = \tilde{A}^\dagger \tilde{A}\, f_0^\dagger f_0 + \tilde{A}\tilde{A}^\dagger (1 - f_0^\dagger f_0)$$

Sum of two commuting Richardson-Gaudin Hamiltonians!

  - Explains the degeneracies observed numerically

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S. Ozaki and H. Katsura,
in preparation

Quantum dynamics of H_4

- Reminder

$$H_4 = -\sum_{i<j<k<l} \gamma_i\gamma_j\gamma_k\gamma_l = \frac{1}{2}\left\{(H_2)^2 - \frac{N(N-1)}{2}\right\}$$

$$H_2 = \mathrm{i}\sum_{i<j}\gamma_i\gamma_j = \sum_{k=1}^{N/2}\epsilon_k\left(f_k^+ f_k^- - \frac{1}{2}\right), \quad \epsilon_k = 2\cot\frac{\theta_k}{2}, \quad \theta_k = \frac{2k-1}{N}\pi$$

- Warm-up: quadratic case

$$[H_2, f_k^\pm] = \pm\epsilon_k f_k^\pm \quad \Rightarrow \quad e^{\mathrm{i}H_2 t} f_k^\pm e^{-\mathrm{i}H_2 t} = \exp\left(\pm\mathrm{i}\epsilon_k t\right) f_k^\pm$$

- Quartic case

$$[H_4, f_k^\pm] = \left(\mp\epsilon_k H_2 - \frac{1}{2}\epsilon_k^2\right) f_k^\pm \quad \Rightarrow \quad e^{\mathrm{i}H_4 t} f_k^\pm e^{-\mathrm{i}H_4 t} = \exp\left(\mp\mathrm{i}\epsilon_k \underline{H_2} t - \frac{1}{2}\epsilon_k^2 t\right) f_k^\pm$$

$$\Rightarrow \quad \gamma_j(t) = e^{\mathrm{i}H_4 t}\gamma_j e^{-\mathrm{i}H_4 t} = \sqrt{\frac{2}{N}}\sum_{s=\pm}\sum_{k=1}^{N/2}\exp\left(\mathrm{i}s(j-1)\theta_k + \mathrm{i}s\epsilon_k H_2 t - \frac{\mathrm{i}}{2}\epsilon_k^2 t\right) f_k^s$$

Out-of-time order correlator (OTOC)

■ Infinite-T OTOC $(i \neq j)$

$$C_{ij}(t) = \mathrm{Tr}[\gamma_i(t)\gamma_j(0)\gamma_i(t)\gamma_j(0)] = -\frac{4}{N^2}\sum_{k,\ell=1}^{N}\cos(\epsilon_k\epsilon_\ell t) + \cdots$$

$$= -\frac{2}{\pi}\left(\sin(4t)\mathrm{Ci}(4t) - \cos(4t)\mathrm{si}(4t) + \cdots\right)$$

Vanish in the large-N limit

$$\mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt$$

$$\mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}dt$$

■ Early-time behavior

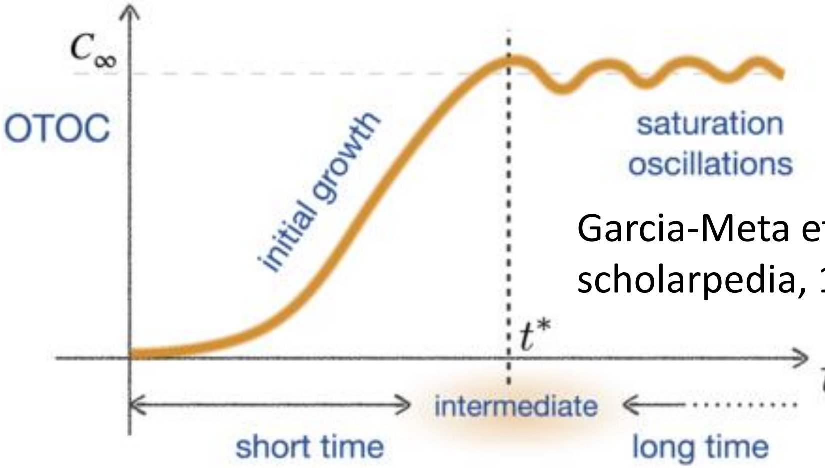
$$C_{ij}(t) \sim -1 + \frac{8}{\pi}(1 - \log(4t) - \gamma)t \qquad (t \ll 1)$$

Euler's constant $\gamma = 0.5772...$

- Different from the exponential growth in the original (disordered) SYK

■ Late-time behavior $C_{ij}(t) \sim \dfrac{1}{2\pi t}$

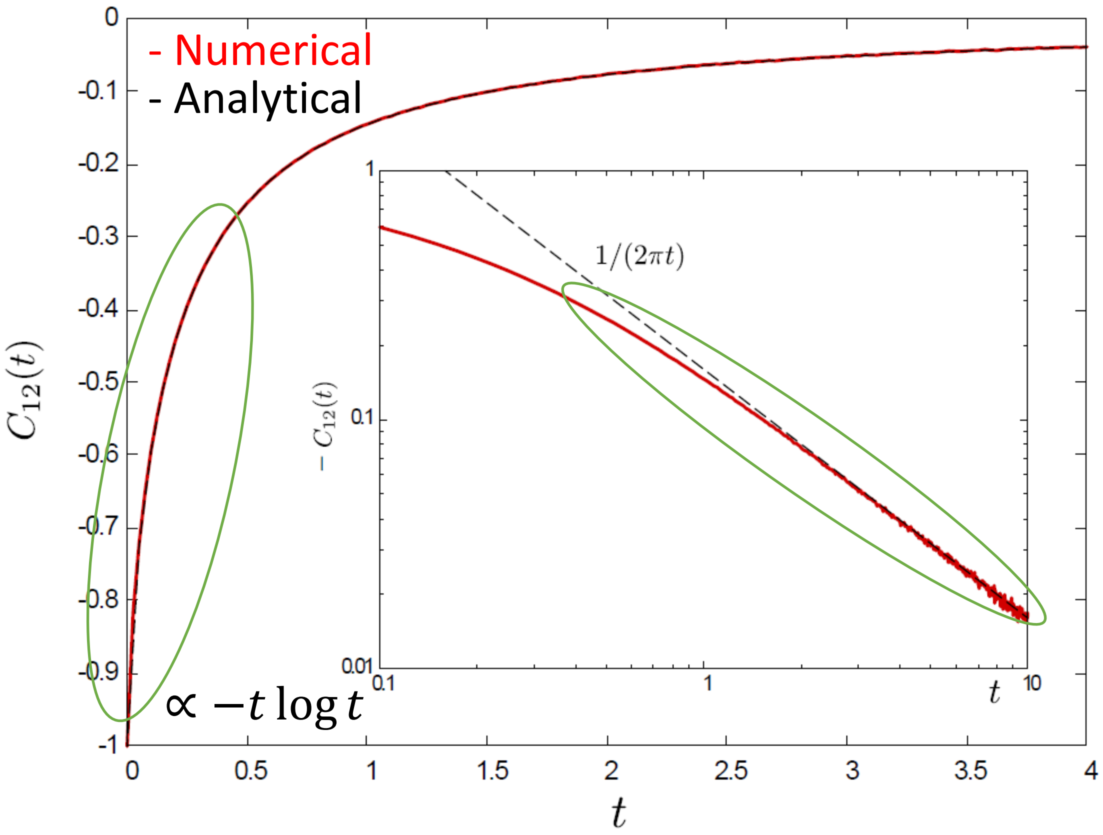
- Similar algebraic decay in other integrable models
  Lin & Motrunich, PRB **97** (2018); Bao & Zhang (2020)



Garcia-Meta et al., scholarpedia, 18(4): 55237

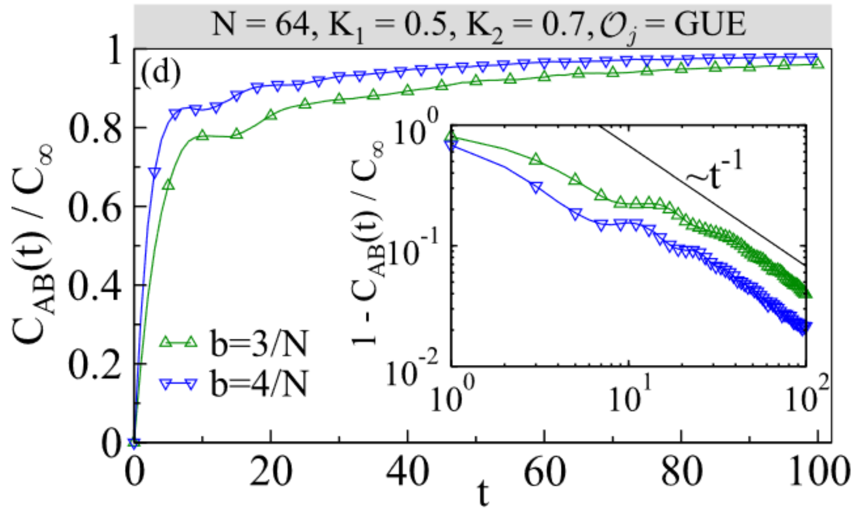
Numerical vs. analytical results

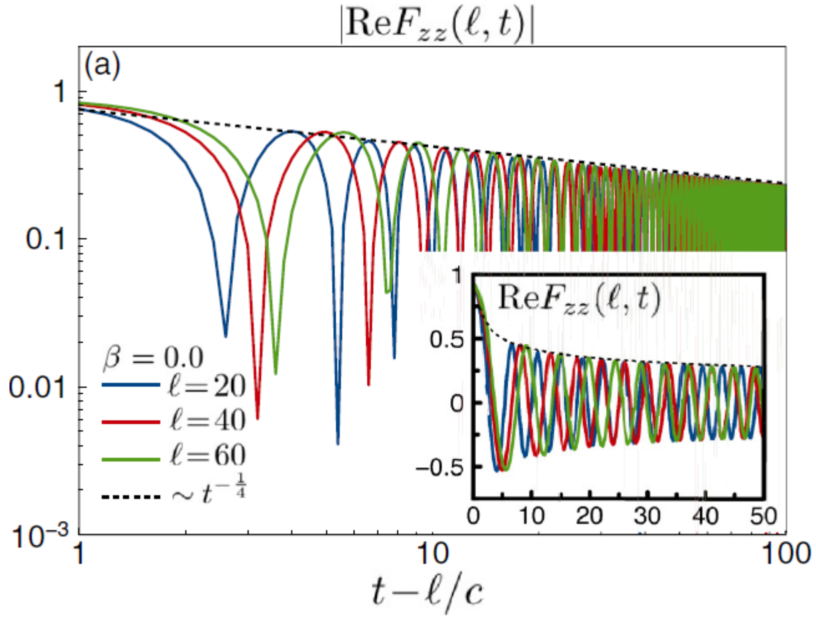
- *N*=1000 particles



Similar behavior in bipartite kicked rotor model

Prakash & Lakshminarayan, PRB **101** (2020)





OTOC in transverse field Ising model

Power-law decay at late times $t^{-1/4}$

Lin & Motrunich, PRB **97** (2018)

Spectral form factor

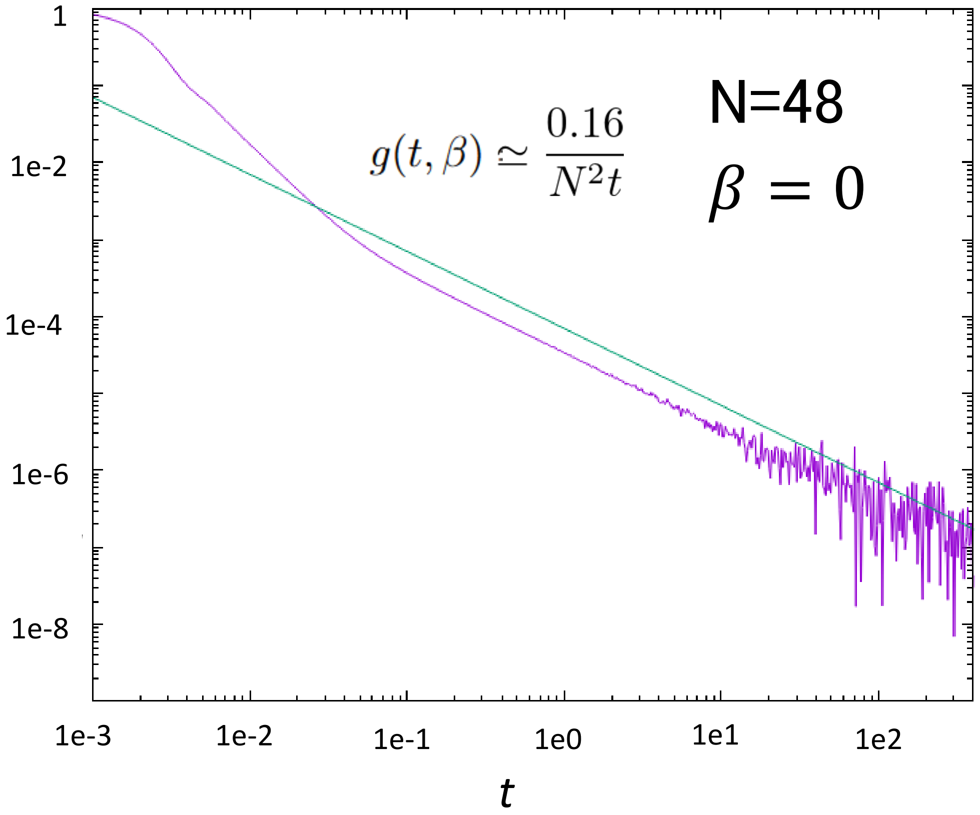
- **Definition**

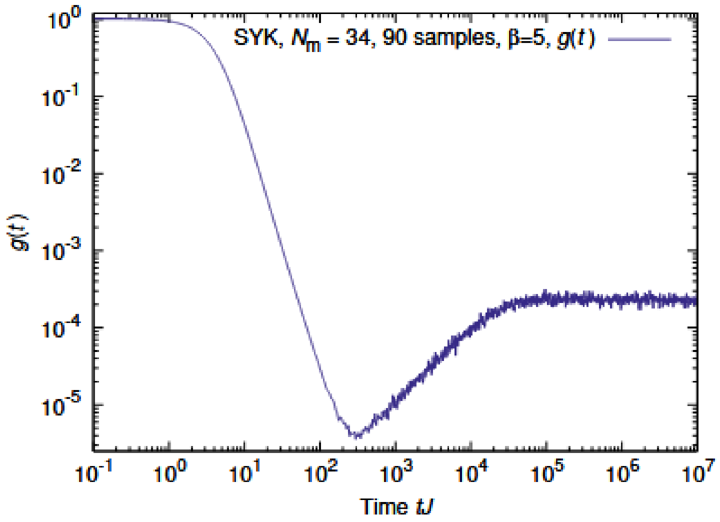
$$g(t,\beta) = \left| \frac{\mathrm{Tr} e^{(\mathrm{i}t-\beta)H_4}}{\mathrm{Tr} e^{-\beta H_4}} \right|^2$$

- **Early-time behavior**
  - can be computed combinatorially

$$g(t,0) = 1 - \binom{N}{4} t^2 + O(t^4)$$

- **Late-time behavior**
  - Yet to get a closed form
  - But likely to be $g(t,0) \simeq \dfrac{0.16}{N^2 t}$
- **Intermediate-time region**
  - Unlike the original SYK, no dip-ramp-plateau structure





Cotler *et al.*, JHEP **5** (2017)

Summary

- (Partial) List of integrable SYK models
  - Clean Majorana SYK
    $$H_4 = -\sum_{i<j<k<l} \gamma_i \gamma_j \gamma_k \gamma_l$$
  - Clean SUSY Majorana SYK
    $$Q_3 = \mathrm{i} \sum_{i<j<k} \gamma_i \gamma_j \gamma_k, \quad H_4^{\mathrm{SUSY}} = (Q_3)^2$$
  - Clean complex SYK
    $$H_{\mathrm{c}4} = \sum_{j<i} \sum_{k<l} c_i^\dagger c_j^\dagger c_k c_l$$
  - Clean SUSY Complex SYK
    $$Q_{\mathrm{c}3} = \sum_{i<j<k} c_i c_j c_k, \quad H_{\mathrm{c}4}^{\mathrm{SUSY}} = \{Q_{\mathrm{c}3}, Q_{\mathrm{c}3}^\dagger\}$$
- Discussed dynamics: OTOC & spectral form factor
- Future directions
  - Clean dissipative SYK? Integrable Lindblad master eq.?
    Kulkarni *et al.*, PRB **106** (2022); Sa *et al.*, PRR **4** (2022); Kawabata *et al.*, PRB **108** (2023)
  - Parafermionic extensions? $\psi_i \psi_j = \omega \psi_j \psi_i \ (i<j), \quad \omega = \exp(2\pi \mathrm{i}/N)$
    Fractional SUSY? $H = Q^3 + (Q^\dagger)^3$
    Bernard & LeClair, PhysLett B **247** (1990); Fendley & O'Brien, SciPost Phys. **9** (2020)