AdS/BCFT from Bootstrap Construction of Gravity with particle & brane

Caltech

Yuya Kusuki

Based on [2206.03035] & [2210.03107], a collaboration with Wei

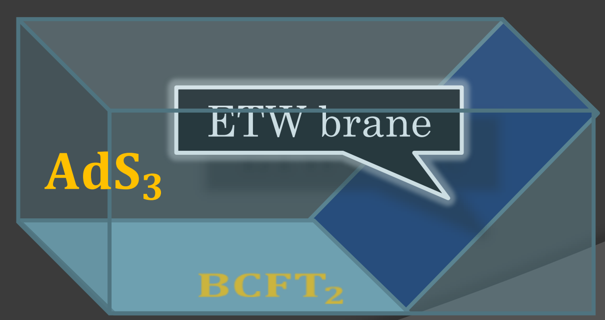
Contents

- Introduction
  - Issues in AdS/BCFT
  - Summary of Results
- Review
- Bootstrapping AdS/BCFT
- Construction of gravity with brane & particle
- Discussion

Gravity with brane

$$I_{grav} = -\frac{1}{16\pi G_N}\int_M d^3x\,\sqrt{g}(R-2\Lambda) + \sum_i m_i \int dl_i - \frac{1}{8\pi G_N}\int_Q d^2x\,\sqrt{h}(K-T)$$

Semiclassical gravity ($c = \frac{3}{2G_N} \gg 1$) with massive particles and ETW branes



AdS with ETW brane

$\partial(\mathrm{ETW}) = \mathrm{bdy.\ of\ CFT}$

Gravity with brane

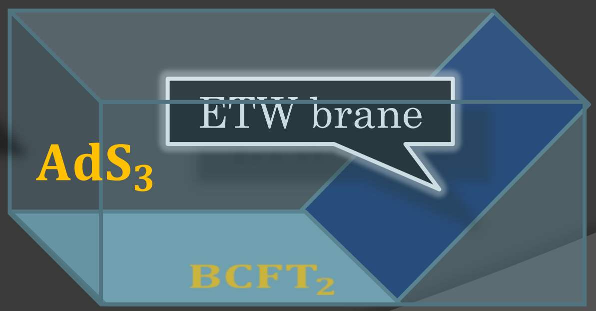
Tension:
Assuming for boundary matter Lagrangian to be constant

$$I_{grav} = -\frac{1}{16\pi G_N}\int_M d^3x\sqrt{g}(R-2\Lambda) + \sum_i m_i \int dl_i - \frac{1}{8\pi G_N}\int_Q d^2x\sqrt{h}(K-T)$$

Induced metric: $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$,
Extrinsic curvature: $K_{\mu\nu} = h_\mu^{\ \rho} h_\nu^{\ \lambda} \nabla_\rho n_\lambda$

Neumann b.c. is imposed on the brane (Einstein eq. of brane).

$$K_{ab} - K h_{ab} = -T h_{ab}$$



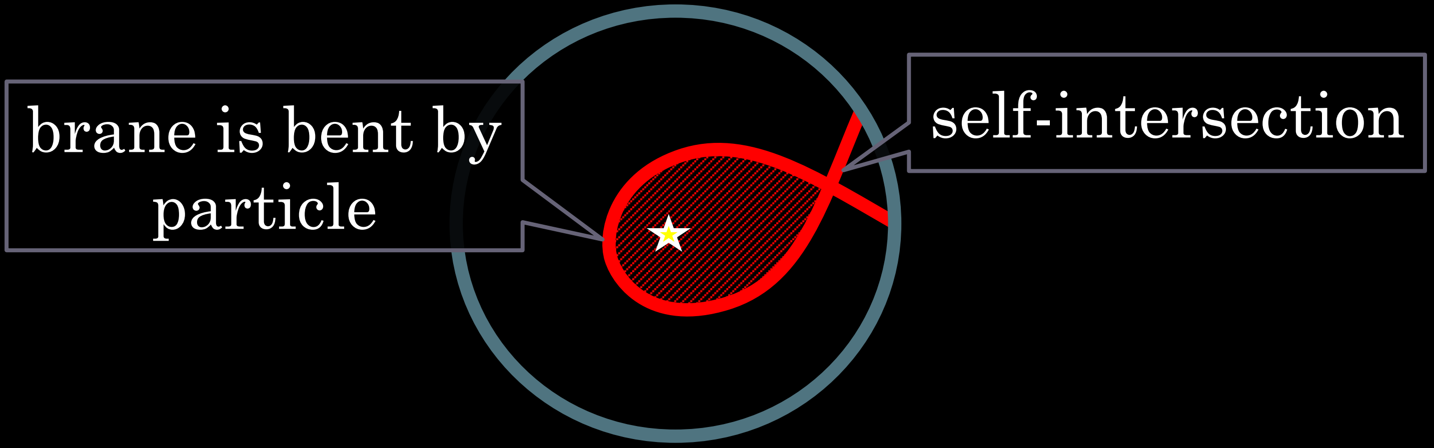
AdS with ETW brane
$\partial(\mathrm{ETW}) = \mathrm{bdy.\ of\ CFT}$

Gravity with brane

What is less understood?

gravity with brane & particle itself

- brane self-intersection (more explained later)

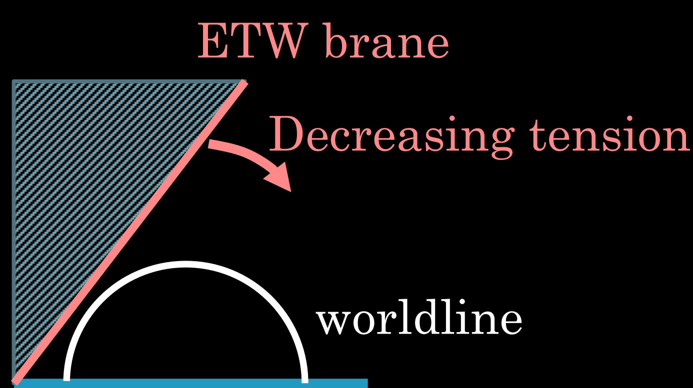


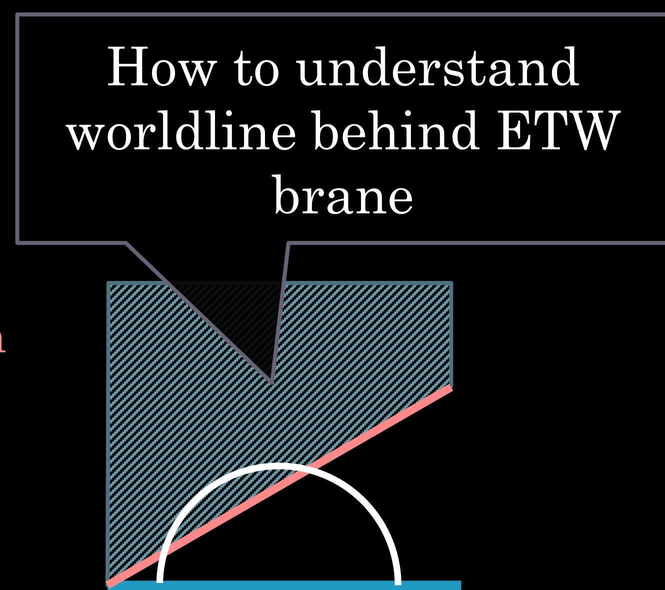
Gravity with brane

What is less understood?

gravity with brane & particle itself

- brane self-intersection
- negative tension brane



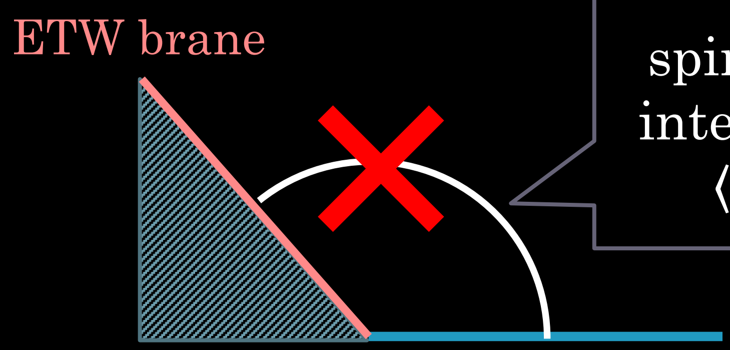


Gravity with brane

What is less understood?

gravity with brane & particle itself

- brane self-intersection
- negative tension brane
- how to deal with spinning particle



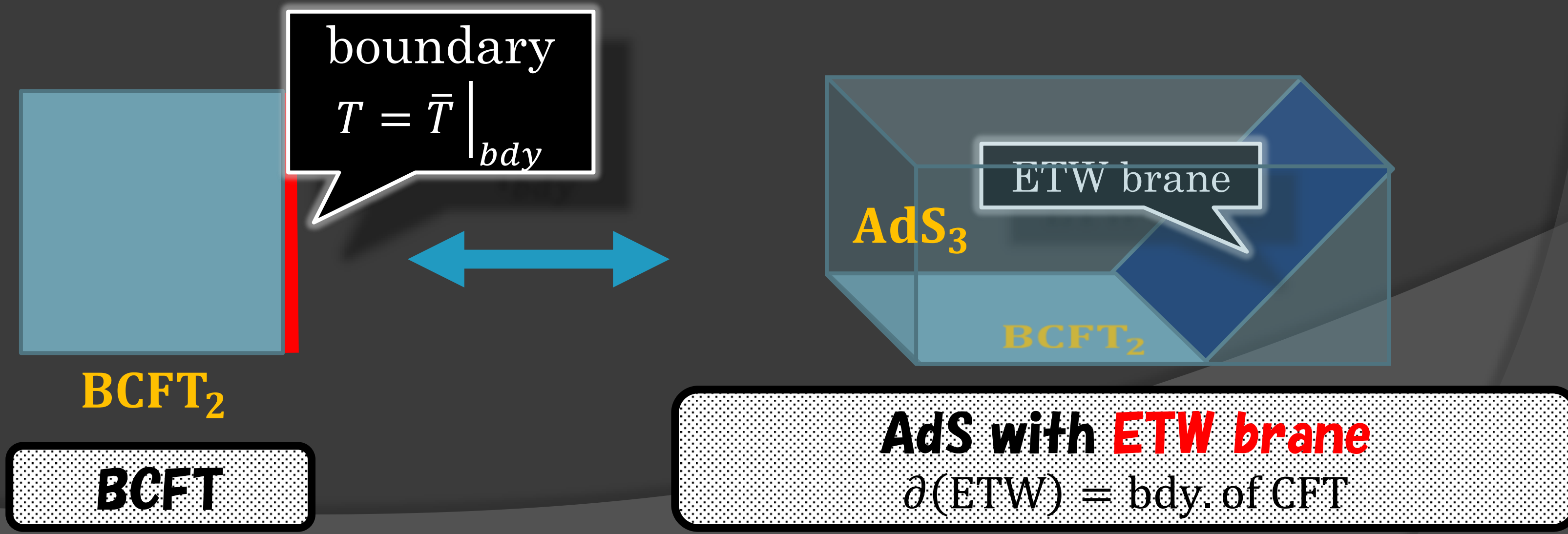
spinning particle cannot interact with brane since $\langle O \rangle_{disk} = 0$ if $h \neq \bar{h}$

AdS / BCFT

[Takayanagi]
[Fujita, Takayanagi, Tonni]

$$I_{grav} = -\frac{1}{16\pi G_N}\int_M d^3x\,\sqrt{g}(R-2\Lambda) + \sum_i m_i \int dl_i - \frac{1}{8\pi G_N}\int_Q d^2x\,\sqrt{h}(K-T)$$

Semiclassical gravity ($c = \frac{3}{2G_N} \gg 1$) with massive particles and ETW branes



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  - Review of BCFT
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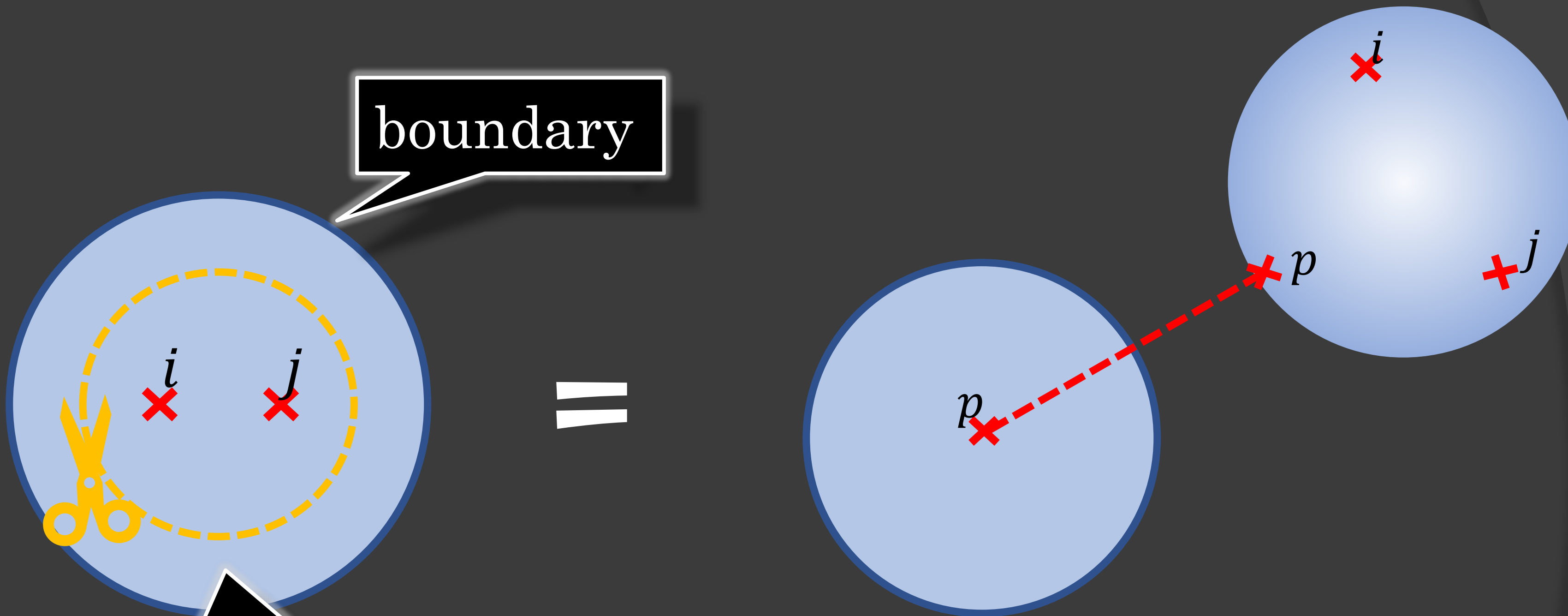
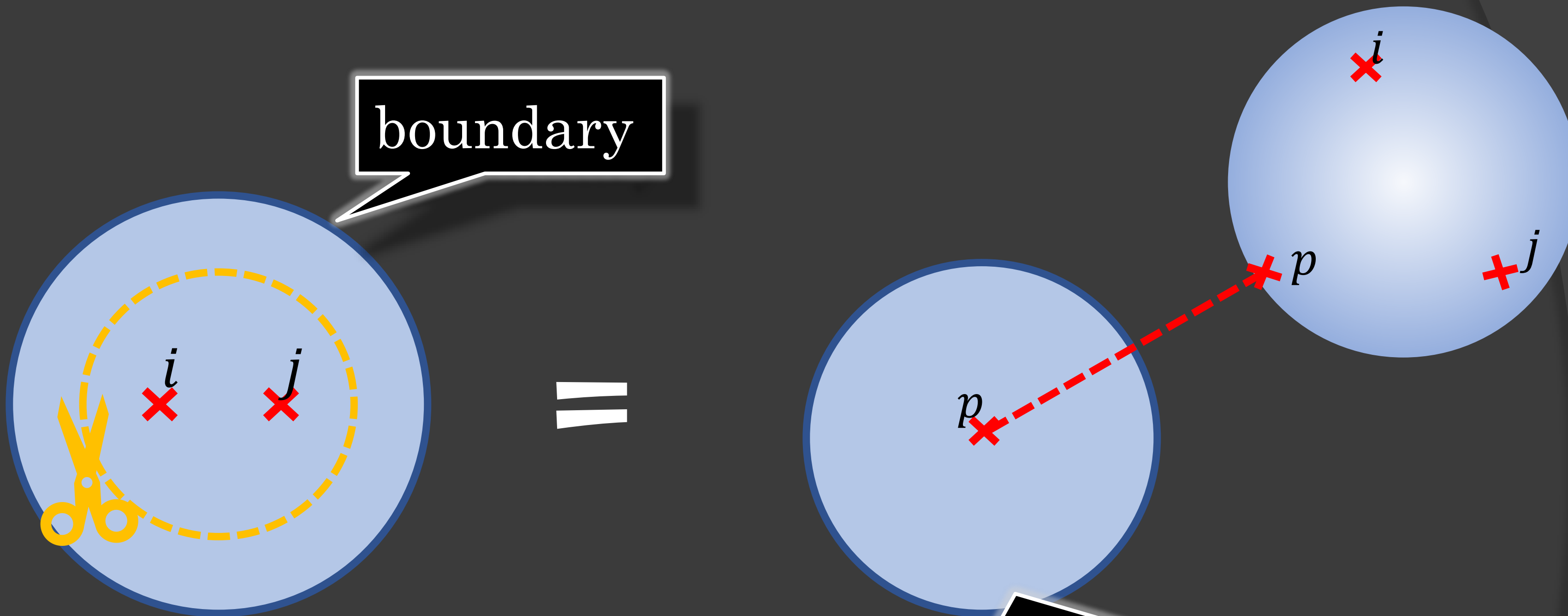
Review of BCFT [Lewellen]



Cutting:

Inserting (bulk operator) complete set

Review of BCFT [Lewellen]



$$\sum_p C_{p0} C_{ijp} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z)$$

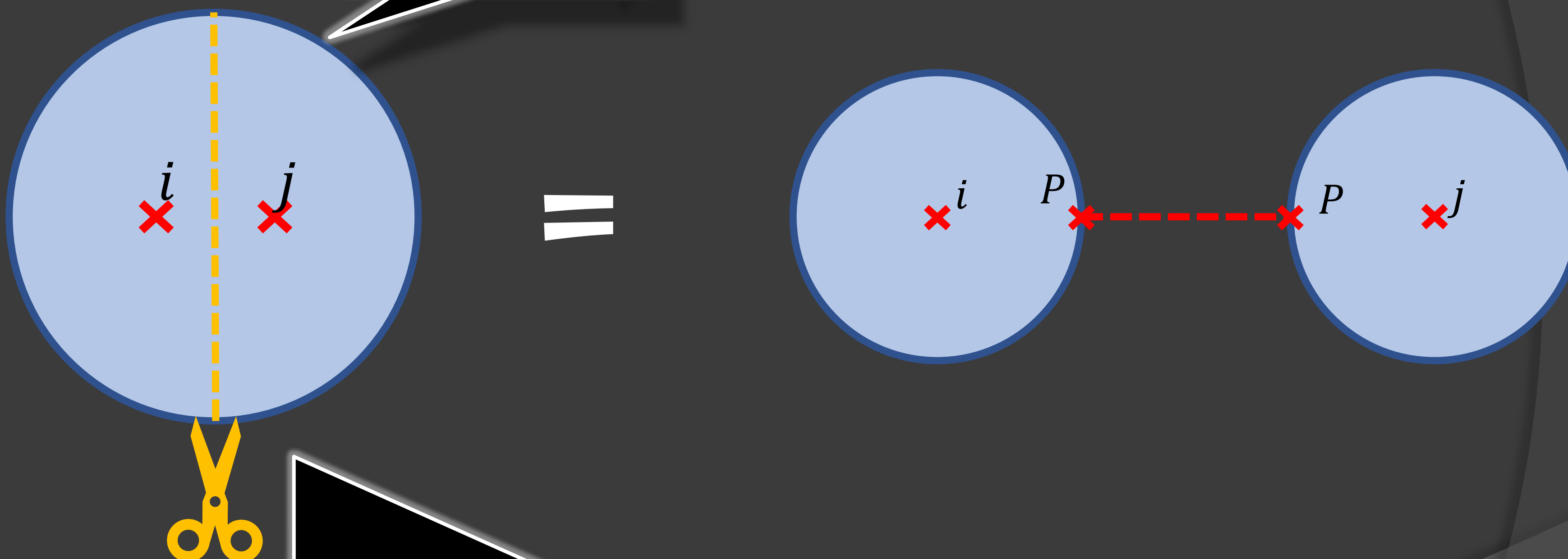
$\mathcal{F}_{\bar{j}\bar{i}}^{ji}$ is fixed by conformal sym. & mirror method

Review

or equivalently, using bulk-boundary OPE

$$\phi_i(z) \sim \sum_P C_{iP} (2\Im z)^{h_P - h_i - \bar{h}_i} \phi_P(\Re z) + \cdots$$

boundary

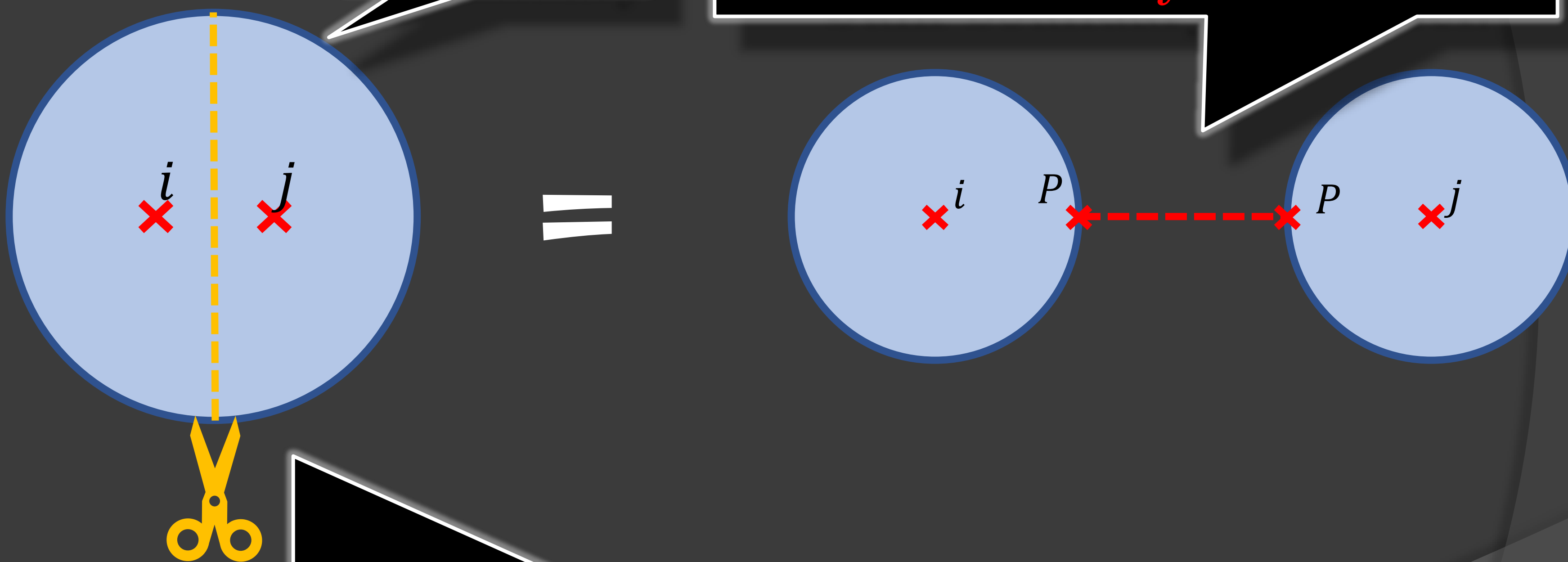


Cutting:

Inserting (boundary operator) complete set

Review of BCFT

boundary



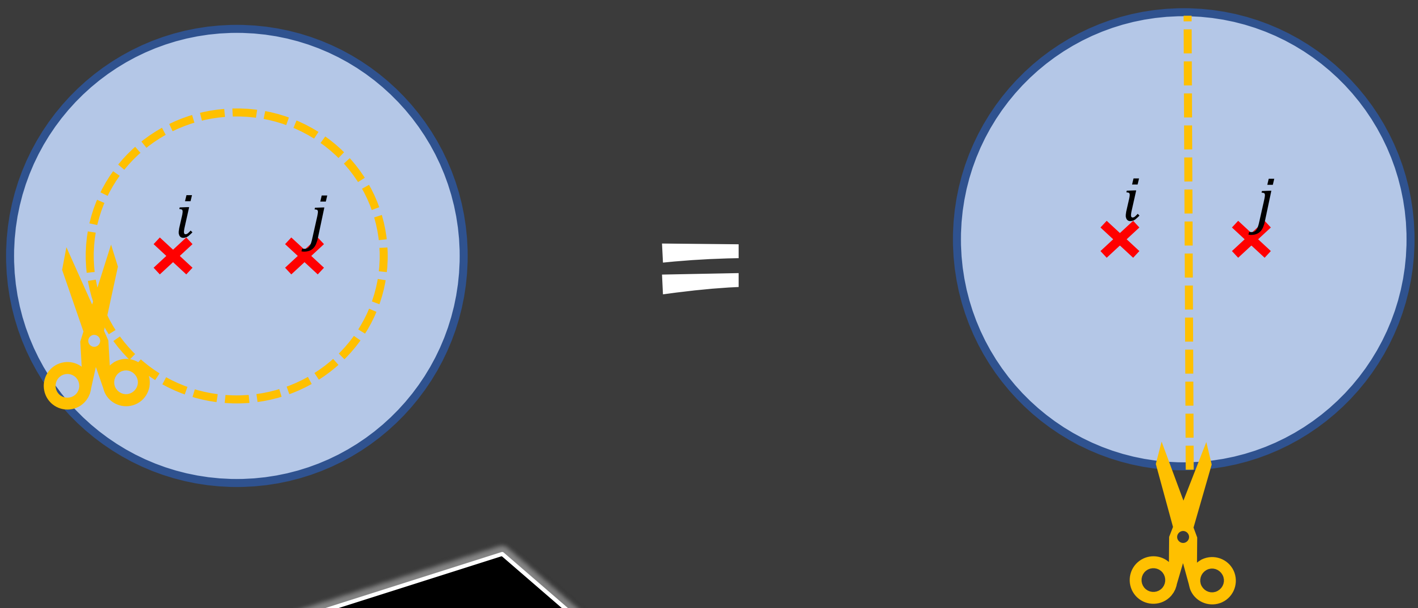
$$\sum_P C_{iP} C_{jP} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(P|z)$$

bulk-boundary OPE coef.

Cutting:

Inserting (boundary operator) complete set

Review of BCFT [Lewellen]



Consistency condition (conformal bootstrap equation)

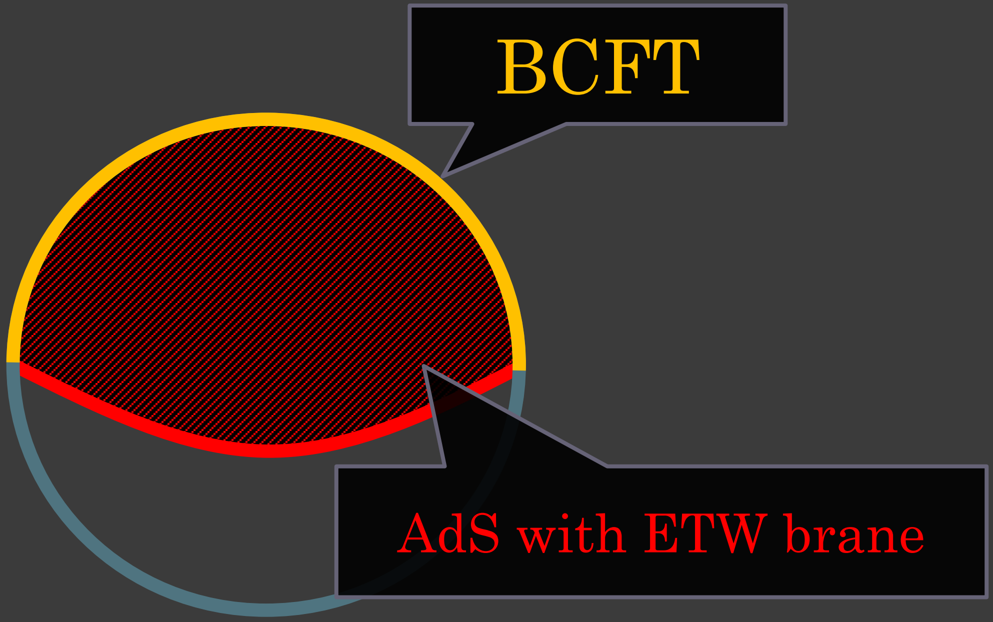
$$\sum_{p} C_{p0} C_{ijp} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z) = \sum_{P} C_{iP} C_{jP} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(P|z)$$

→ constraints on CFT data

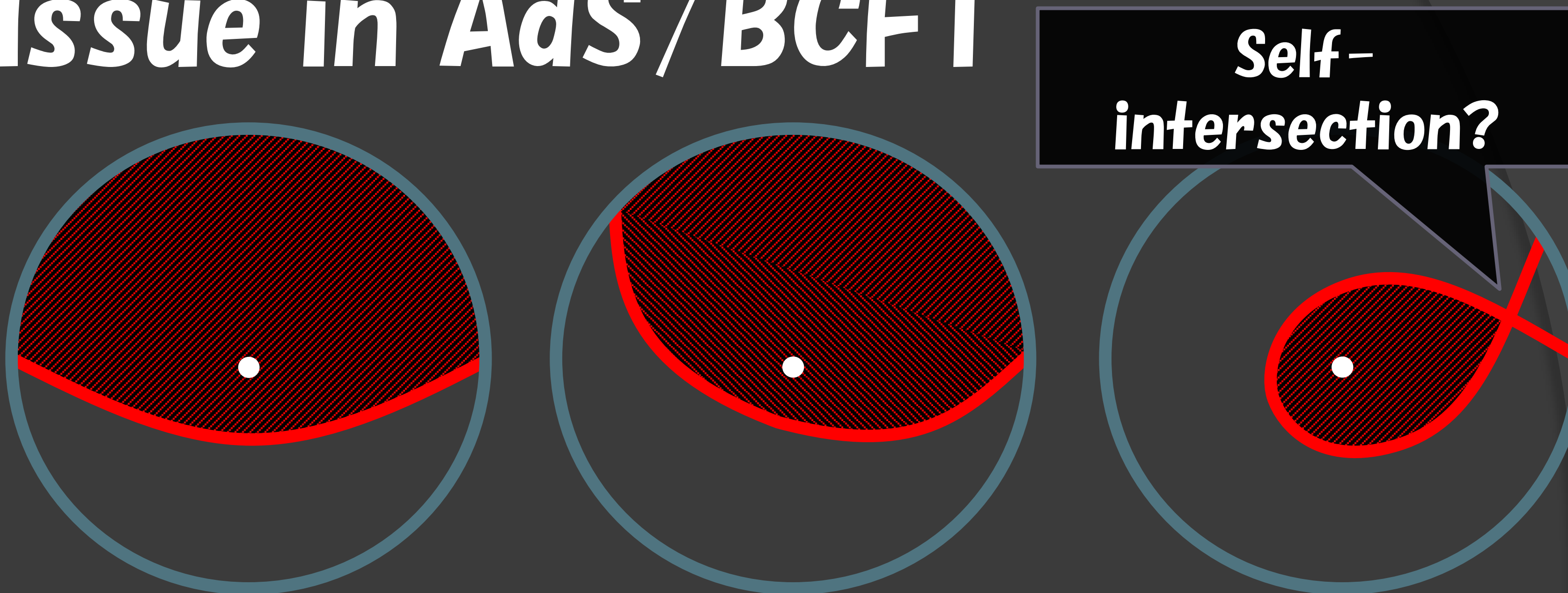
Contents

- Introduction
- Review
- Bootstrapping AdS/BCFT
  - How to bootstrap AdS/BCFT?
  - Results from bootstrap
- Construction of gravity with brane & particle
- Discussion

Issue in AdS/BCFT



Issue in AdS/BCFT



$h_i = 0$ $\quad 0 < h_i < \frac{c}{32}$ $\quad \frac{c}{32} < h_i$

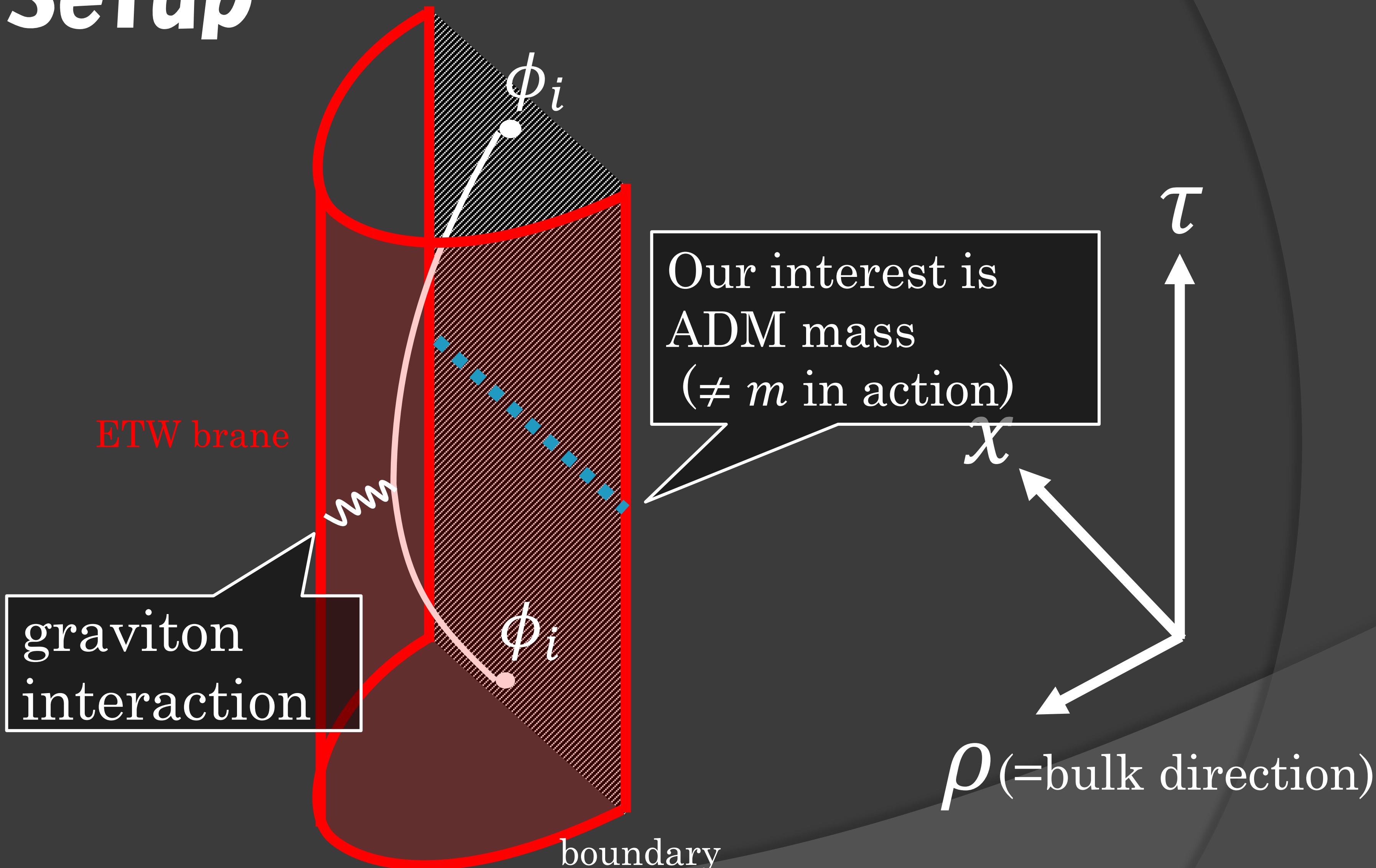
Massive particle produces deficit angle

$$\delta\theta = 8\pi G_N m$$
$$= 2\pi\left(1 - \sqrt{1 - \frac{c}{24}h_i}\right)$$

Pointed out by
[Geng, Lust, Mishra, Wakeham]
[Kawamoto, Mori, Suzuki, Takayanagi]
[Bianchi, De Angelis, Meineri]

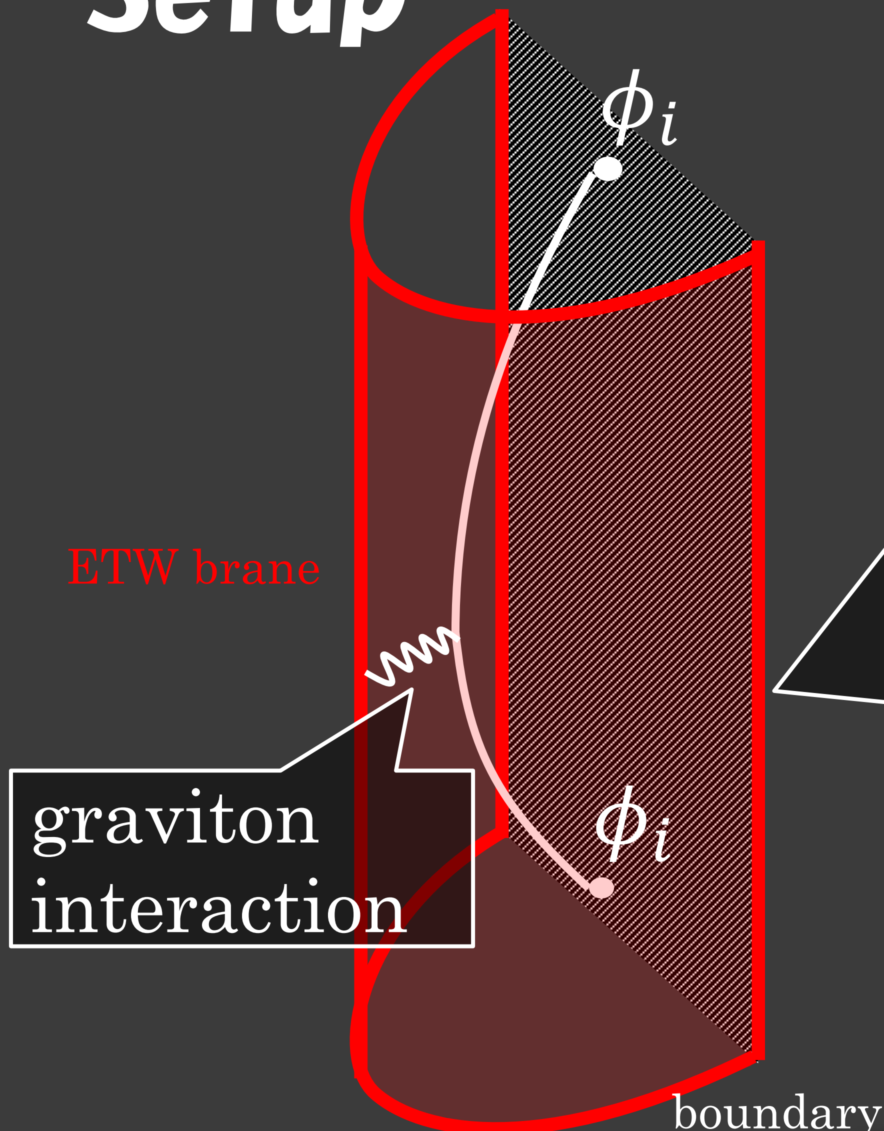
The first one proposed that $h_i \in \left[\frac{c}{32}, \frac{c}{24}\right)$ should be excluded in holographic CFT

Setup



ϕ_i
ϕ_i
Our interest is ADM mass ($\neq m$ in action)
ETW brane
graviton interaction
boundary
τ
x
ρ (=bulk direction)

Setup



Q. What is input to solve bootstrap?

A. No interaction between particle and brane, except for gravitons. (No matter-brane interaction term in action)

Bootstrap

Property of this solution to Einstein's equation:

No interaction between particle and brane, except for gravitons.
[Takayanagi], [Fujita, Takayanagi, Tonni], [Suzuki, Takayanagi]

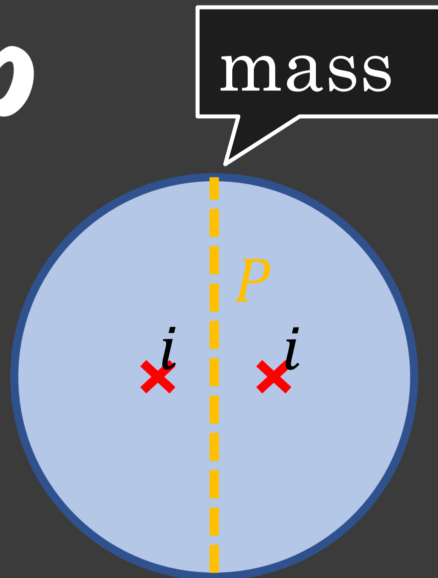
CFT counterpart:

For states $\{p\}$ in OPE between ϕ_is, (in large c)

$$C^{a}_{p\mathbb{I}} = \delta_{p\mathbb{I}}$$

Note: This is possible at least in the case $p \neq \bar{p}$.

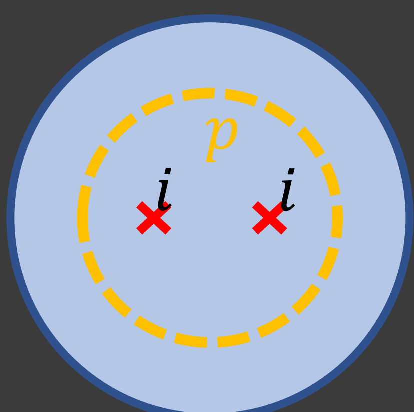
Bootstrap



$$= \sum_{P} C_{iP} C_{jP} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(P|z)$$

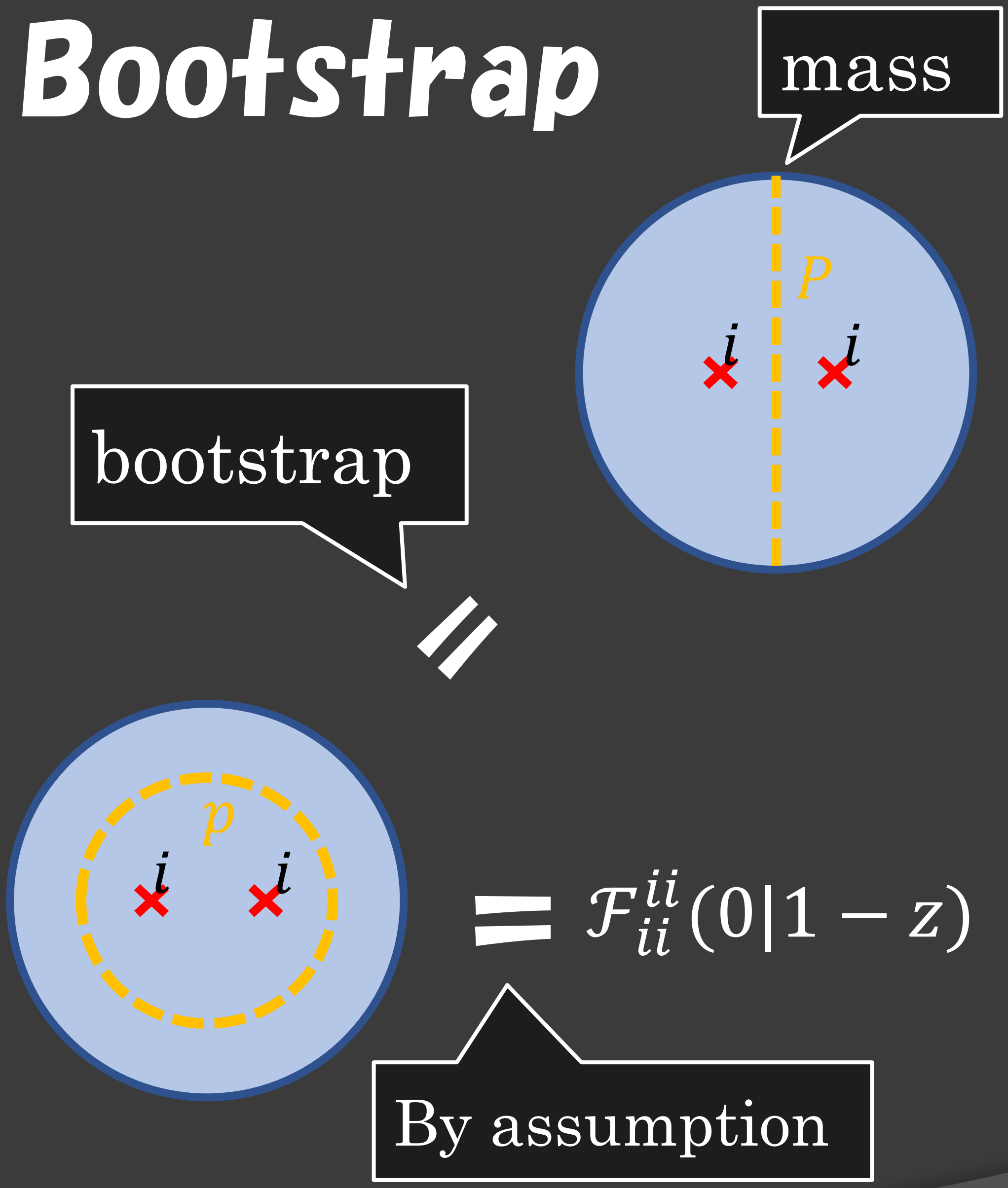
bootstrap

=



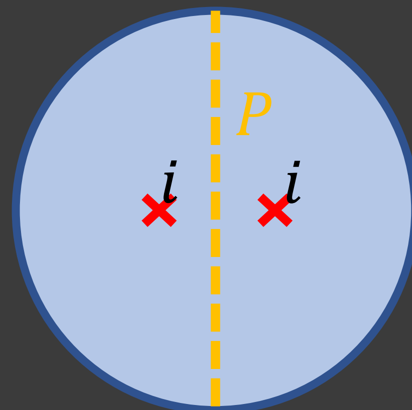
$$= \sum_{p} C_{p0} C_{iip} \mathcal{F}^{ii}_{ii}(p|1-z)$$

Bootstrap



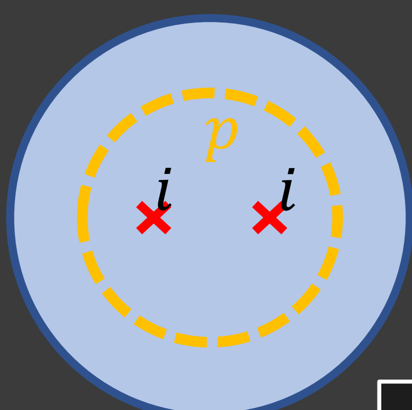
Bootstrap

mass



bootstrap

$$\parallel$$



$$= \mathcal{F}_{ii}^{ii}(0|1-z) = \int \mathrm{d}\alpha_P \, F_{0P}\begin{bmatrix} i & i \\ i & i \end{bmatrix} \mathcal{F}_{ii}^{ii}(P|z)$$

Liouville momentum

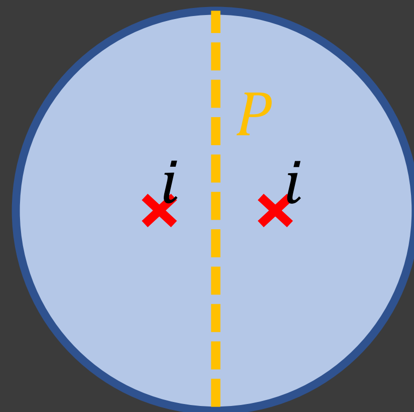
$c = 1 + 6Q^2,$
$h_i = \alpha_i(Q - \alpha_i)$

By assumption

Fusion transformation

Bootstrap

mass



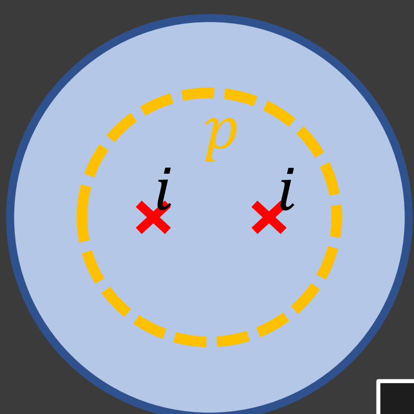
Now it is expressed in terms of the same basis

$$\sum_P C_{iP} C_{jP} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(P|z)$$

It is possible to extract OPE coef. by the coefficient comparison.

bootstrap

$=$ $=$



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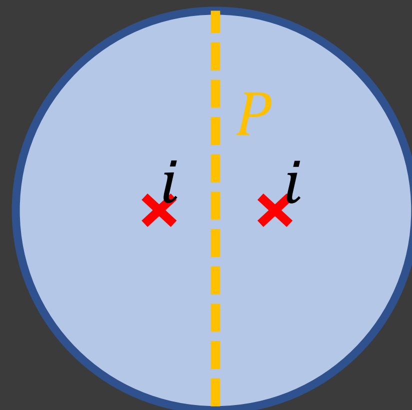
By assumption

Fusion transformation

Bootstrap

mass



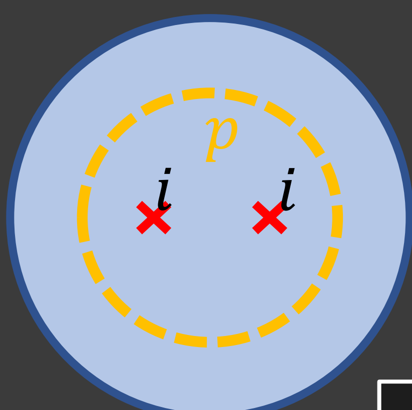
Now it is expressed in terms of the same basis

$$\sum_{P} C_{iP} C_{jP} \mathcal{F}^{ji}_{\bar{j}\bar{i}}(P|z)$$

It is possible to extract the spectrum from the support of the fusion kernel.

bootstrap

$\|$ $\|$



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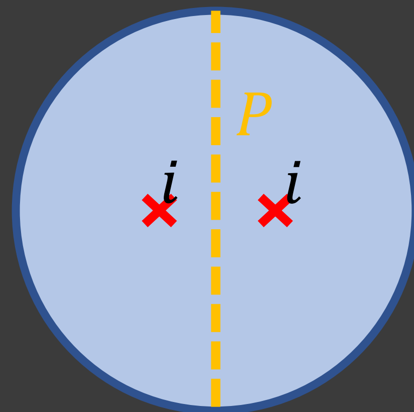
Fusion transformation

Bootstrap

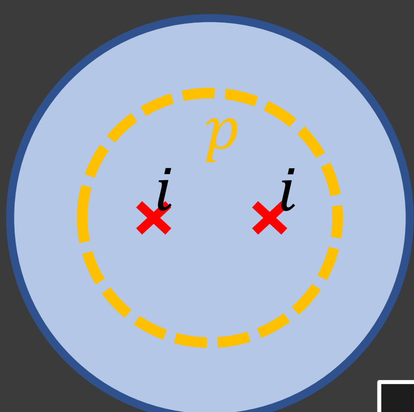
mass



ADM mass = lowest primary dimension

$$\alpha_P = 2\alpha_i$$

bootstrap



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By assumption

Fusion transformation

Implication [YK]

$c = 1 + 6Q^2$,
$h_i = \alpha_i(Q - \alpha_i)$

Relation between ADM mass & mass of particle

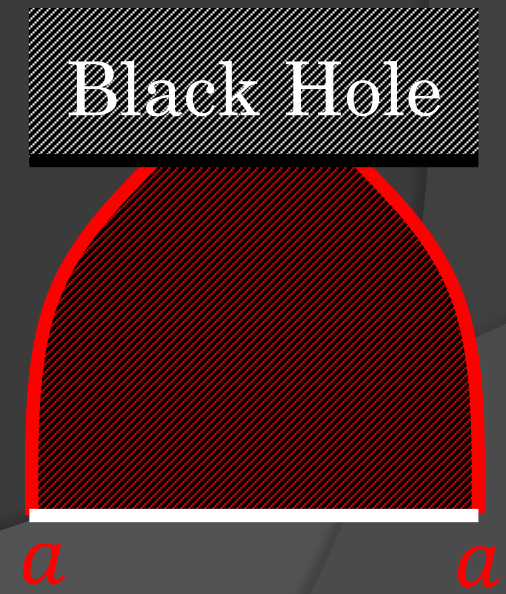
$$h_{ADM} = \alpha_P(Q - \alpha_P), \qquad \alpha_P = 2\alpha_i$$

It implies that black hole forms when

$$h_i \geq \frac{c}{32} \quad \Leftrightarrow \quad h_P \geq \frac{c}{24} \text{ (BTZ threshold)}$$

This completely matches self-intersection bound

→ self-intersection can be avoided by blackhole formation



More results [YK, Wei]

The bootstrap also tells us the following theorems,

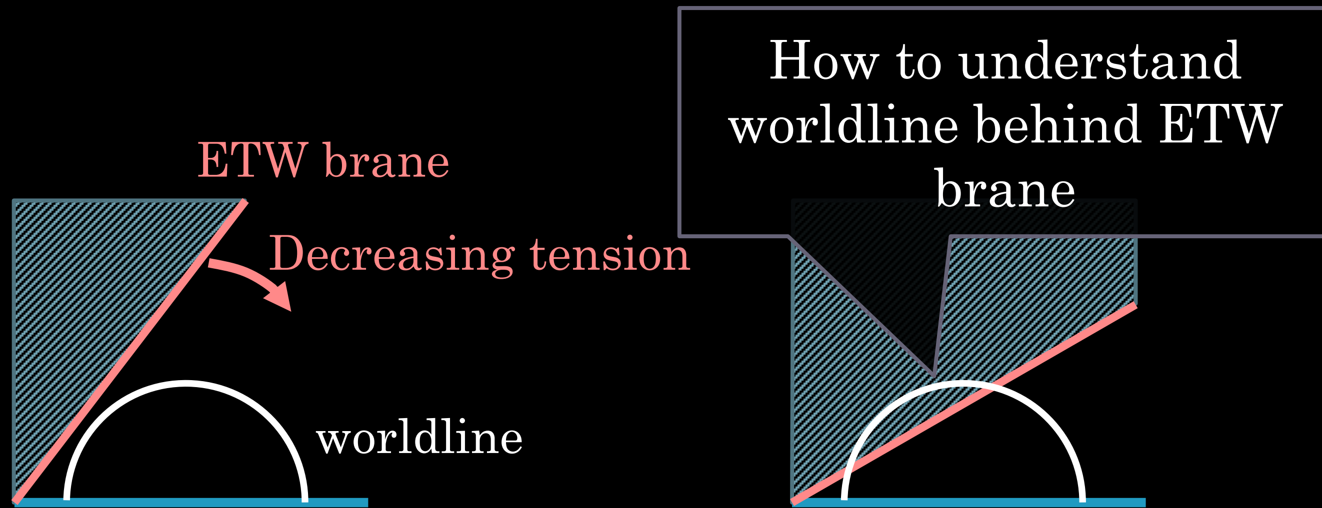
Relation between ADM mass & mass of spinning particle

$$\alpha_P = \alpha_i + \bar{\alpha}_i$$

Non-sensitivity to brane tension

The relation between ADM mass & particle mass is true even if brane tension is negative.

Negative tension brane



Transition?

[Bianchi, De Angelis, Meineri] has proposed that the boundary primary spectrum should be changed if the tension is negative,
and also proposed that this transition can be found by bootstrap.

$\Rightarrow$ Bootstrap answers “no transition”

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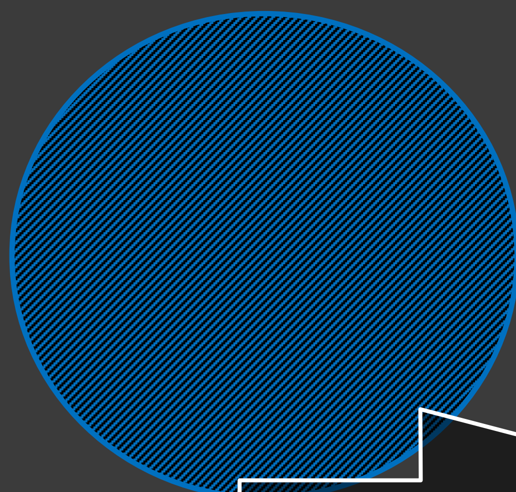
  - Gravity with spinning particle
  - Gravity with negative tension brane
- Discussion

Cut & Paste construction

How can we construct a conical defect geometry?
→ very simple way by cut & paste



Time slice on

$$ds^2 = (1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2$$

Cut & Paste construction

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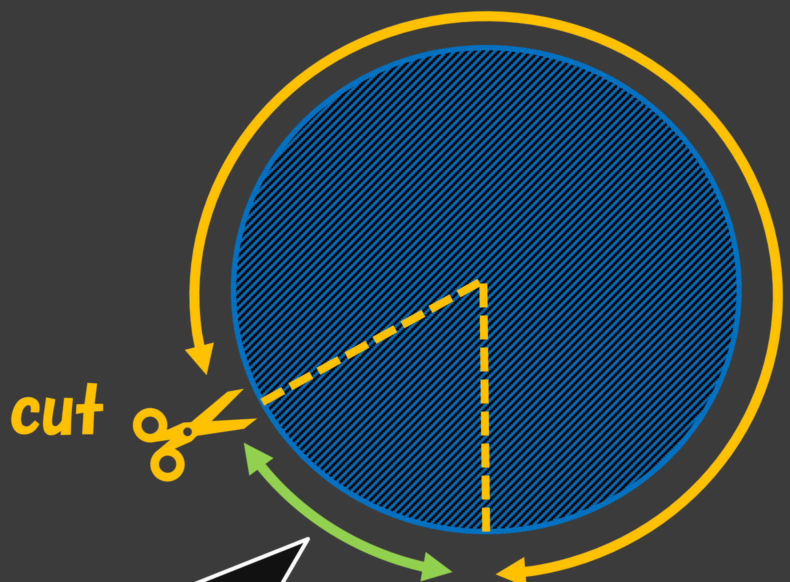
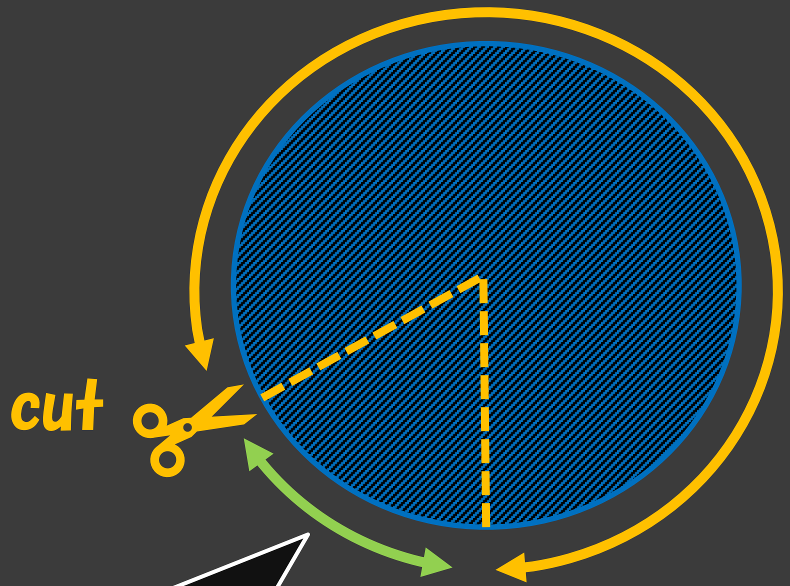


Circumference of asymptotic boundary
$2\pi\chi$

Deficit angle
$\delta\theta = 8\pi G_N m$

Cut & Paste construction

$$I_{grav} = -\frac{1}{16\pi G_N}\int_M d^3x\,\sqrt{g}(R-2\Lambda) + \sum_i m_i \int dl_i - \frac{1}{8\pi G_N}\int_Q d^2x\,\sqrt{h}(K-T)$$



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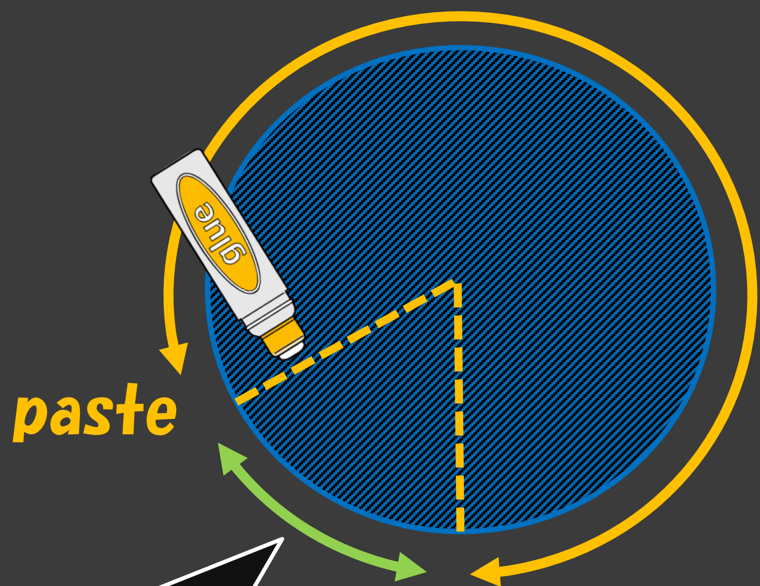

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Deficit angle
$\delta\theta = 8\pi G_N m$

Cut & Paste construction

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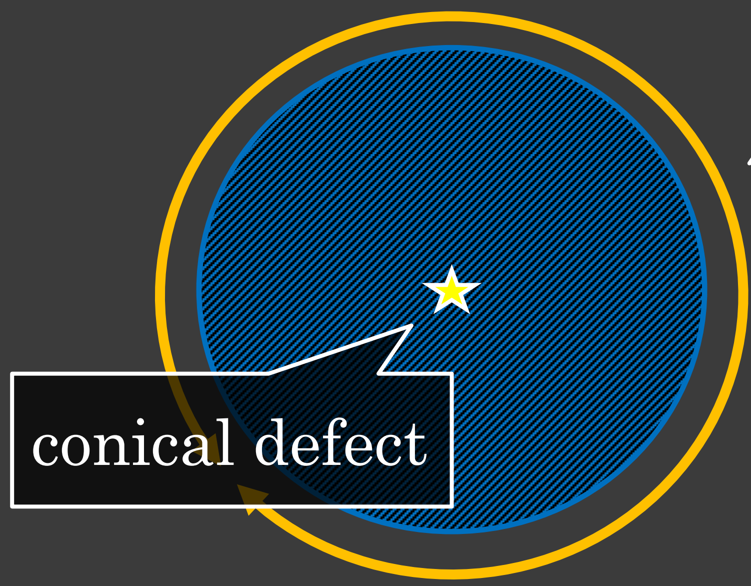
Circumference of asymptotic boundary

2π



Rescale to compare with conformal dimension

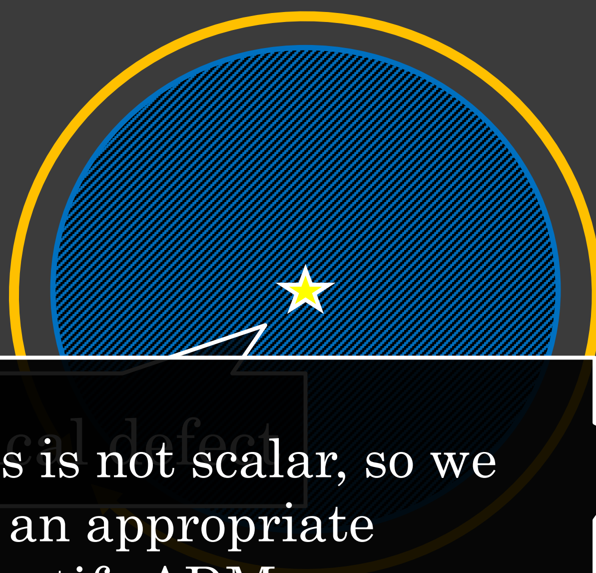
$$\theta \to \theta' = \frac{1}{\chi}\theta$$

$$t \to t' = \frac{1}{\chi}t$$

Cut & Paste construction

How can we construct a conical defect geometry?
→ very simple way by cut & paste

Circumference of asymptotic boundary
2π



Note: ADM mass is not scalar, so we should consider an appropriate coordinate to identify ADM mass as conformal dimension

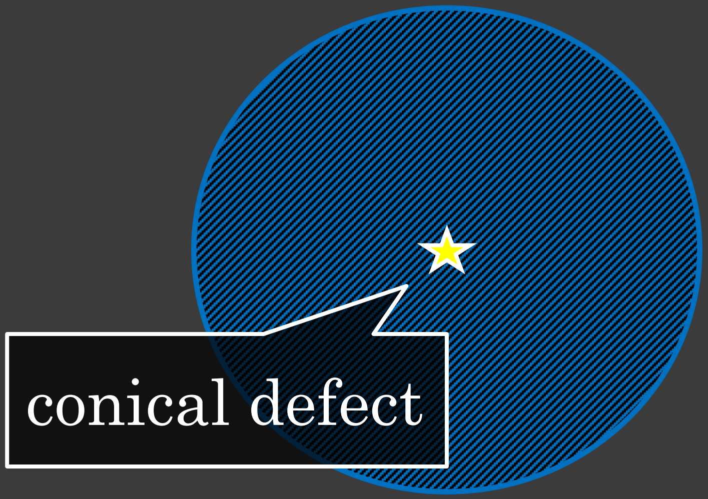
Rescale to compare with conformal dimension

$$\theta \to \theta' = \frac{1}{\chi}\theta$$

$$t \to t' = \frac{1}{\chi}t$$

Cut & Paste construction

How can we construct a conical defect geometry?
→ very simple way by cut & paste



$$E_{ADM} = \int_0^{2\pi} d\theta \, T_{tt} = -\frac{\chi^2}{8G_N}$$

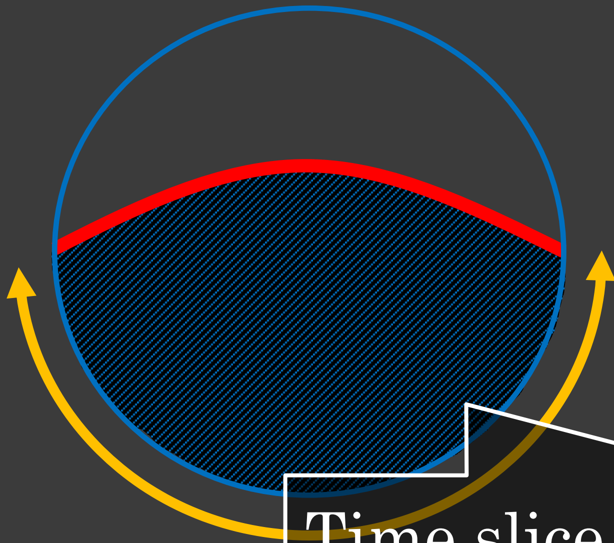
This leads to the well-known relation,

$$E_{ADM} + E_{Casimir} = 2h_i$$

Cut & Paste construction

How can we construct a conical defect geometry with a brane?

→ cut & paste in AdS/BCFT



Circumference of asymptotic boundary

π

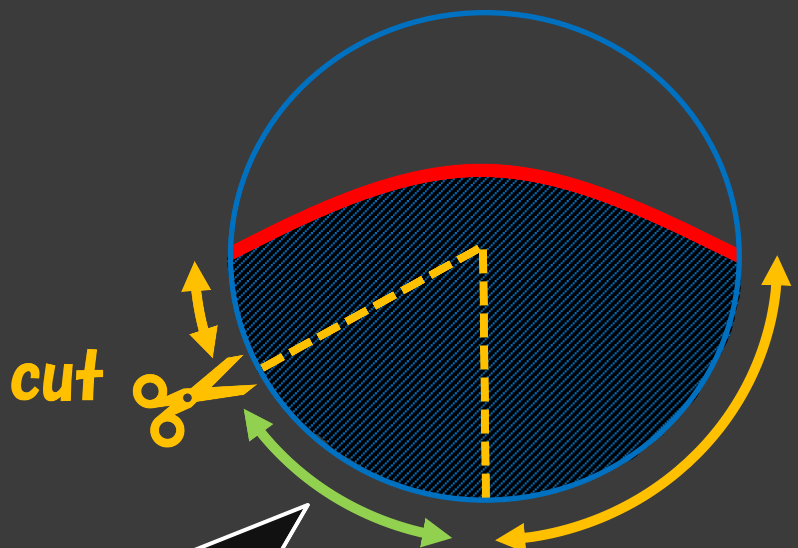
Time slice on

$$ds^2 = (1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2$$

Cut & Paste construction

How can we construct a conical defect geometry with a brane?

→ cut & paste in AdS/BCFT



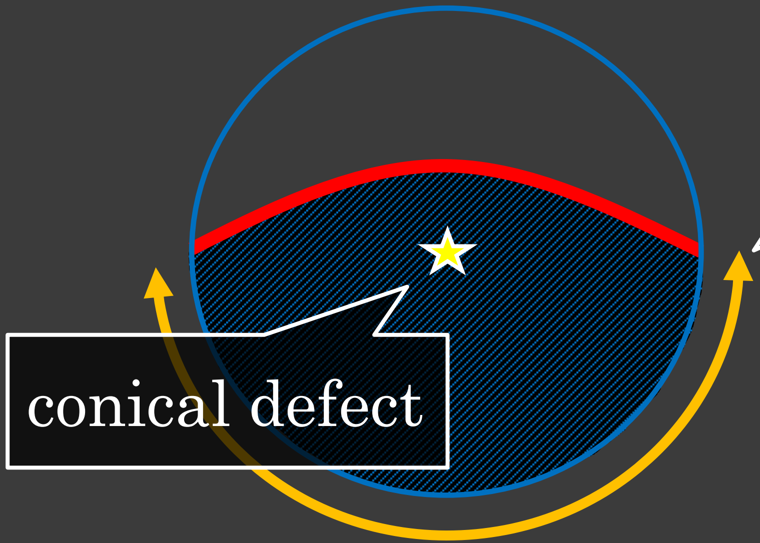
Circumference of asymptotic boundary
$\pi(2\chi - 1)$

Deficit angle
$\delta\theta = 8\pi G_N m$

Cut & Paste construction

How can we construct a conical defect geometry with a brane?

$\rightarrow$ cut & paste in AdS/BCFT



Circumference of asymptotic boundary

$$\pi$$

Rescale to compare with conformal dimension

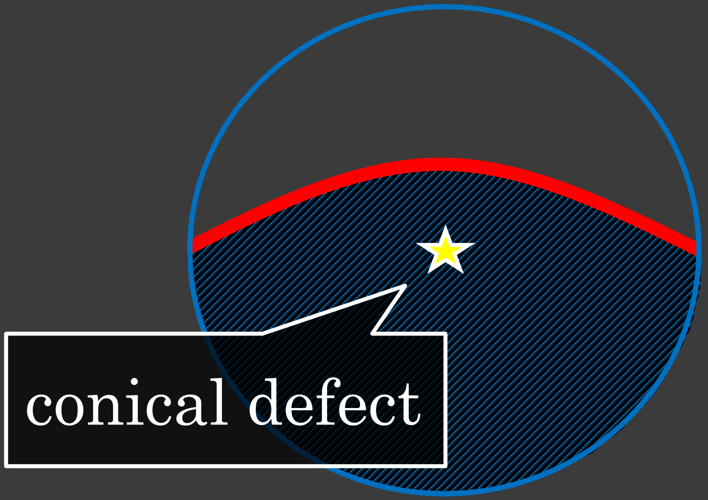
$$\theta \rightarrow \theta' = \frac{1}{2\chi - 1}\theta$$

$$t \rightarrow t' = \frac{1}{2\chi - 1}t$$

Cut & Paste construction

How can we construct a conical defect geometry with a brane?

→ cut & paste in AdS/BCFT



$$E_{ADM} = \int_0^{2\pi} d\theta \, T_{tt} = -\frac{(2\chi - 1)^2}{16G_N}$$

This leads to

$$E_{ADM} + E_{Casimir} = 2\alpha_i(Q - 2\alpha_i) \neq 2h_i$$

Particle is attracted close to brane by gravity force. This interaction changes the ADM mass.

Implication [YK]

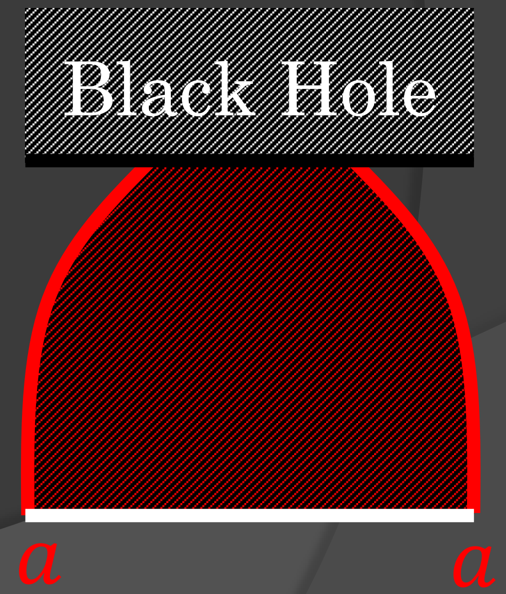
$c = 1 + 6Q^2,$
$h_i = \alpha_i(Q - \alpha_i)$

Relation between ADM mass & mass of particle

$$h_{ADM} = \alpha_P(Q - \alpha_P), \qquad \alpha_P = 2\alpha_i$$

It implies that black hole forms when

$$h_i \geq \frac{c}{32} \quad \Leftrightarrow \quad h_P \geq \frac{c}{24} \text{ (BTZ threshold)}$$



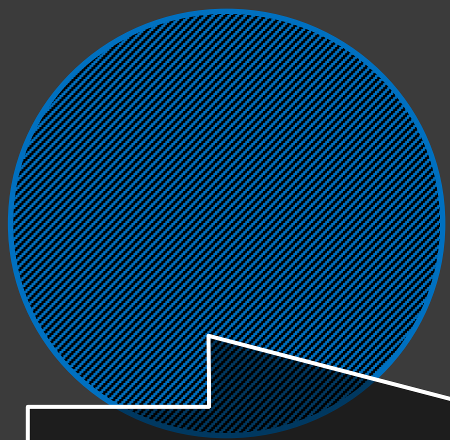
This completely matches self-intersection bound

→ self-intersection can be avoided by blackhole formation

Cut & Paste construction

How can we construct a spinning defect geometry?

→ cut & twisted paste



Time slice on

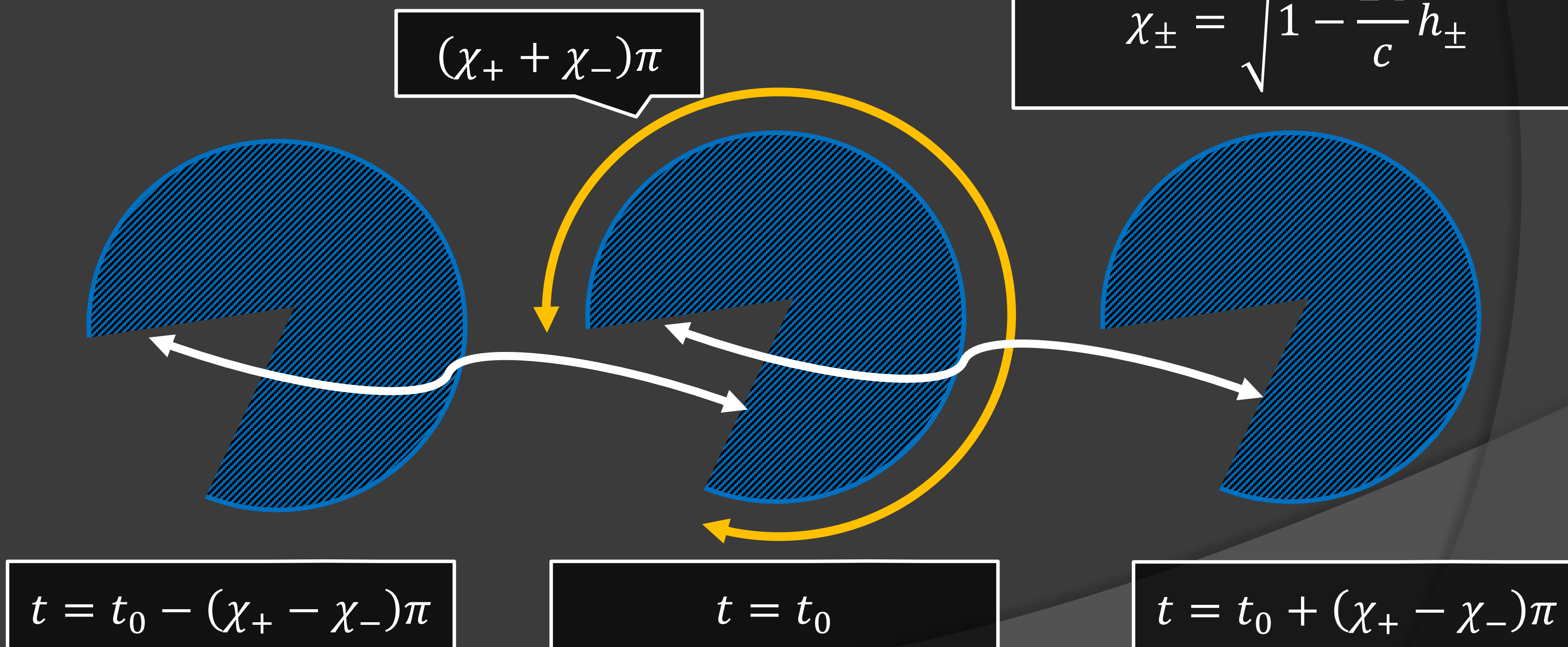
$$ds^2 = (1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2$$

Cut & Paste construction

How can we construct a spinning defect geometry?

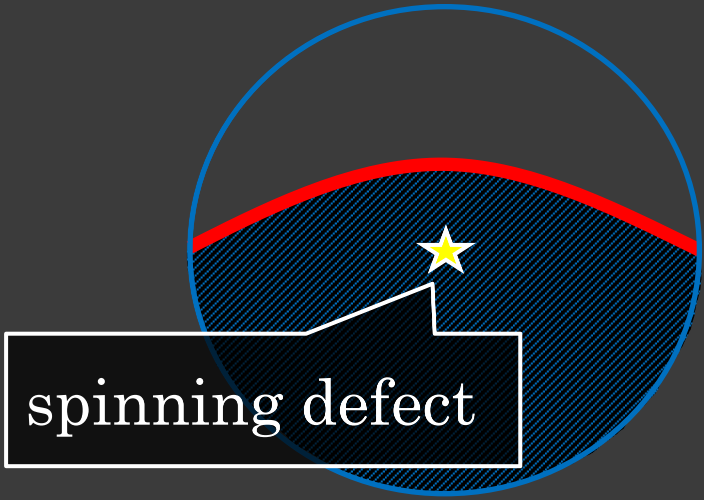
→ cut & twisted paste

$$\chi_\pm = \sqrt{1 - \frac{24}{c} h_\pm}$$



Cut & Paste construction

How can we construct a spinning defect geometry?
→ cut & twisted paste



By this construction, we obtain the self-intersection bound in the spinning defect geometry,

$$(\chi_+ + \chi_i)\pi < \pi$$

This matches the black hole threshold predicted from bootstrap.

More results [YK, Wei]

The bootstrap also tells us the following theorems,

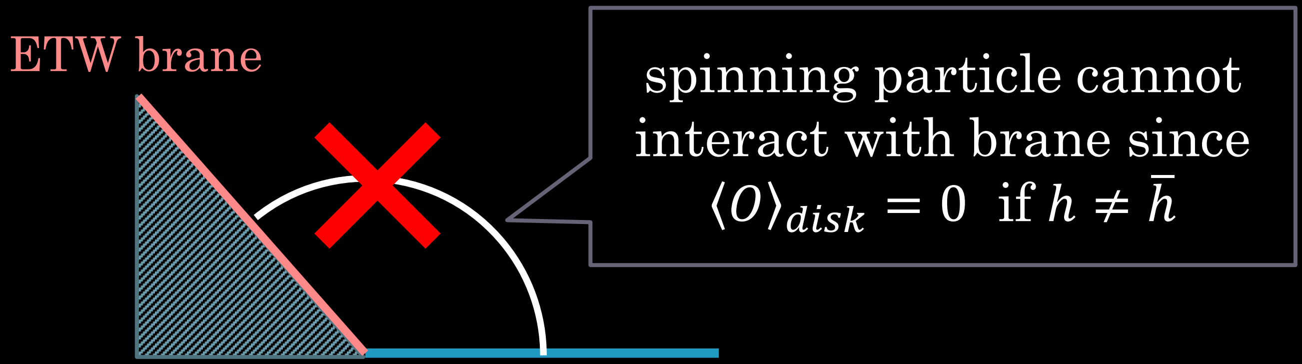
Relation between ADM mass & mass of spinning particle

$$\alpha_P = \alpha_i + \bar{\alpha}_i$$

Then, the black hole threshold is

$$\alpha_i + \bar{\alpha}_i = \frac{Q}{2}$$

One-point function

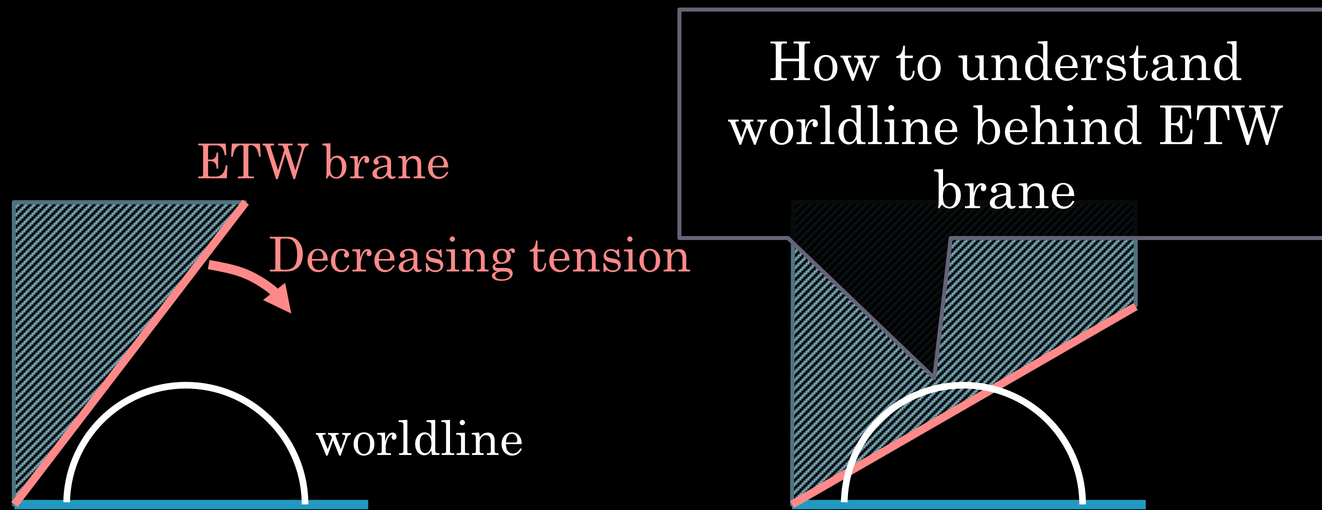


Twisted identification leads to mismatch of brane.
Such a singular configuration is not a solution.
This explains

$$\langle O \rangle_{disk} = 0 \quad \text{if } h \neq \bar{h}$$

from the gravity side.

Negative tension brane



Transition?

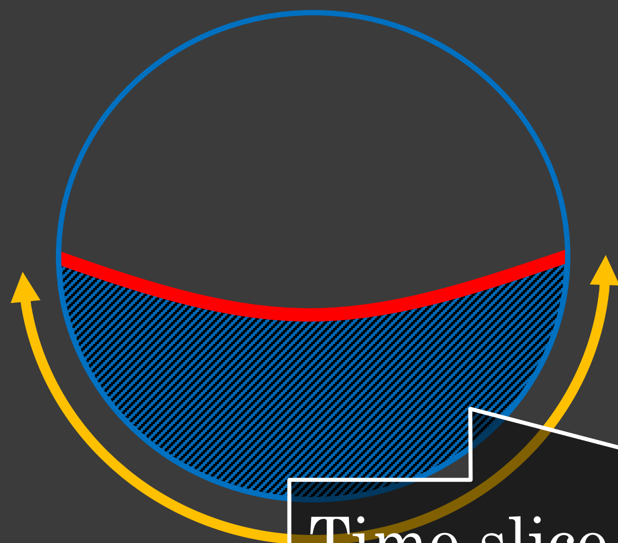
[Bianchi, De Angelis, Meineri] has proposed that the boundary primary spectrum should be changed if the tension is negative,
and also proposed that this transition can be found by bootstrap.

$\Rightarrow$ Bootstrap answers "no transition"

Cut & Paste construction

How can we construct a conical defect geometry with a negative tension brane?

$\to$ cut & paste in AdS/BCFT



Circumference of asymptotic boundary

π

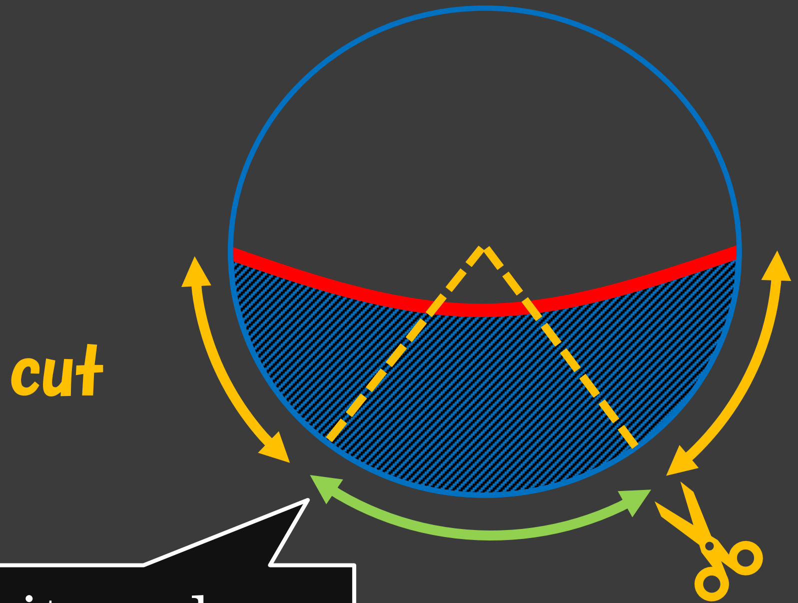
Time slice on

$$ds^2 = (1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2$$

Cut & Paste construction

How can we construct a conical defect geometry with a negative tension brane?

→ cut & paste in AdS/BCFT



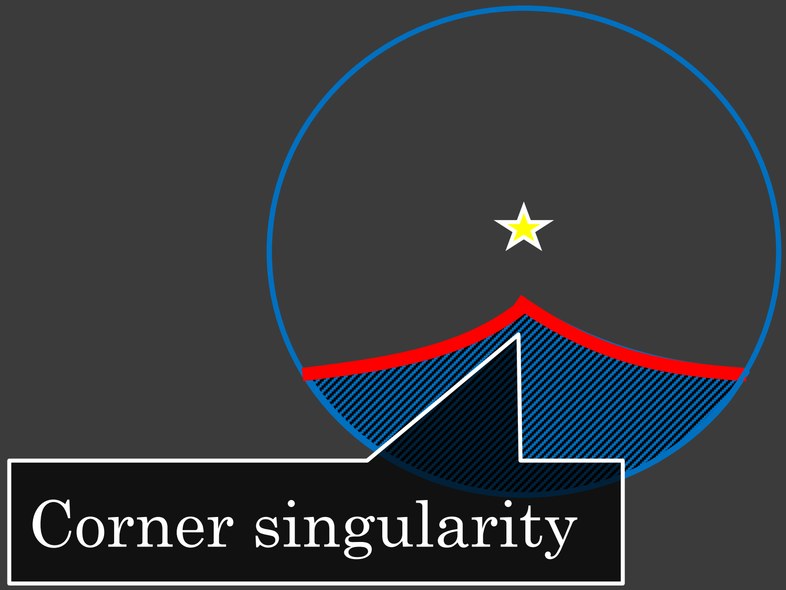
Circumference of asymptotic boundary
$\pi(2\chi - 1)$

Deficit angle
$\delta\theta = 8\pi G_N m$

Cut & Paste construction

How can we construct a conical defect geometry with a negative tension brane?

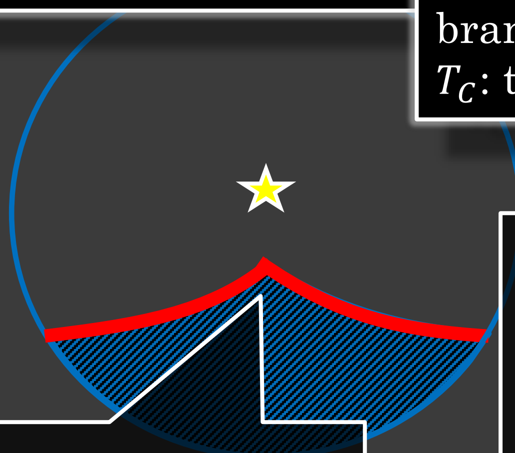
→ cut & paste in AdS/BCFT



Cut & Paste construction

$$I_{grav} = -\frac{1}{16\pi G_N}\int_M d^3x\,\sqrt{g}(R-2\Lambda) + \sum_i m_i \int dl_i - \frac{1}{8\pi G_N}\int_Q d^2x\,\sqrt{h}(K-T)$$
$$-\frac{1}{8\pi G_N}\int_C \sqrt{\eta}(\Theta - T_C)$$

$\eta_{\mu\nu}$: induced metric on C
Θ: internal angle between branes
T_C: tension of corner defect



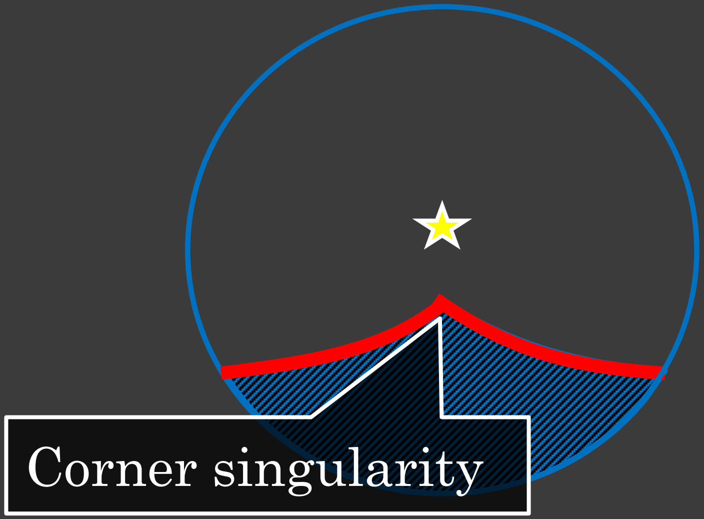
Note:
Generalized Hayward term has additional parameter T_C.

However, for the action to give solutions, T_C is dynamically determined by T and m_i

Cut & Paste construction

How can we construct a conical defect geometry with a negative tension brane?

→ cut & paste in AdS/BCFT



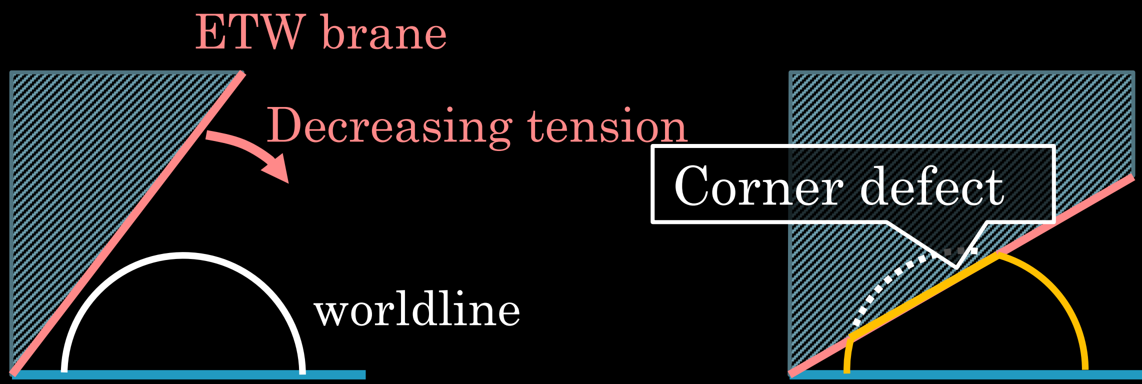
$$E_{ADM} = \int_0^{2\pi} d\theta \, T_{tt} = -\frac{(2\chi - 1)^2}{16 G_N}$$

This leads to

$$E_{ADM} + E_{Casimir} = 2\alpha_i (Q - 2\alpha_i)$$

While the brane configuration looks sensitive to sign of tension, ADM mass is not sensitive to whether tension is positive or negative.

Negative tension brane



The singularity behind the ETW brane appears as a corner defect on the ETW brane.

This construction gives results consistent with conformal bootstrap.

Contents

Discussion

- More bootstrapping AdS/BCFT ?

  We have six fundamental bootstrap equations in BCFT, but we only use one of them. We may be able to give more consistency conditions on branes from others.

- Spinning particle

  We present a way to induce spinning defects on gravity with branes. This can be applied to study more various setups including spinning particles

- Wormholes in AdS/BCFT
- Insights into braneworld holography
- Higher dimensional generalization

Appendix

AdS/BCFT

What is less understood?

gravity with brane & particle itself

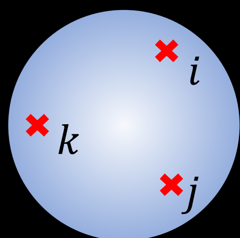
- brane self-intersection
- negative tension brane
- how to deal with spinning particle

Why less understood?

We need details deep into the bulk,
unlike a common case where FG expansion works.
$\rightarrow$ we need to solve Einstein eq. explicitly.
$\rightarrow$ this is difficult & complicated.

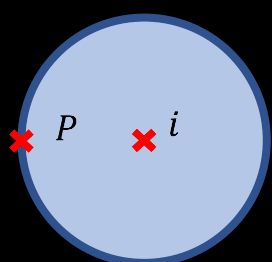
Review of BCFT

BCFT data (information to evaluate correlators)



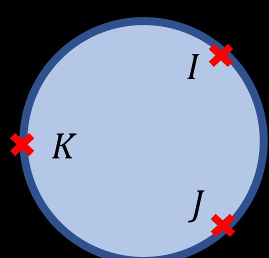
$\equiv C_{ijk}$

Bulk-bulk-bulk OPE coefficient



$\equiv C_{iP}$

Bulk-boundary OPE coefficient



$\equiv C_{IJK}$

Bdy-bdy-bdy OPE coefficient

$\rho(h,h)$

Bulk primary spectrum

$\rho^{bdy}(h)$

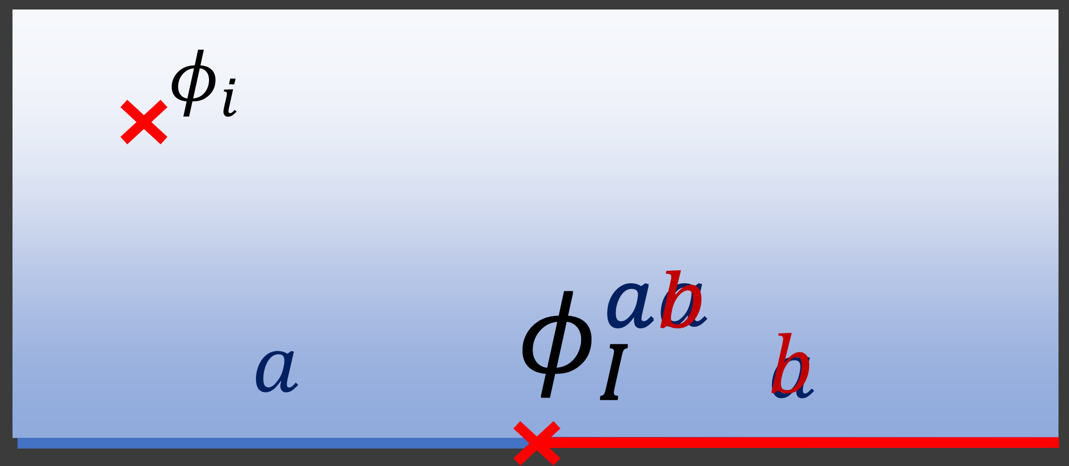
Bdy primary spectrum

g

Boundary entropy

Review of BCFT

i: *bulk*
I: *boundary*



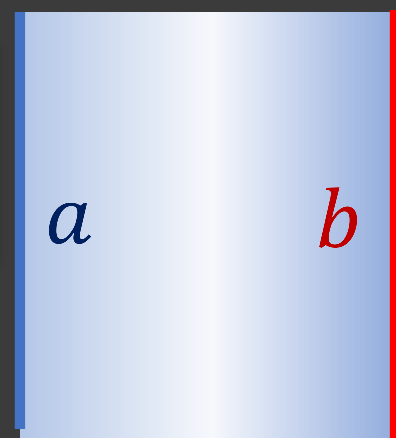
boundary

New ingredient (boundary primary)

Primary operator living on boundary,
which can change boundary condition.
Same transformation law under conformal mapping.

Review of BCFT

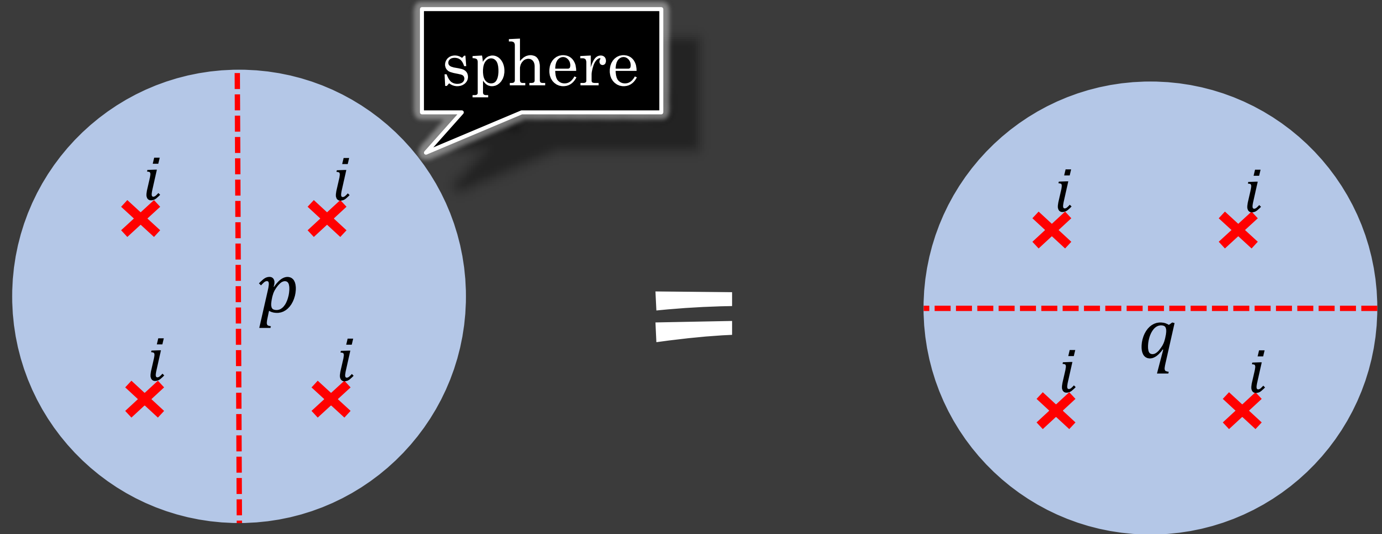
state – operator like mapping



Conformal weight of ϕ_I^{ab}

= Energy corresponding to the state on the strip

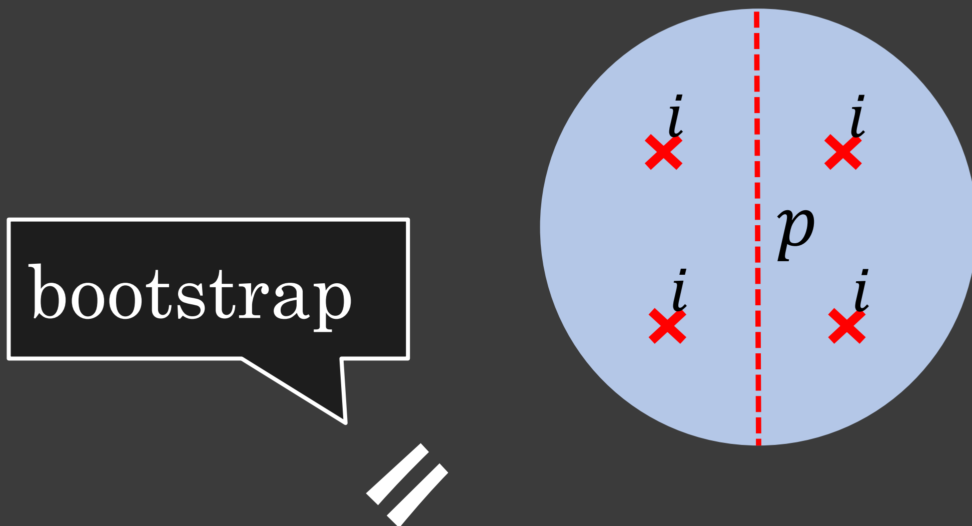
Bootstrap

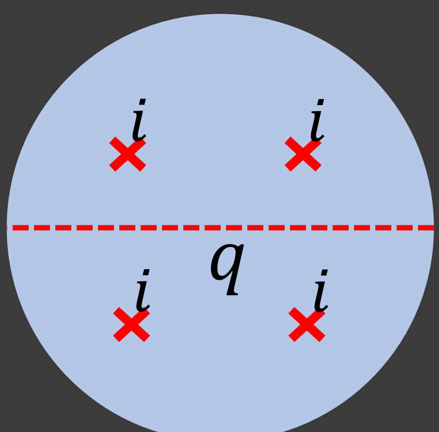


$$\sum_p C_{iip}^2 \left|\mathcal{F}_{ii}^{ii}(p|z)\right|^2 = \sum_q C_{iiq}^2 \left|\mathcal{F}_{ii}^{ii}(q|1-z)\right|^2$$

→ constraints on CFT data

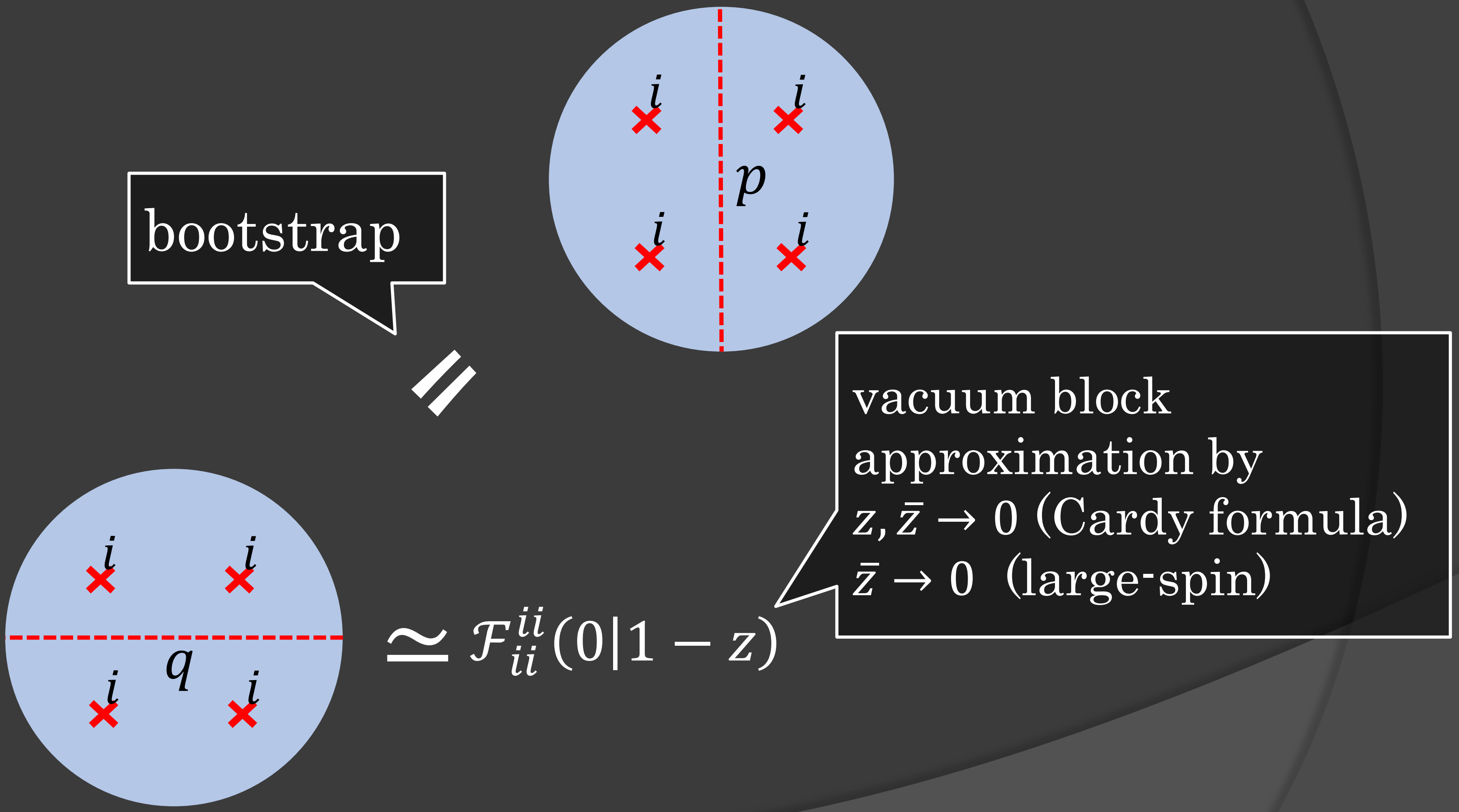
Analytic Bootstrap



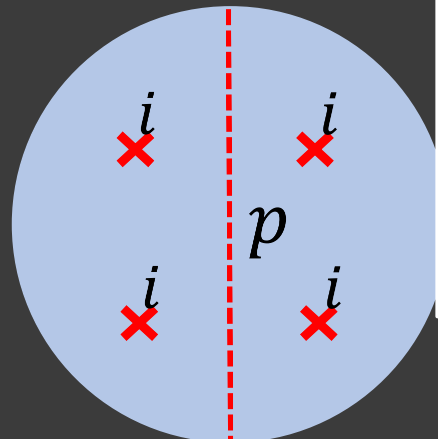


$$= \sum_q C_{iiq}^2 \left| \mathcal{F}_{ii}^{ii}(q|1-z) \right|^2$$

Analytic Bootstrap



Analytic Bootstrap



Now it is expressed in terms of the same basis

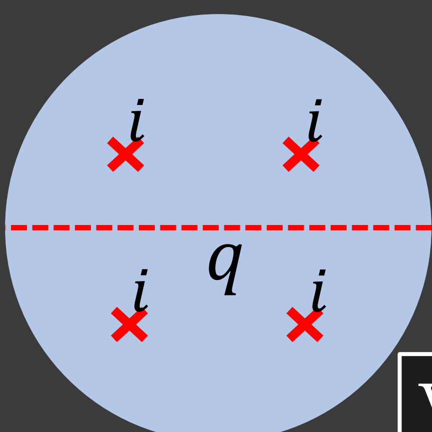
$$\int \mathrm{d}\alpha_q \, C_{iiq}^2 \left|\mathcal{F}_{ii}^{ii}(q|z)\right|^2$$

It is possible to extract OPE coef. by the coefficient comparison.

bootstrap

=

=



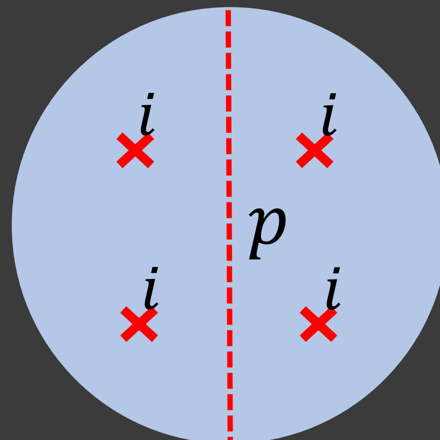
$$\simeq \mathcal{F}_{ii}^{ii}(0|1-z) = \int \mathrm{d}\alpha_q \, F_{0q}\begin{bmatrix} i & i \\ i & i \end{bmatrix} \mathcal{F}_{ii}^{ii}(q|z)$$

Vacuum block approximation

Fusion transformation

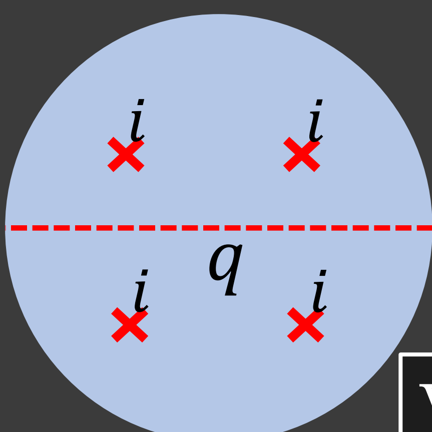
Analytic Bootstrap

[YK]
[Collier, Gobeil, Maxfield, Perlmutter]



bootstrap

$$C_{iip}^2 \simeq F_{0q}\begin{bmatrix} i & i \\ i & i \end{bmatrix}$$



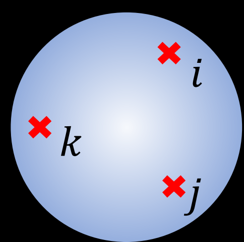
$$\simeq \mathcal{F}^{ii}_{ii}(0|1-z) = \int \mathrm{d}\alpha_q\, F_{0q}\begin{bmatrix} i & i \\ i & i \end{bmatrix} \mathcal{F}^{ii}_{ii}(q|z)$$

Vacuum block approximation

Fusion transformation

Analytic Bootstrap in BCFT

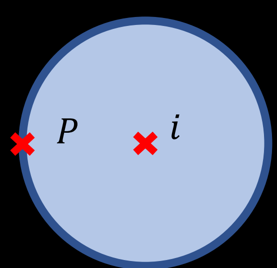
Universal formula in BCFT



[Collier, Maloney, Maxfield, Tsiares]

$\equiv C_{ijk}$

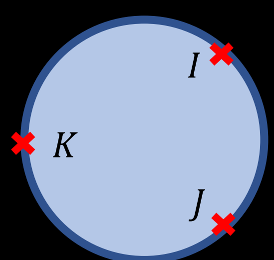
Bulk-bulk-bulk OPE coefficient



[YK], [Numasawa, Tsiares]

$\equiv C_{iP}$

Bulk-boundary OPE coefficient



[YK], [Numasawa, Tsiares]

$\equiv C_{IJK}$

Bdy-bdy-bdy OPE coefficient

[Cardy]

$\rho(h,h)$

Bulk primary spectrum

[YK], [Numasawa, Tsiares]

$\rho^{bdy}(h)$

Bdy primary spectrum

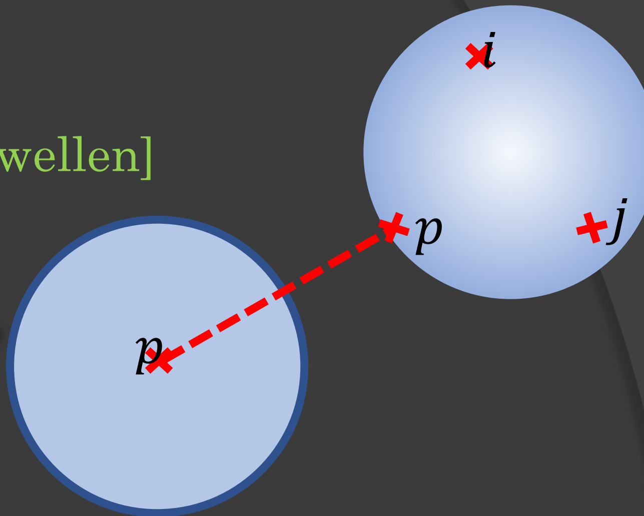
[Collier, Mazac, Wang]

g

Boundary entropy

Review of BCFT [Lewellen]

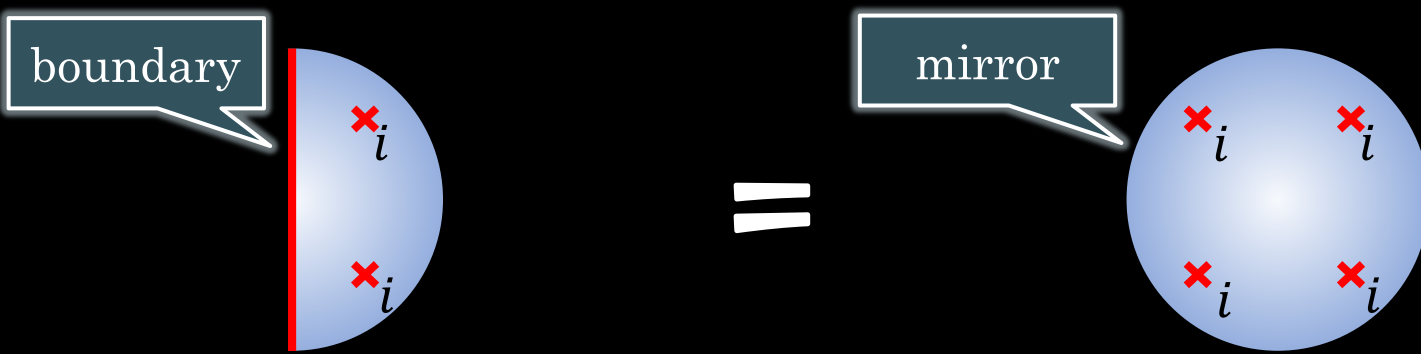
$$\sum_p C_{p0} C_{ijp} \mathcal{F}_{\bar{j}i}^{ji}(p|z)$$



Note:

$\mathcal{F}_{\bar{j}i}^{ji}(p|z)$ = Virasoro block.

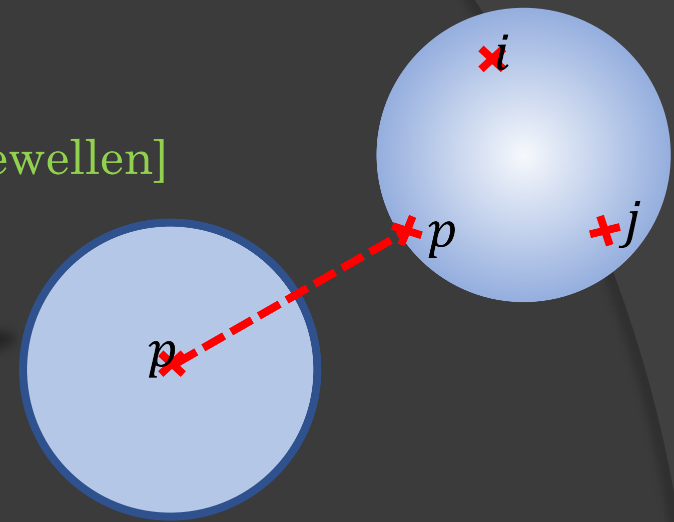
Because Ward id (with bdy) is equivalent to Ward id (without bdy) by mirror method



kinematic part = conformal block

Review of BCFT [Lewellen]

$$\sum_p C_{p0} C_{ijp} \mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z)$$



Note:

$\mathcal{F}_{\bar{j}\bar{i}}^{ji}(p|z)$ = Virasoro block.

Because Ward id (with bdy) is equivalent to Ward id (without bdy) by mirror method

$$\sum_{p,\bar{p},N,\bar{N}} \langle \phi_i | \phi_j | L_{-N}\phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-\bar{N}}\phi_{\bar{p}} \rangle \langle L_{-N} L_{-\bar{N}} \phi_{p,\bar{p}} \rangle_{disk}$$

$$= \sum_{p,\bar{p},N,\bar{N}} \langle \phi_i | \phi_j | L_{-N}\phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-\bar{N}}\phi_{\bar{p}} \rangle \langle L_{-N}\phi_p \, | L_{-\bar{N}}\phi_{\bar{p}} \rangle$$

$$= \sum_{p,N} \langle \phi_i | \phi_j | L_{-N}\phi_p \rangle \langle \phi_{\bar{i}} | \phi_{\bar{j}} | L_{-N}\phi_p \rangle$$