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Symmetry classification of typical quantum entanglement

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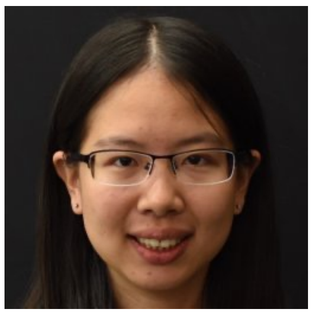
Phys. Rev. B 108, 085109 (2023)

Collaborators



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(IAS, Princeton)

Outline

1. Introduction

2. Symmetry classification of typical quantum entanglement

3. Weingarten calculus

4. Connection with mesoscopic condensed-matter physics

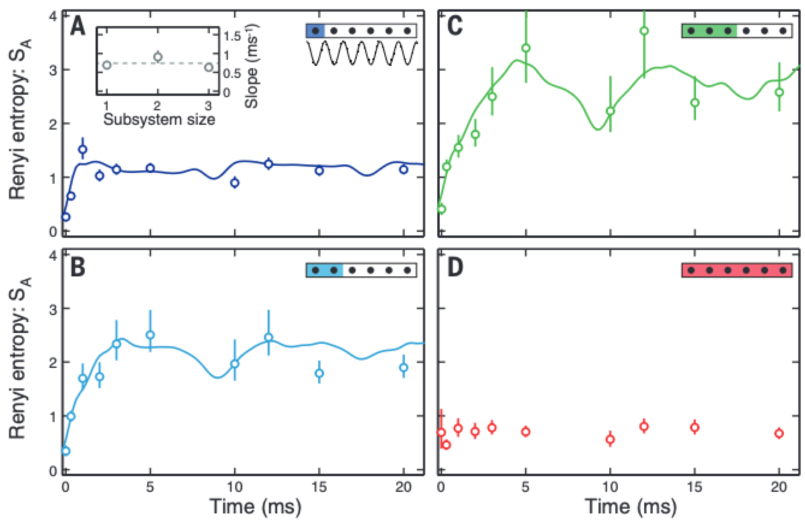
Quantum entanglement

Quantum entanglement lies at the heart of quantum physics.

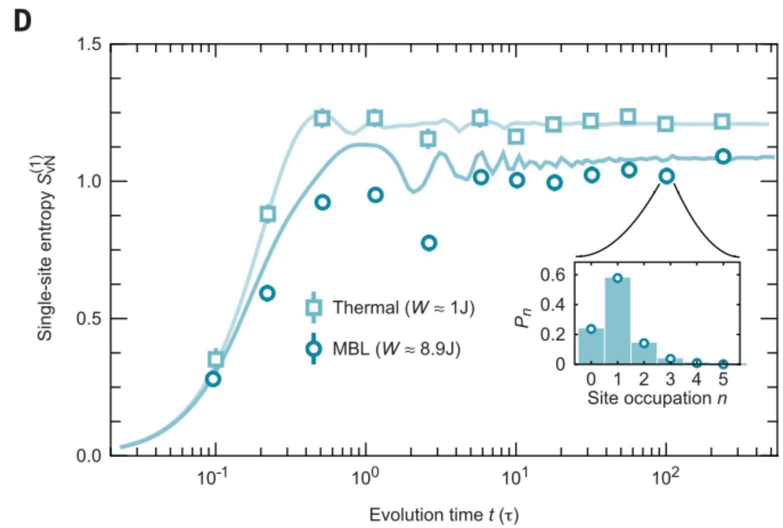
Entanglement characterizes quantum phases of matter:

- Quantum critical phenomena
- Topological phases

Entanglement also provides a foundation of thermalization or lack thereof.



Kaufman *et al.*, Science **353**, 794 (2016)



Lukin *et al.*, Science **364**, 256 (2019)

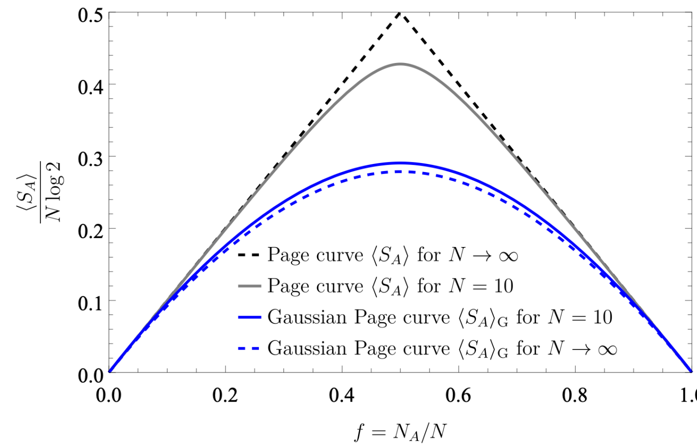
Typical entanglement: Page curve

☆ **Entanglement entropy of typical states in chaotic (nonintegrable) systems:**

maximal and proportional to the volume of the subsystem (**volume law**)

$$\langle S \rangle = \frac{\log 2}{2} N - \frac{1}{2}$$

→ thermalization



Bianchi *et al.*, PRX Quantum **3**, 030201 (2022)

☆ **Relevant to black-hole physics ("Page curve")**

Page, PRL **71**, 1291 (1993)

☆ **Entanglement of typical Gaussian states in free fermions**
"free-fermion (Gaussian) Page curve"

single-particle quantum chaos and thermalization in free fermions

Liu *et al.*, PRB **97**, 245126 (2018); Bianchi *et al.*, PRB **103**, L241118 (2021)

Quantum chaos

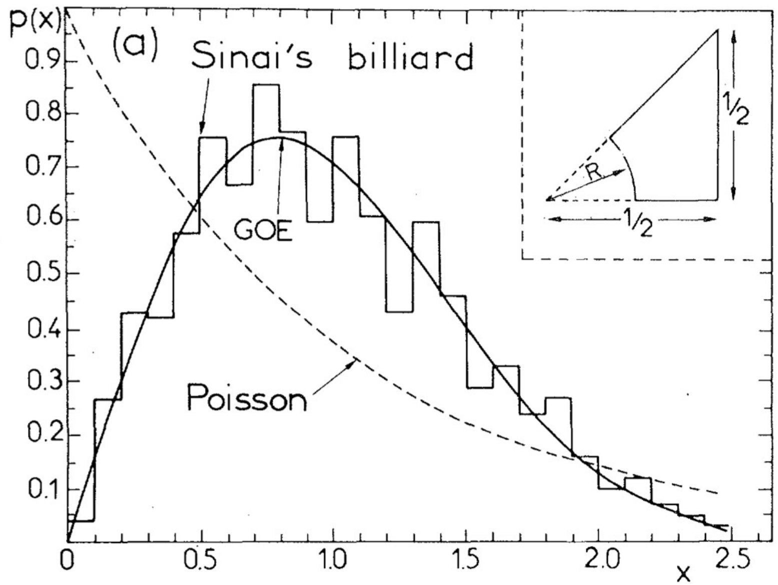
☆ **Quantum chaos manifests itself in spectral statistics.**

- Integrable quantum systems: **Poisson statistics**

Berry & Tabor, Proc. R. Soc. A **356**, 375 (1977)

- Nonintegrable quantum systems: **random-matrix statistics**

→ quantum chaos & thermalization



Bohigas *et al.*, PRL **52**, 1 (1984)

Altland-Zirnbauer symmetry

☆ **Quantum chaos (random matrices) is classified by the tenfold AZ symmetry.**

time reversal $T^{-1}H^{*}T = H$

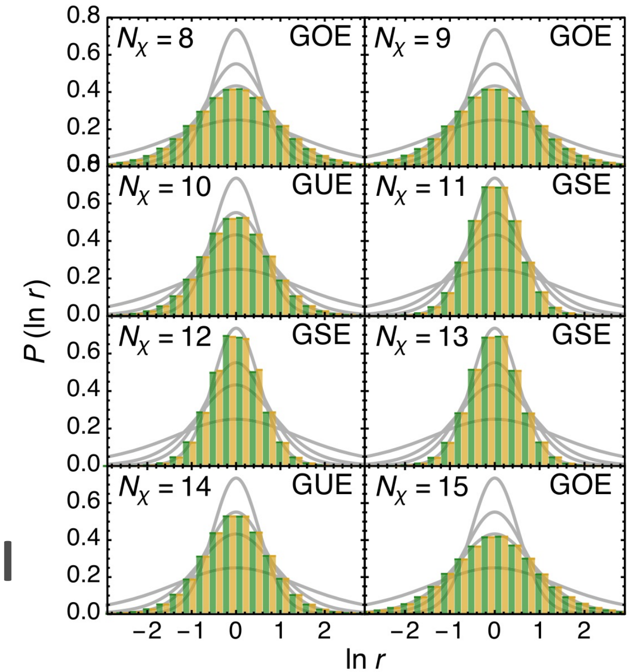
particle hole $C^{-1}H^{*}C = -H$

chiral $S^{-1}HS = -H$

Altland & Zirnbauer, PRB **55**, 1142 (1997)

e.g., Tenfold symmetry classification of the SYK model

symmetry-enriched behavior of quantum chaos



You *et al.*, PRB **95**, 115150 (2017)
Cotler *et al.*, JHEP **2017**, 118

☆ **AZ symmetry is also relevant to the physics of free fermions.**

- Anderson localization and transition
- Topological insulators and superconductors

Motivation

Typical entanglement characterizes quantum chaos.

Quantum chaos (spectral statistics) is classified according to the tenfold Altland-Zirnbauer symmetry.

→ How does symmetry affect typical entanglement?

In general, little has been understood about the role of symmetry in entanglement theory.

Results

☆ **We develop the tenfold classification of typical entanglement (free-fermion Page curves) based on AZ symmetry!**

AZ class	TRS	PHS	CS		Classifying space	β	α	$\langle S_0 \rangle$	$\langle (\Delta S)^2 \rangle$
A	0	0	0	$\mathcal{C}_0$	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	2	N/A	0	σ_0^2
AIII	0	0	1	$\mathcal{C}_1$	$\mathrm{U}(N)$	2	1	0	$2\sigma_0^2$
AI	+1	0	0	$\mathcal{R}_0$	$\mathrm{O}(2N)/\mathrm{O}(N)\times\mathrm{O}(N)$	1	N/A	$-(\log 2-1/2)$	$2\sigma_0^2$
BDI	+1	+1	1	$\mathcal{R}_1$	$\mathrm{O}(N)$	1	0	$-((3/2)\log 2-1)$	$4\sigma_0^2$
D	0	+1	0	$\mathcal{R}_2$	$\mathrm{O}(2N)/\mathrm{U}(N)$	2	0	$(1-\log 2)/2$	$2\sigma_0^2$
DIII	−1	+1	1	$\mathcal{R}_3$	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	4	1	$(\log 2)/4$	σ_0^2
AII	−1	0	0	$\mathcal{R}_4$	$\mathrm{Sp}(2N)/\mathrm{Sp}(N)\times\mathrm{Sp}(N)$	4	N/A	$(\log 2-1/2)/2$	$\sigma_0^2/2$
CII	−1	−1	1	$\mathcal{R}_5$	$\mathrm{Sp}(N)$	4	3	$((3/2)\log 2-1)/2$	σ_0^2
C	0	−1	0	$\mathcal{R}_6$	$\mathrm{Sp}(N)/\mathrm{U}(N)$	2	2	$-(1-\log 2)/2$	$2\sigma_0^2$
CI	+1	−1	1	$\mathcal{R}_7$	$\mathrm{U}(N)/\mathrm{O}(N)$	1	1	$-(\log 2)/2$	$4\sigma_0^2$

Relevant to symmetry-enriched quantum chaos and thermalization in free fermions.

Liu, Kudler-Flam & **Kawabata**, PRB **108**, 085109 (2023)

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Altland-Zirnbauer (AZ) symmetry

- **Random free fermion (two-body complex Sachdev-Ye-Kitaev model)**

$$\hat{H} = \sum_{ij} \hat{c}_i^\dagger H_{ij} \hat{c}_j$$

Altland-Zirnbauer (AZ) symmetry classification

Altland & Zirnbauer, PRB **55**, 1142 (1997)

time reversal: $T^{-1} H^* T = H \quad (T^* T = \pm 1)$ (antiunitary)

particle hole: $C^{-1} H^* C = -H \quad (C^* C = \pm 1)$ (antiunitary)

chiral: $S^{-1} H S = -H \quad (S^2 = 1)$ (unitary)

$T, C, S:$ unitary matrices

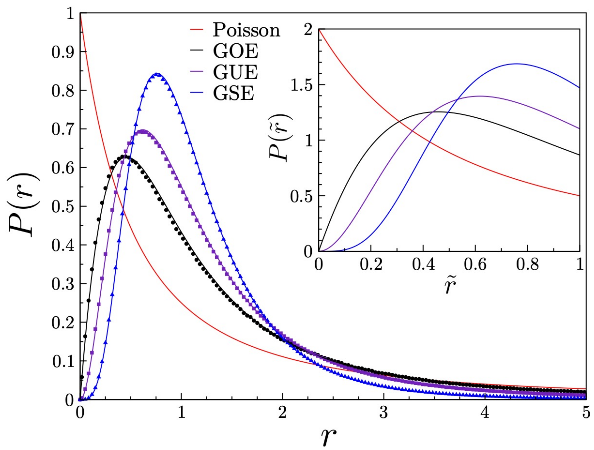
Tenfold internal symmetry classes for Hermitian matrices

Relevant to band insulators and superconductors

Random matrix theory

☆ AZ symmetry classifies the universality classes of Hermitian random matrices.

- **Time-reversal symmetry changes the bulk spectral correlations.**



Atas *et al.*, PRL **110**, 084101 (2013)

Threefold spectral correlations
(Wigner & Dyson)

- **Particle-hole or chiral symmetry changes the spectral correlations around zero eigenvalue.**

cf. chiral symmetry breaking in QCD — Verbaarschot, PRL **72**, 2531 (1994)

cf. stability of zero modes in superconductors — Beenakker, RMP **87**, 1037 (2015)

Random matrix theory

β: bulk (time-reversal symmetry)

AZ class	TRS	PHS	CS		Classifying space	β	α	$\langle S_0 \rangle$	$\langle (\Delta S)^2 \rangle$
A	0	0	0	$\mathcal{C}_0$	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	2	N/A	0	σ_0^2
AIII	0	0	1	$\mathcal{C}_1$	$\mathrm{U}(N)$	2	1	0	$2\sigma_0^2$
AI	+1	0	0	$\mathcal{R}_0$	$\mathrm{O}(2N)/\mathrm{O}(N)\times\mathrm{O}(N)$	1	N/A	$-(\log 2 - 1/2)$	$2\sigma_0^2$
BDI	+1	+1	1	$\mathcal{R}_1$	$\mathrm{O}(N)$	1	0	$-((3/2)\log 2 - 1)$	$4\sigma_0^2$
D	0	+1	0	$\mathcal{R}_2$	$\mathrm{O}(2N)/\mathrm{U}(N)$	2	0	$(1-\log 2)/2$	$2\sigma_0^2$
DIII	−1	+1	1	$\mathcal{R}_3$	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	4	1	$(\log 2)/4$	σ_0^2
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CII	−1	−1	1	$\mathcal{R}_5$	$\mathrm{Sp}(N)$	4	3	$((3/2)\log 2 - 1)/2$	σ_0^2
C	0	−1	0	$\mathcal{R}_6$	$\mathrm{Sp}(N)/\mathrm{U}(N)$	2	2	$-(1-\log 2)/2$	$2\sigma_0^2$
CI	+1	−1	1	$\mathcal{R}_7$	$\mathrm{U}(N)/\mathrm{O}(N)$	1	1	$-(\log 2)/2$	$4\sigma_0^2$

α: zero (chiral or particle-hole symmetry)

Topological insulators

☆ Periodic table of topological insulators and superconductors

AZ Symmetry				Dimension							
Class	TRS	PHS	CS	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	-1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	-1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	-1	1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Schnyder, Ryu, Furusaki & Ludwig, PRB **78**, 195125 (2008); NJP **12**, 065010 (2010)

Kitaev, AIP Conf. Proc. **1134**, 22 (2009)

Free-fermion Page curve (1)

We study typical entanglement entropy of random free fermions $\hat{H} = \sum_{ij} \hat{c}_i^\dagger H_{ij} \hat{c}_j$

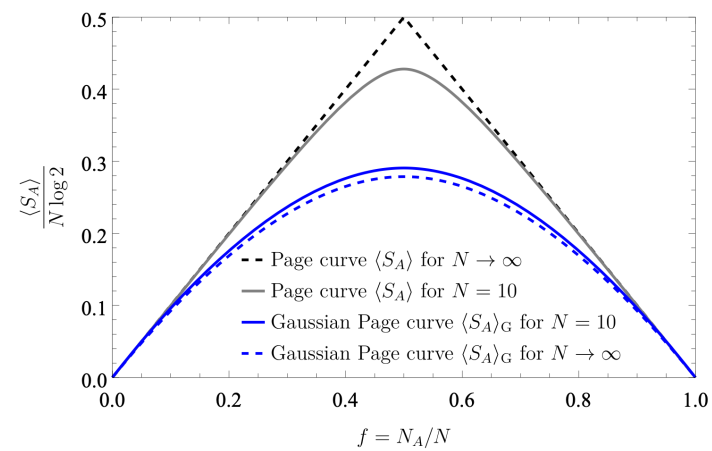
Single-particle eigenstates are randomly chosen by the Haar measure

- **Free-fermion (Gaussian) Page curve**

Liu, Chen & Balents, PRB **97**, 245126 (2018)

Bianchi, Hackl & Kieburg, PRB **103**, L241118 (2021)

Bianchi *et al.*, PRX Quantum **3**, 030201 (2022)



Free-fermion Page curve was studied without symmetry (i.e., class A or D)

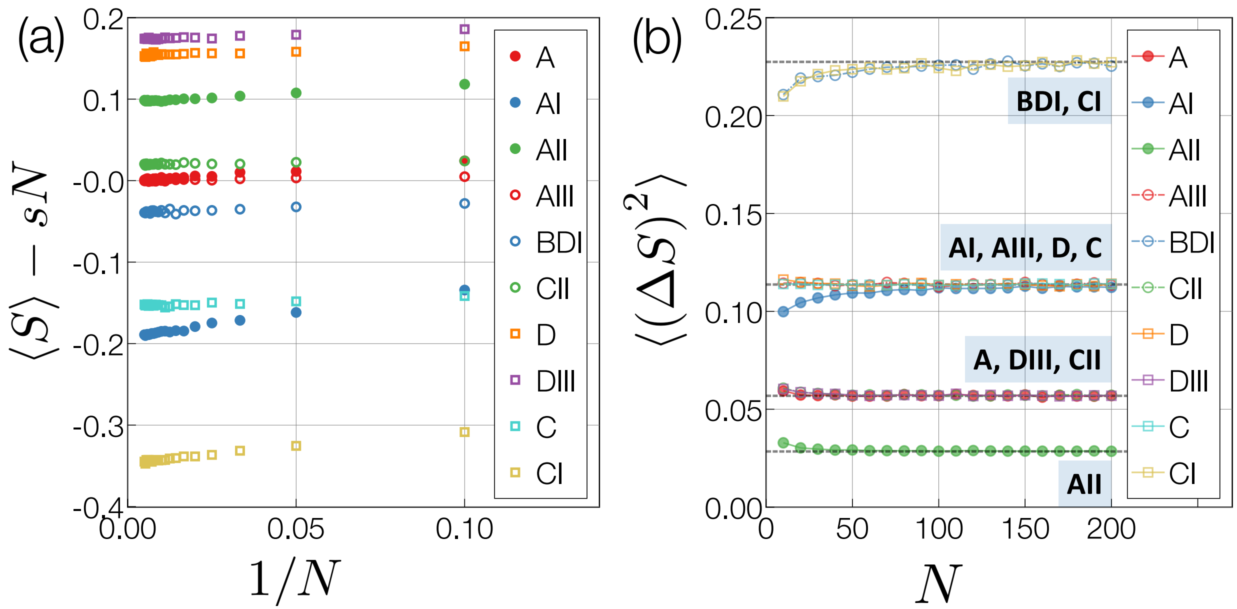
→ Effect of symmetry?

→ **We develop the tenfold classification of free-fermion Page curve based on AZ symmetry!**

Liu, Kudler-Flam & **Kawabata**, PRB **108**, 085109 (2023)

Free-fermion Page curve (2)

☆ **We numerically calculate the average and variance of entanglement entropy.**
(we focus on the half filling and half bipartition)



Tenfold different universal behavior!

☆ **We derive the tenfold Page curves analytically, by the Weingarten calculus for different symmetry classes.**

Standard (Wigner-Dyson) class (1)

☆ **Time-reversal symmetry** $T^{-1}H^*T = H \quad (T^*T = \pm 1)$

Class A: no symmetry → **unitary U(N)**

Class AI: TRS with +1 → **orthogonal O(N)**

Class AII: TRS with -1 → **symplectic Sp(N)**

(classifying spaces of single-particle eigenstates)

We take eigenstates Haar-randomly from U(N), O(N), and Sp(N)

$$\langle S \rangle = \left(\log 2 - \frac{1}{2} \right) \left(N + 1 - \frac{2}{\beta} \right) + o(1) \qquad \beta = \begin{cases} 1 & \text{(class AI)} \\ 2 & \text{(class A)} \\ 4 & \text{(class AII)} \end{cases}$$

The leading term ∝ N does not change even in the presence of TRS.

The constant term does depend on TRS.

Standard (Wigner-Dyson) class (2)

- **Constant term of the average entanglement entropy**

$$\langle S_0 \rangle = \left(1 - \frac{2}{\beta}\right)\left(\log 2 - \frac{1}{2}\right) \begin{cases} < 0 & (\text{class AI}) \\ = 0 & (\text{class A}) \\ > 0 & (\text{class AII}) \end{cases}$$

TRS with +1 decreases average entanglement

TRS with -1 increases average entanglement

- **Variance of entanglement entropy**

$$\langle (\Delta S)^2 \rangle = \frac{2}{\beta}\left(\frac{3}{4} - \log 2\right) + o(1)$$

$$\beta = \begin{cases} 1 & (\text{class AI}) \\ 2 & (\text{class A}) \\ 4 & (\text{class AII}) \end{cases}$$

TRS with +1 (-1) increases (decreases) the variance of entanglement entropy

Consistent with the level repulsion in random matrices

Chiral class

☆ **Chiral symmetry** $S^{-1}HS = -H \quad \left(S^2 = 1\right)$

$$H = \begin{pmatrix} 0 & h \\ h^{\dagger} & 0 \end{pmatrix} \qquad h \in \begin{cases} \mathrm{U}\left(N\right) & (\text{class AIII}) \\ \mathrm{O}\left(N\right) & (\text{class BDI}) \\ \mathrm{Sp}\left(N\right) & (\text{class CII}) \end{cases}$$

(depending on time-reversal symmetry)

- **Average entanglement entropy**

  The volume-law term does not change.

  The constant term does depend on chiral symmetry:

$$\langle S_0 \rangle = \left(1 - \frac{2}{\beta}\right)\left(\frac{3}{2}\log 2 - 1\right)$$

- **Variance of entanglement entropy**

$$\langle (\Delta S)^2 \rangle = \frac{4}{\beta}\left(\frac{3}{4} - \log 2\right) + o\left(1\right)$$ (twice larger than the standard class)

Bogoliubov-de Gennes (BdG) class

☆ **Particle-hole symmetry** $C^{-1} H^* C = -H \quad (C^* C = \pm 1)$

Classifying spaces: $\begin{cases} \mathrm{O}(2N)/\mathrm{U}(N) & (\text{class D}) \\ \mathrm{Sp}(N)/\mathrm{U}(N) & (\text{class C}) \\ \mathrm{U}(N)/\mathrm{O}(N) & (\text{class CI}) \\ \mathrm{U}(2N)/\mathrm{Sp}(N) & (\text{class DIII}) \end{cases}$

- **Classes D & C**

$$\langle S_0 \rangle = \frac{1}{2} (1 - \alpha)(1 - \log 2) \qquad \alpha = \begin{cases} 0 & (\text{class D}) \\ 2 & (\text{class C}) \end{cases}$$

(random-matrix index for zero eigenvalue)

- **Classes CI & DIII**

$$\langle S_0 \rangle = \frac{1}{2} \left(1 - \frac{2}{\beta} \right) \log 2 \qquad \beta = \begin{cases} 1 & (\text{class CI}) \\ 4 & (\text{class DIII}) \end{cases}$$

Liu, Kudler-Flam & **Kawabata**, PRB **108**, 085109 (2023)

☆ We develop the tenfold classification of free-fermion Page curves based on AZ symmetry!

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CII	−1	−1	1	$\mathcal{R}_5$	$\mathrm{Sp}(N)$	4	3	$((3/2)\log 2 - 1)/2$	σ_0^2
C	0	−1	0	$\mathcal{R}_6$	$\mathrm{Sp}(N)/\mathrm{U}(N)$	2	2	$-(1-\log 2)/2$	$2\sigma_0^2$
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Relevant to symmetry-enriched quantum chaos and thermalization in free fermions.

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Weingarten calculus (1)

The analytical approaches in the literature are not straightforward to generalize in the presence of symmetry.

→ **We use the symmetry-enriched versions of Weingarten calculus.**

Collins, Int. Math. Res. Not. **2003**, 953 (2003)

Collins & Śniady, Commun. Math. Phys. **264**, 773 (2006)

As a special feature of free fermions, the entanglement entropy is obtained from the single-particle correlation matrix: Peschel, J. Phys. A **36**, L205 (2003)

$$C_{ij} := \langle \Psi | \hat{c}_i^{\dagger} \hat{c}_j \underline{| \Psi \rangle}$$

(many-body eigenstate)

λ_i : eigenspectrum of C

(i.e., single-particle entanglement spectrum)

→ **average entanglement** $\langle S \rangle = \int_0^1 d\lambda \; s(\lambda) \, \underline{\langle D(\lambda) \rangle}$

density of the ES

$$s(\lambda) := -\lambda \log \lambda - (1-\lambda) \log (1-\lambda) \qquad D(\lambda) := \sum_i \delta(\lambda - \lambda_i)$$

Weingarten calculus (2)

The average density is obtained from the resolvent:

$$\langle D(\lambda) \rangle = -\frac{1}{\pi} \mathrm{Im} \lim_{\varepsilon \to 0^+} \underline{\langle R(\lambda + \mathrm{i}\varepsilon) \rangle}$$

resolvent

$$R(z) := \mathrm{Tr}\left(\frac{I}{zI - C}\right) = \mathrm{Tr}\left(\frac{I}{z}\right) + \sum_{n=1}^{\infty} \frac{\mathrm{Tr}\, C^n}{z^{n+1}}$$

Thus, the calculations of average entanglement reduce to $\underline{\langle \mathrm{Tr}\, C^n \rangle}$

$$\int dU \; U_{i_1, j_1} \cdots U_{i_n, j_n}$$

systematically calculable by the Weingarten calculus

☆ Depending on different classifying spaces in different symmetry classes, different types of the Weingarten functions are relevant.

→ Tenfold different values of typical entanglement entropy

Entanglement spectra

Standard class (A, AI & AII):

$$\langle D(\lambda)\rangle = \frac{N+1-2/\beta}{2\pi\sqrt{\lambda(1-\lambda)}}$$

Chiral class (AIII, BDI & CII):

$$\langle D(\lambda)\rangle = \frac{N+1-2/\beta}{\pi\sqrt{\lambda(1-\lambda)}} + \frac{1}{2}\left(1-\frac{2}{\beta}\right)\delta\left(\lambda-\frac{1}{2}\right)$$

BdG class (D, DIII, C & CI):

$$\langle D(\lambda)\rangle = \frac{N-(1-\alpha)/2}{\pi\sqrt{\lambda(1-\lambda)}} - \left(1-\frac{\alpha}{2}-\frac{1}{\beta}\right)\delta\left(\lambda-\frac{1}{2}\right)$$

Time-reversal symmetry:

global scaling of the entanglement spectrum

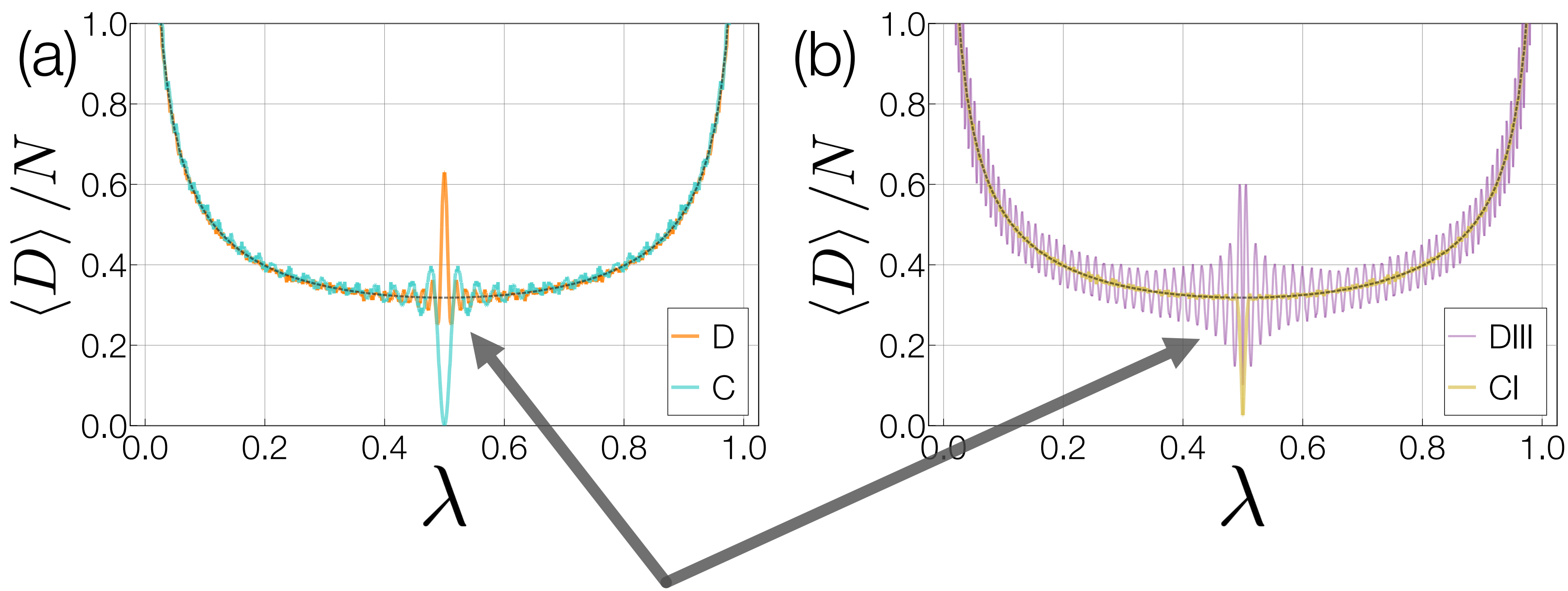
Chiral and particle-hole symmetries:

singular peaks at the center of the entanglement spectrum

(entanglement zero modes $\lambda = 1/2$)

Sachdev-Ye-Kitaev model

Numerical results of the density of the single-particle entanglement spectrum in the quadratic SYK model (N=100)



The singular peaks and dips are controlled by particle-hole symmetry

$$\begin{cases} +\left(1/2\right)\delta\left(\lambda-1/2\right) & [\text{class D } (\beta=2, \alpha=0)] \\ -\left(1/2\right)\delta\left(\lambda-1/2\right) & [\text{class C } (\beta=2, \alpha=2)] \end{cases}$$

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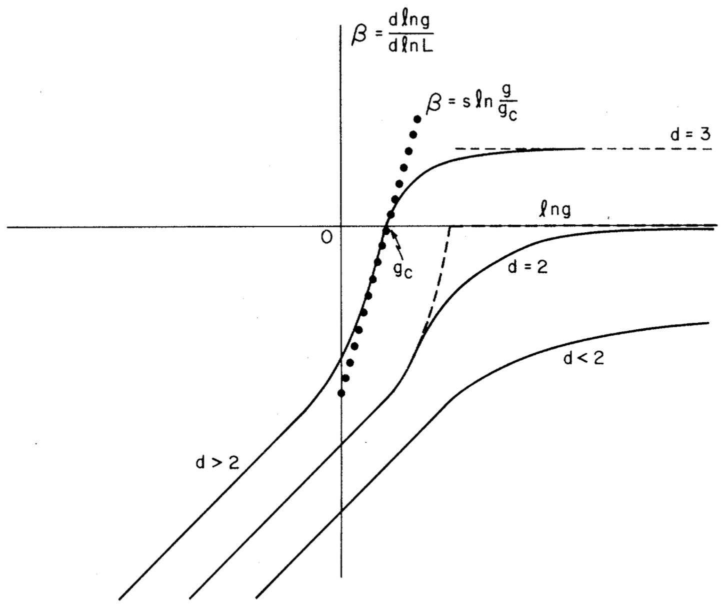
Anderson localization

☆ Typical entanglement has a similarity to mesoscopic transport phenomena!

- **Anderson localization and transition**

disorder induces localization of coherent waves Anderson, PR **109**, 1492 (1958)

Anderson localization also leads to a continuous phase transition



Abrahams *et al.*, PRL **42**, 673 (1979)

[scaling theory]

$$\beta(G) := \frac{d \log G}{d \log L}$$

G : conductance

L : length scale

Weak localization (quantum correction)

Perturbative expansion of the beta function around $G \to \infty$

$$\beta = d - 2 + \left(1 - \frac{2}{\beta}\right)\frac{1}{\pi^2 G} + o\left(1/G\right)$$

Drude term (semiclassical)

Quantum correction (relevant to Anderson transition)

Gor'kov *et al.*, JETP Lett. **30**, 228 (1979)
Altshuler *et al.*, PRB **22**, 5142 (1980)
Hikami *et al.*, PTP **63**, 707 (1980)

$$\beta = \begin{cases} 1 & (\text{class AI}) \\ 2 & (\text{class A}) \\ 4 & (\text{class AII}) \end{cases}$$

Average entanglement entropy (our present result):

$$\langle S \rangle = \left(\log 2 - \frac{1}{2}\right)\left(N + 1 - \frac{2}{\beta}\right) + o\left(1\right)$$

Same dependence! Originating from the same theoretical mechanism?

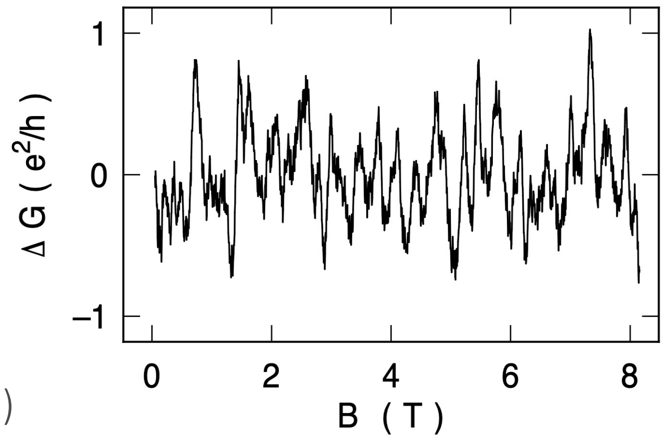
Universal conductance fluctuations

- **Universal conductance fluctuations**

  Conductance fluctuations in the diffusive regime:

$$\langle (\Delta G)^2 \rangle \propto \frac{1}{\beta}$$

Lee & Stone, PRL **55**, 1622 (1985)
Altshuler, JETP Lett. **41**, 648 (1985)



Washburn & Webb, Adv. Phys. **35**, 375 (1986)

Variance in entanglement entropy:

$$\langle (\Delta S)^2 \rangle = \frac{2}{\beta}\left(\frac{3}{4} - \log 2\right) \propto \frac{1}{\beta}$$

Same 1/β dependence!

We find that typical quantum entanglement has a similarity to mesoscopic transport phenomena in old condensed matter physics.

There, the transmission probability seems to hold a parallel role to the single-particle entanglement spectrum.

Mesoscopic transport phenomena: field theory of nonlinear sigma model

$$S[X] = \frac{1}{t} \int d^d r \; G_{AB} \underline{[X]} \partial_\mu X^A \partial_\mu X^B$$

target space: classified by AZ symmetry

Efetov, *Supersymmetry in Disorder and Chaos* (1996)

We may have a nonlinear sigma model description of typical entanglement.

Summary Phys. Rev. B 108, 085109 (2023)

- We develop the tenfold classification of typical quantum entanglement (free-fermion Page curve) based on Altland-Zirnbauer symmetry.
- It is relevant to characterization of symmetry-enriched quantum chaos.
- Typical entanglement exhibits a similarity to mesoscopic transport.

AZ class	TRS	PHS	CS		Classifying space	β	α	$\langle S_0 \rangle$	$\langle (\Delta S)^2 \rangle$
A	0	0	0	$\mathcal{C}_0$	$\mathrm{U}(2N)/\mathrm{U}(N)\times\mathrm{U}(N)$	2	N/A	0	σ_0^2
AIII	0	0	1	$\mathcal{C}_1$	$\mathrm{U}(N)$	2	1	0	$2\sigma_0^2$
AI	+1	0	0	$\mathcal{R}_0$	$\mathrm{O}(2N)/\mathrm{O}(N)\times\mathrm{O}(N)$	1	N/A	$-(\log 2 - 1/2)$	$2\sigma_0^2$
BDI	+1	+1	1	$\mathcal{R}_1$	$\mathrm{O}(N)$	1	0	$-((3/2)\log 2 - 1)$	$4\sigma_0^2$
D	0	+1	0	$\mathcal{R}_2$	$\mathrm{O}(2N)/\mathrm{U}(N)$	2	0	$(1-\log 2)/2$	$2\sigma_0^2$
DIII	−1	+1	1	$\mathcal{R}_3$	$\mathrm{U}(2N)/\mathrm{Sp}(N)$	4	1	$(\log 2)/4$	σ_0^2
AII	−1	0	0	$\mathcal{R}_4$	$\mathrm{Sp}(2N)/\mathrm{Sp}(N)\times\mathrm{Sp}(N)$	4	N/A	$(\log 2 - 1/2)/2$	$\sigma_0^2/2$
CII	−1	−1	1	$\mathcal{R}_5$	$\mathrm{Sp}(N)$	4	3	$((3/2)\log 2 - 1)/2$	σ_0^2
C	0	−1	0	$\mathcal{R}_6$	$\mathrm{Sp}(N)/\mathrm{U}(N)$	2	2	$-(1-\log 2)/2$	$2\sigma_0^2$
CI	+1	−1	1	$\mathcal{R}_7$	$\mathrm{U}(N)/\mathrm{O}(N)$	1	1	$-(\log 2)/2$	$4\sigma_0^2$