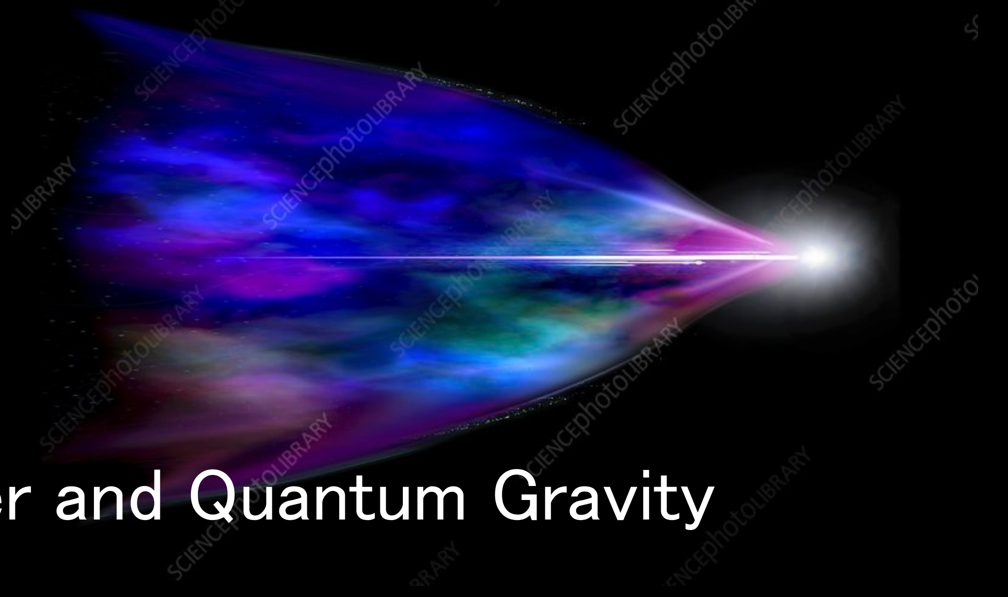
Optimal light cone and digital quantum simulation of interacting bosons



Tomotaka Kuwahara
RIKEN, RQC & CPR

Quantum Information, Quantum Matter and Quantum Gravity (QIMG2023)
21th Jul. 2023

T.K. and K. Saito, PRL, 127, 070403 (2021) , Presented in TQC2021
T.K., T. V. Vu, and K. Saito, arXiv:2206.14736v2, Presented in QIP2023

Contents

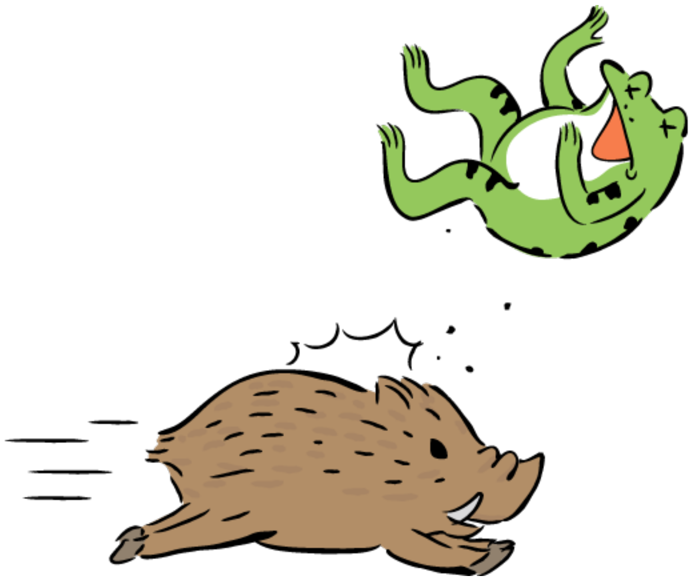
1 : Lieb-Robinson bound

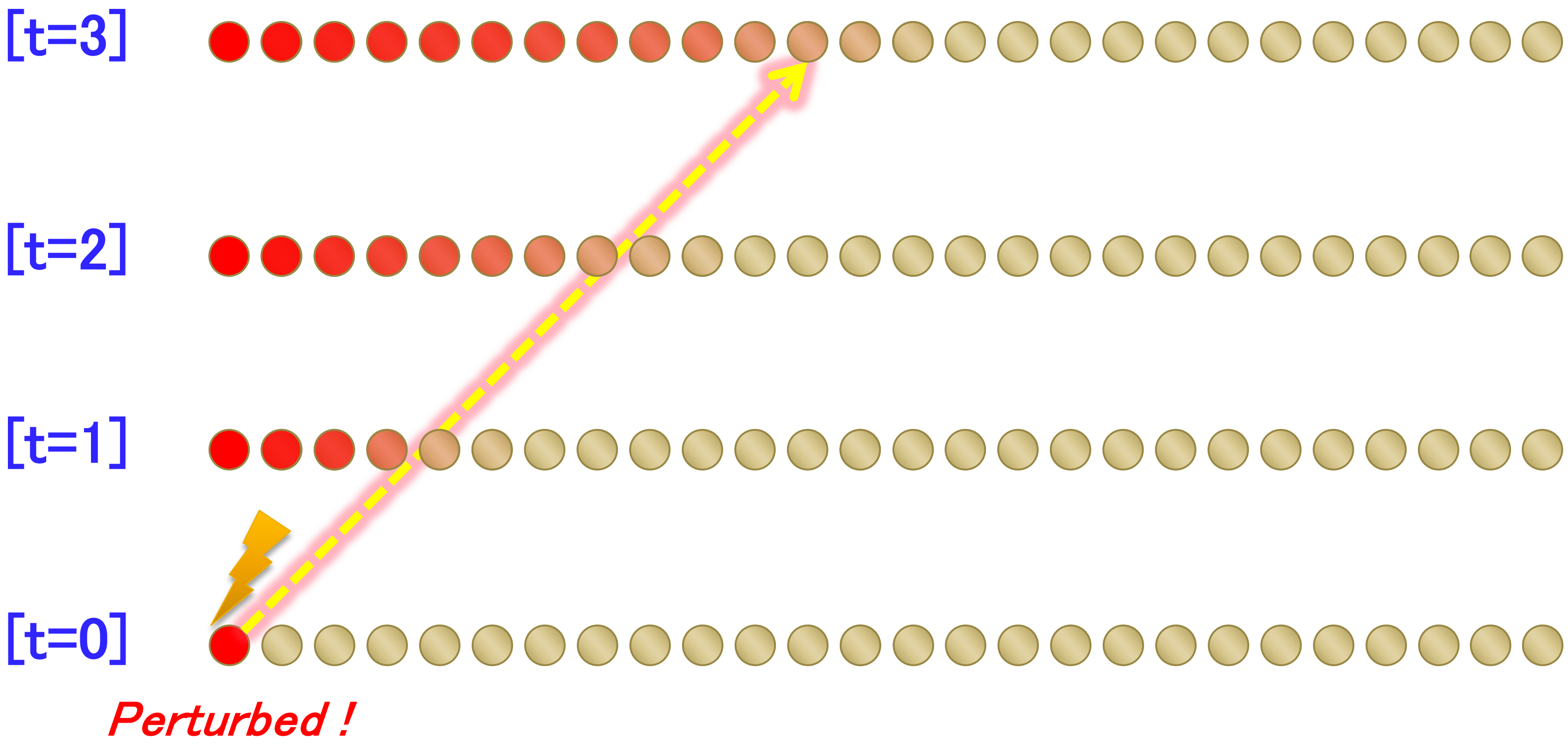
Summary:

Quantifying the speed limit to generate correlations by many-body dynamics

Speed limit of information propagation

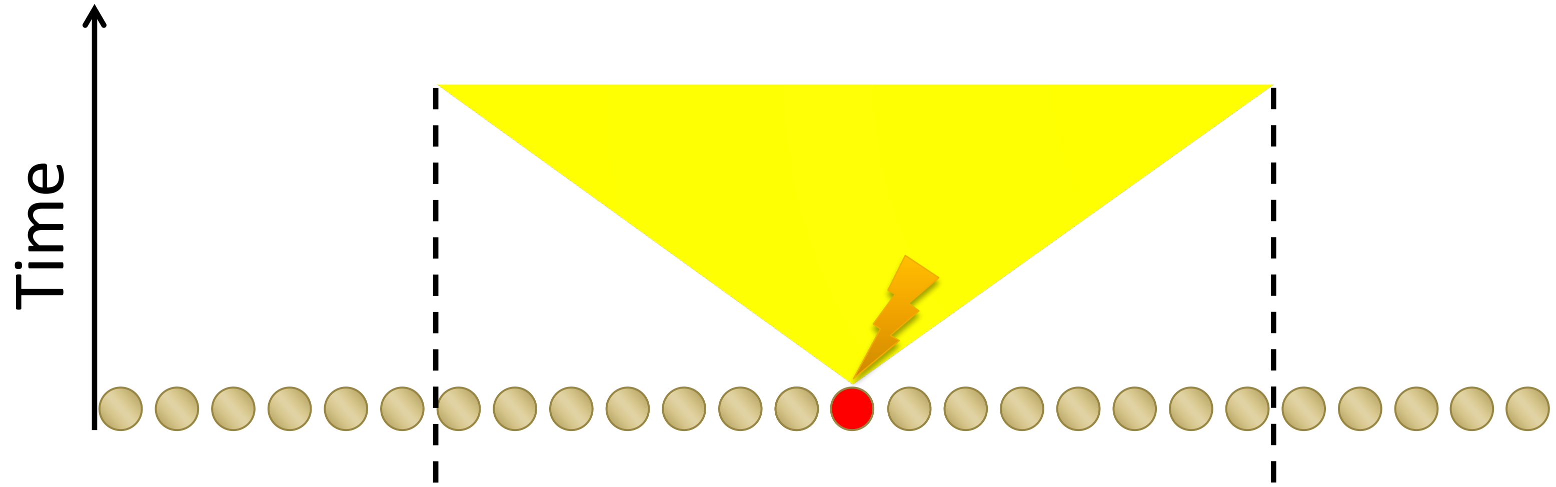
Perturbation is added to the left-edge spin





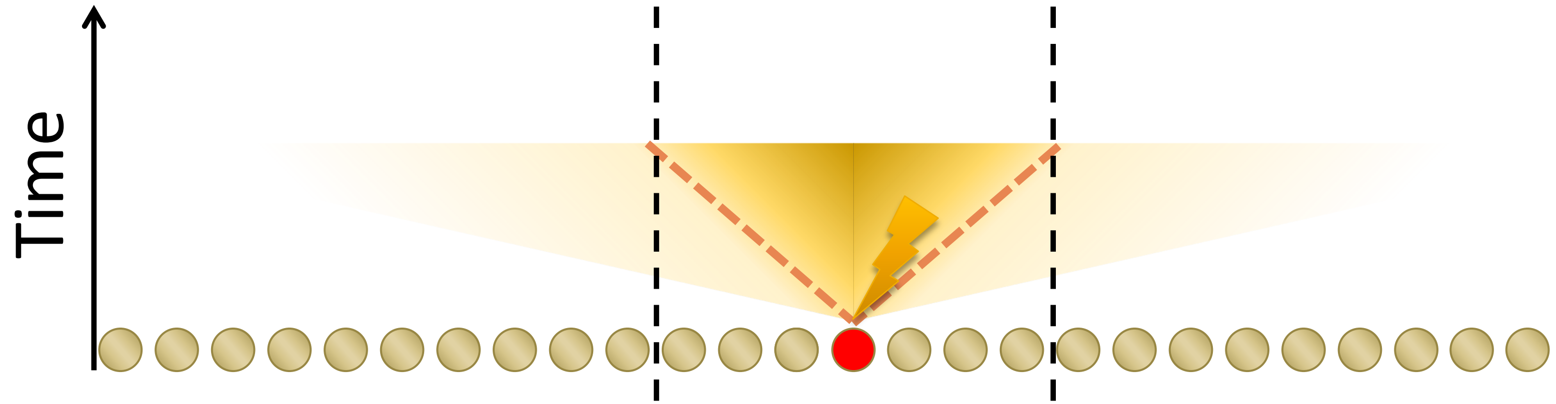
Effective light cone

[Light cone in the relativity theory]



[General limit on the information propagation]

[Effective light cone in the non-relativistic cases]

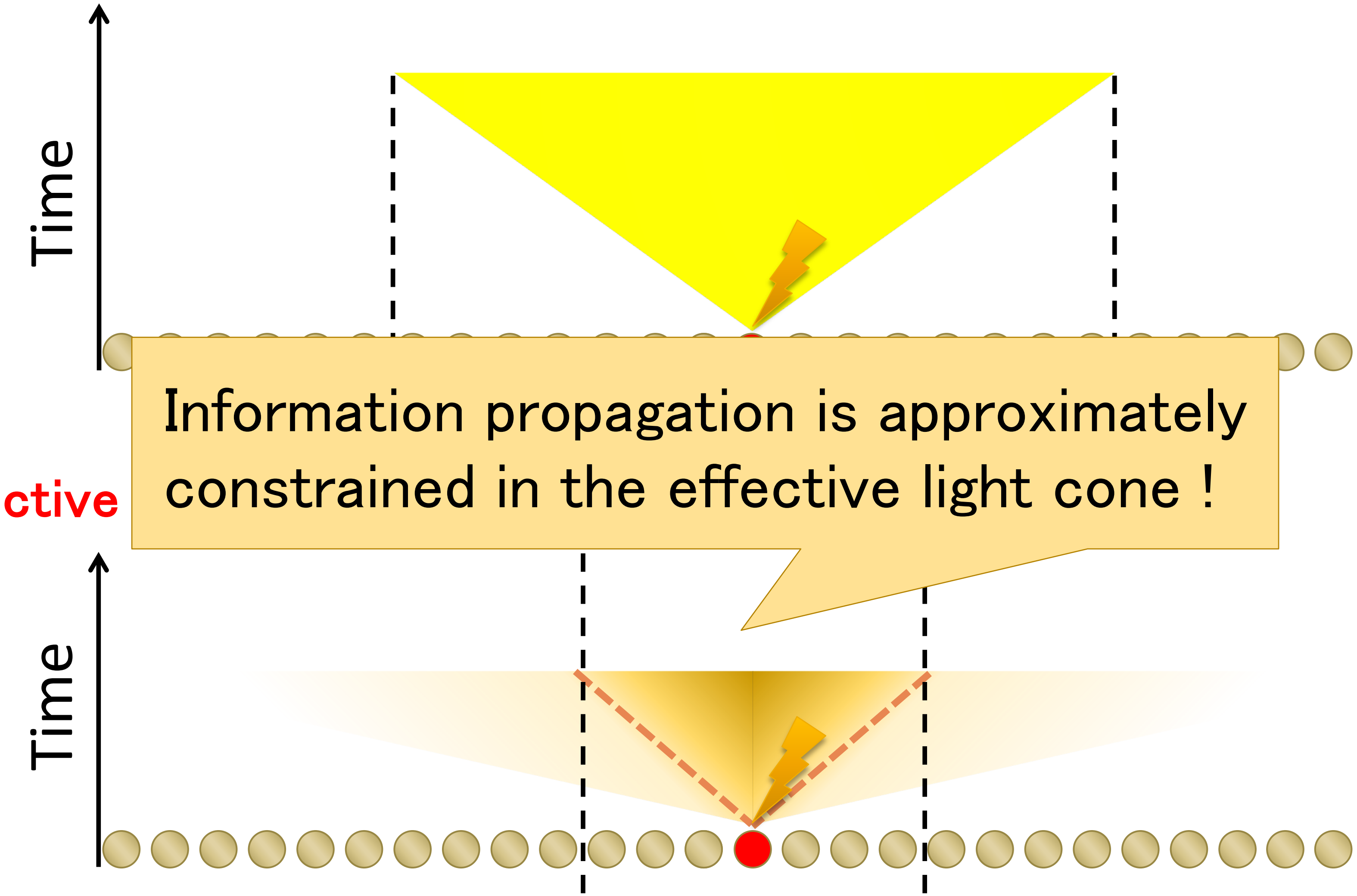


Lieb–Robinson bound

Effective light cone

[General limit on the information propagation]

[Light cone in the relativity theory]



Information propagation is approximately constrained in the effective light cone !

[Effective

Lieb–Robinson bound

Lieb-Robinson bound

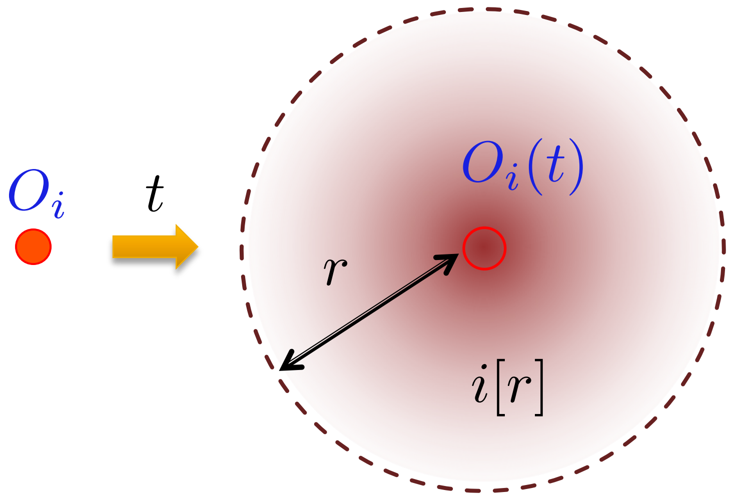
- Time evolution: $O_i(t) = e^{iHt} O_i e^{-iHt}$

- Local approximation : $O_i(t, i[r])$

$$O_i(t, i[r]) := \frac{1}{\mathrm{tr}_{i[r]^{\mathrm{c}}}(\hat{1})} \mathrm{tr}_{i[r]^{\mathrm{c}}} \left[O_i(t)\right] \otimes \hat{1}_{i[r]^{\mathrm{c}}},$$

- **Estimation of the approximation error**

$$O_i(t) \overset{?}{\simeq} O_i(t, i[r])$$



$i[r]$: extended region from i with a radius r

➡ Lieb-Robinson bound gives a general upper bound on the error

Two basic conditions

- **Two conditions for Hamiltonian**

e.g., two body Hamiltonian $H = \sum_{i,j} h_{i,j} + \sum_i h_i$

➡ **(1) Interaction is short range**

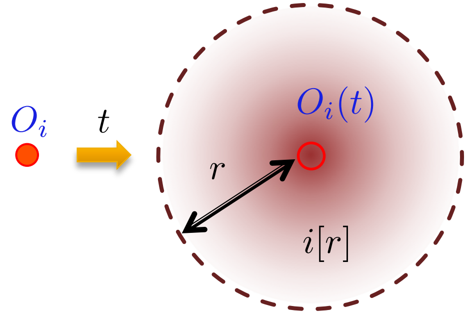
$$h_{i,j} = 0 \quad \text{for} \quad d_{i,j} \geq \text{const.}$$

$d_{i,j}$: distance between i and j

$\| \cdots \|$: operator norm

➡ **(2) one-site energy is finite**

$$\sum_{j \neq i} \|h_{i,j}\| + \|h_i\| = \mathcal{O}(1) \quad \text{for} \quad \forall i \in \Lambda$$



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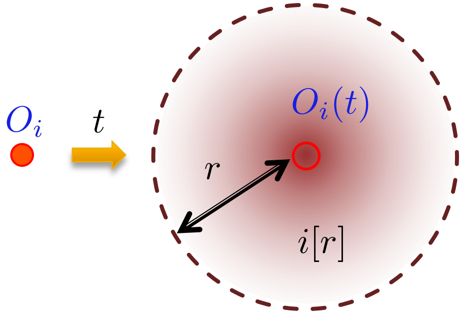
➡ (2) one-site energy is finite

$\|O\|$:= Square root of the maximum eigenvalue of $O^\dagger O$

➡ **For $\forall\, |\psi\rangle$, we have $\langle\psi|O^\dagger O|\psi\rangle \leq \|O\|^2$**



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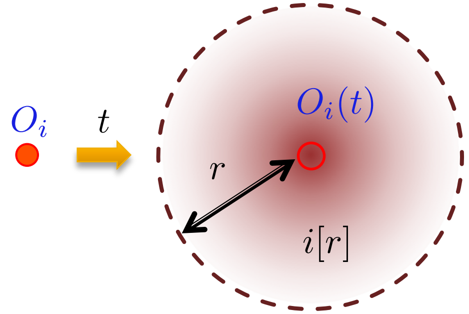


[Lieb-Robinson bound]

Lieb and Robinson, Commun. Math. Phys. **28**, 251 (1972).

$$\|[O_i(t), O_j]\| \leq e^{-c(d_{i,j} - vt)} \quad \Rightarrow \quad \|O_i(t) - O_i(t, i[r])\| \leq c' r^{D-1} e^{-c(r - vt)}$$

Lieb-Robinson bound : non-trivial cases

- What happens when one of the conditions breaks down ?

- Effective light cone : $R = vt$

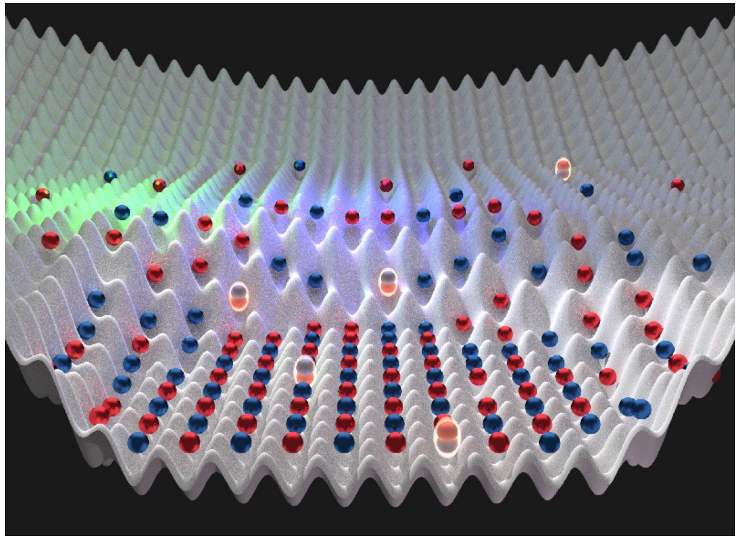
$$O_i(t) \simeq O_i(t, i[r]) \quad \text{for} \quad r \gg R$$

➡ **(1) Interaction is short range**

$$h_{i,j} = 0 \quad \text{for} \quad d_{i,j} \geq \text{const.}$$

➡ **(2) one-site energy is finite**

$$\sum_{j \neq i} \|h_{i,j}\| + \|h_i\| = \mathcal{O}(1) \quad \text{for} \quad \forall i \in \Lambda$$



Cold atom experiment
（C. Chiu/Harvard University）

Lieb-Robinson bound : non-trivial cases

- What happens when one of the conditions breaks down ?

- Effective light cone :

$$O_i(t) \simeq O_i(t, i[r]) \quad \text{fo}$$

[Long-range interactions]

Recent progress

Chen and Lucas, PRL 123, 250605 (2019).
Kuwahara and Saito, PRX 10, 031010 (2020)
Tran, et al., PRX, 11, 031016 (2021)
Kuwahara and Saito, PRL **126**, 030604 (2021)
Tran, et al., PRL, 127, 160401 (2021)

➡ **(1) Interaction is short range**

$$h_{i,j} = 0 \quad \text{for} \quad d_{i,j} \geq \text{const.}$$

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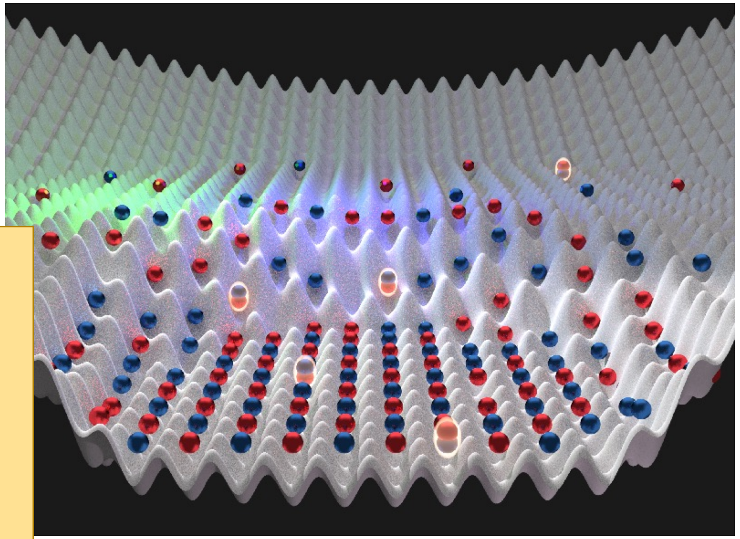
$$\sum_{j \neq i} \|h_{i,j}\| + \|h_i\| = \mathcal{O}(1) \quad \text{for} \quad \forall i$$

[interacting Bosons]

Ubiquitous in cold atom experiment
→ Notoriously difficult !



Cold atom experiment
(C. Chiu/Harvard University)

2: Basic problems for bosonic systems

Summary:

In bosonic systems, we need to consider two problems:

(1) Speed of boson transport
(2) Speed of total information propagation

Remarks on boson systems

- Restricting the boson number on one site (e.g., Hard-core boson)

M. B. Hastings and T. Koma, Communications in Mathematical Physics **265**, 781 (2006).

➡ Practically OK, but no theoretical guarantee …

- Non-interacting bosons

(e.g., quantum harmonic oscillator)

M. Cramer, A. Serafini, and J. Eisert, arXiv:0803.0890
B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, Commun. Math. Phys., **286**, 1073 (2009).

- Systems with bosonic bath

(Spin systems connected to non-interacting bosons)

J. Jünemann, et al., Phys. Rev. Lett. 111, 230404 (2013).
M. P. Woods, M. Cramer, and M. B. Plenio,, Phys. Rev. Lett. **115**, 130401 (2015).

Remarks on boson systems

- Restricting

M. B. Hastings a

Practically

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(e.g., quantu

Phys., **286**, 1073 (2009).

- Systems w

(Spin system

In general cases ? **NO!**

Counterexample by Eisert and Gross (QIP2009)

" There exists a 1D boson model with translation invariance which allows exponential speed of information propagation in arbitrary low-energy states" J. Eisert and D. Gross, Phys. Rev. Lett. 102, 240501 (2009).

However, the model is unphysical.
What happens in more realistic models?

J. Jünemann, et al., Phys. Rev. Lett. 111, 230404 (2013).
M. P. Woods, M. Cramer, and M. B. Plenio,, Phys. Rev. Lett. **115**, 130401 (2015).

Set up :Bose-Hubbard type Hamiltonian

The first experiment for the Lieb-Robinson bound M. Cheneau, et al., Nature **481**, 484 (2012).

- Generalized Bose-Hubbard model (Spatial dimension: D)

Boson-boson interaction

$$H = \sum_{\langle i,j \rangle} J_{i,j}(b_i b_j^\dagger + \mathrm{h.c.}) + \sum_i f(\hat{n}_i, \hat{n}_{i_1}, \hat{n}_{i_2}, \ldots, \hat{n}_{i_k}) \qquad \hat{n}_i = b_i^\dagger b_i$$

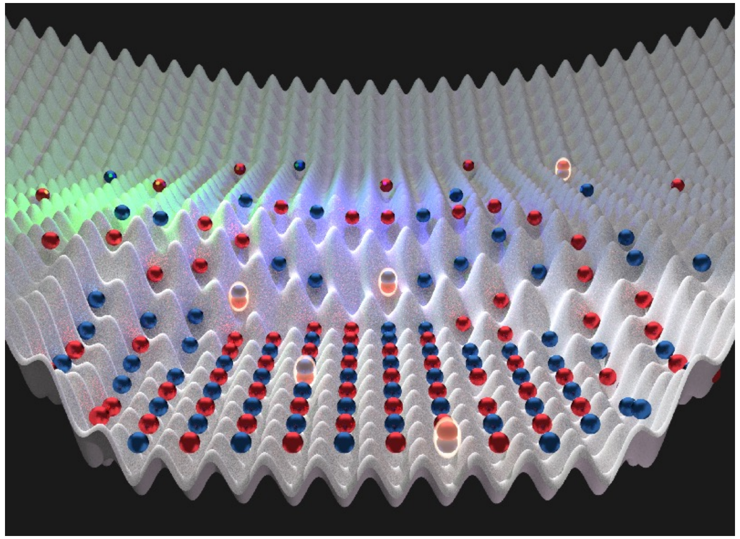
$$f(\hat{n}_i, \hat{n}_{i_1}, \hat{n}_{i_2}, \ldots, \hat{n}_{i_k}) = \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu \hat{n}_i$$

[Bose-Hubbard]

$$H = \sum_{\langle i,j \rangle} J(b_i b_j^\dagger + \mathrm{h.c.}) + \frac{U}{2}\sum_{i \in \Lambda} \hat{n}_i(\hat{n}_i - 1) - \mu \sum_{i \in \Lambda} \hat{n}_i$$

→ **One-site energy:** $\sum_{j:d_{i,j}=1} J(b_i b_j^\dagger + \mathrm{h.c.}) + \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu \hat{n}_i$

→ N bosons on the site i : **local energy is proportional to N^2**



Cold atom experiment

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Our goal: establishing Lieb–Robinson bound for Bose–Hubbard model

[Bose-Hu

$$H = \sum_{\langle i,j\rangle}$$

➡ **One-site energy:** $\sum_{j:d_{i,j}=1} J(b_i b_j^\dagger + \text{h.c.}) \quad \frac{}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i$

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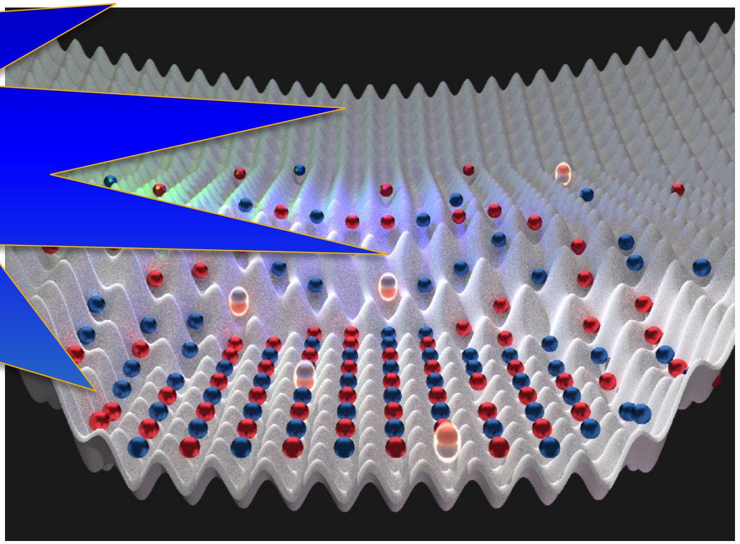


Cold atom experiment
(C. Chiu/Harvard University)

Two problems in bosonic systems

- **We need to consider two problems**

➡ **(1) Speed of boson transport**

$$[\hat{n}_X(t)]^s \preccurlyeq \left[\hat{n}_{X[R]} + \ \varepsilon_{R,s,t}\right]^s \quad (s \in \mathbb{N})$$

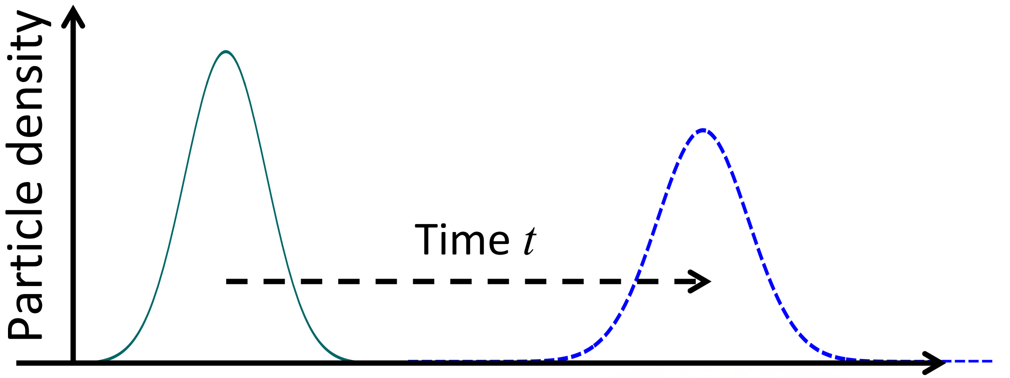
$$\hat{n}_X := \sum_{i\in X} \hat{n}_i \qquad\qquad \lim_{R\to\infty} \varepsilon_{R,s,t} = 0$$

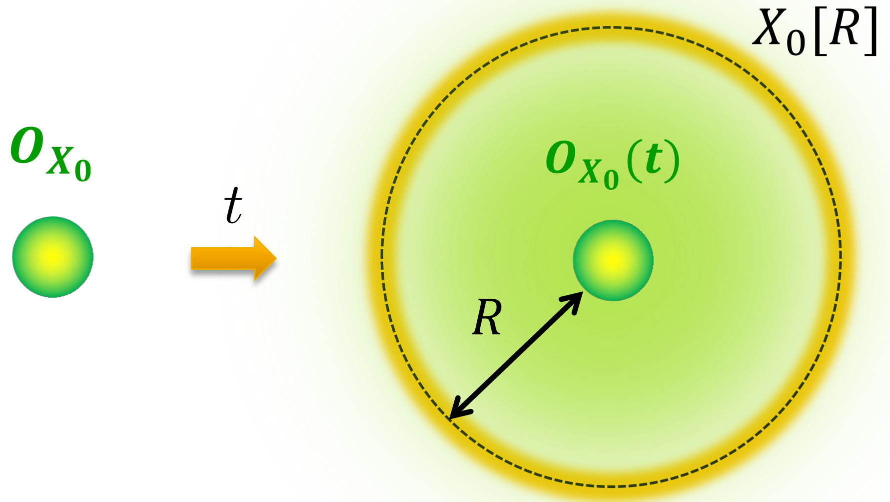
➡ **(2) Speed of total information propagation**

$$\left\|\left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right)\right]\rho_0\right\|_1 \le \delta_{R,t}$$

$$\lim_{R\to\infty} \delta_{R,t} = 0$$

$H_{X_0[R]}$: subset Hamiltonian on extended region $X_0[R]$





$$O_{X_0}(t) \ \approx O_{X_0}\left(H_{X_0[R]}, t\right)$$

Two problems in bosonic systems

- **We need to consider two problems**

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$$\hat{n}_X := \sum_{i \in X} \hat{n}_i \qquad \lim_{R \to \infty} \varepsilon_{R,s,t} = 0$$

$$\mathrm{tr}[\rho_0(t)\hat{n}_X^s] \lesssim \mathrm{tr}\left[\rho_0 \hat{n}_{X[R]}^s\right]$$

↓

Probability distribution for boson number after time evolution

→ **(2) Speed of total information propagation**

$$\left\|\left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right)\right] \rho_0\right\|_1 \le \delta_{R,t}$$

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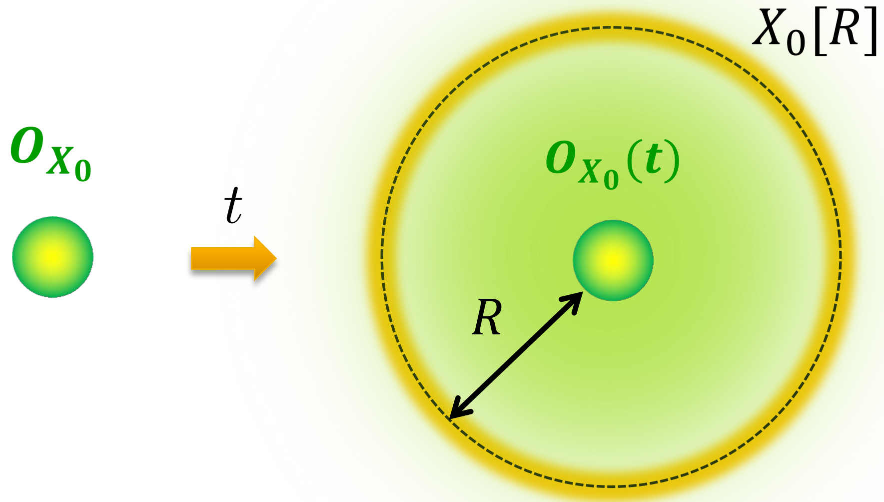

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Two problems in bosonic systems

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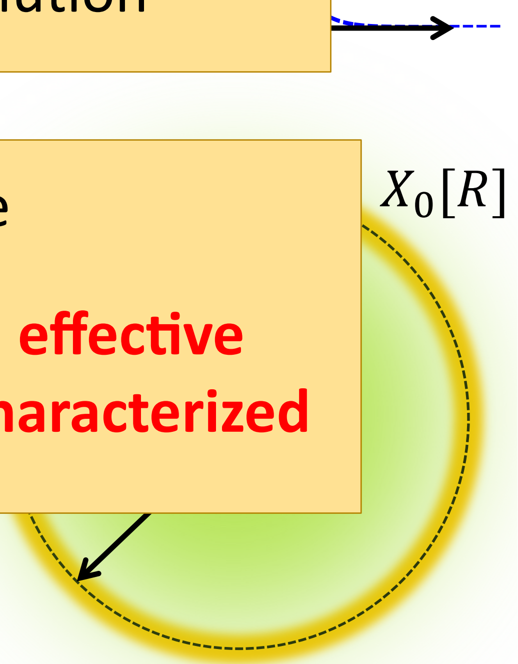
$H_{X_0[R]}$: subset Hamiltonian on extended region $X_0[R]$

ρ_0: initial state

Lieb-Robinson effective light cone is characterized



$$O_{X_0}(t) \approx O_{X_0}\left(H_{X_0[R]}, t\right)$$

Outline of the strategy

- **(1) Speed of boson transport** → $\mathrm{tr}[\rho_0(t)\hat{n}_X^s] \lesssim \mathrm{tr}\left[\rho_0\hat{n}_{X[R]}^s\right]$
- The upper bound of the boson number distribution
- Truncation of the boson number → Effective Hamiltonian (Bounded local energy)

Kuwahara and Saito, PRL, 127, 070403 (2021)

- Deriving the Lieb-Robinson bound for the effective Hamiltonian

(2) Speed of total information propagation

3: Previous results and our new results

Summary:

Previous results:

Schuch, Harrison, Osborne, Eisert, PRA **84**, 032309 (2011).
Faupin, Lemm, and Sigal, PRL **128**, 150602 (2022).

Wang and Hazzard, PRX Quantum **1**, 010303 (2020)
Kuwahara and Saito, PRL, 127, 070403 (2021) .
Yin and Lucas, PRX, **12**, 021039 (2022).

Our main results: optimal light cone and gate complexity of the digital quantum simulation

Particle transport in Bose-Hubbard model

- Schuch-Harrison-Osborne-Eisert (QIP2011)

N. Schuch, S. K. Harrison, T. J. Osborne, and J. Eisert, Phys. Rev. A **84**, 032309 (2011).

- **Diffusion of the concentrated bosons in the vacuum**

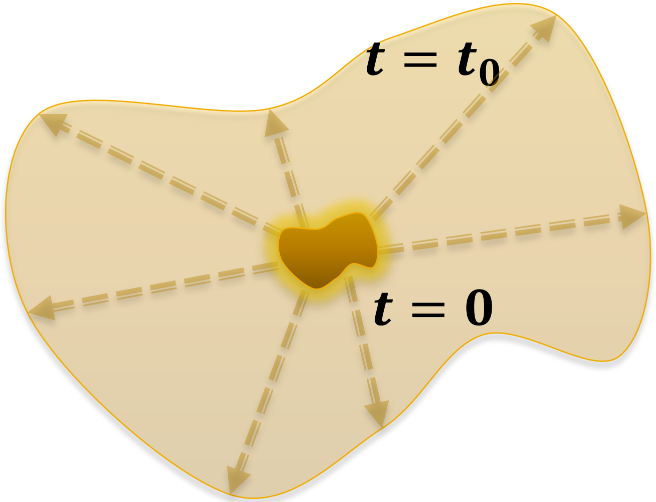
$|\psi_X\rangle$: $\langle\psi_X|\hat{n}_i|\psi_X\rangle = 0$ for $i \notin X$

Bosons exist only in the region *X*

$\Big\downarrow\ e^{-iHt}$

$$\langle\psi_X|\hat{n}_j|\psi_X\rangle \le N_{\mathrm{tot}} e^{-c\,(d_{j,X}-vt)}$$

(N_{tot} : total boson number、$d_{j,X}$: distance from X



Particle transport in Bose-Hubbard model

- Schuch-Harrison-Osborne-Eisert (QIP2011)

N. Schuch, S. K. Harrison, T. J. Osborne, and J. Eisert,

- **Diffusion of**

$|\psi_X\rangle$: $\langle \psi_X | \hat{n}_i |$

$\Big\downarrow e^{-iHt}$

$$\langle \psi_X | \hat{n}_j | \psi_X \rangle \le N_{\rm tot} e^{-c\,(d_{j,X} - vt)}$$

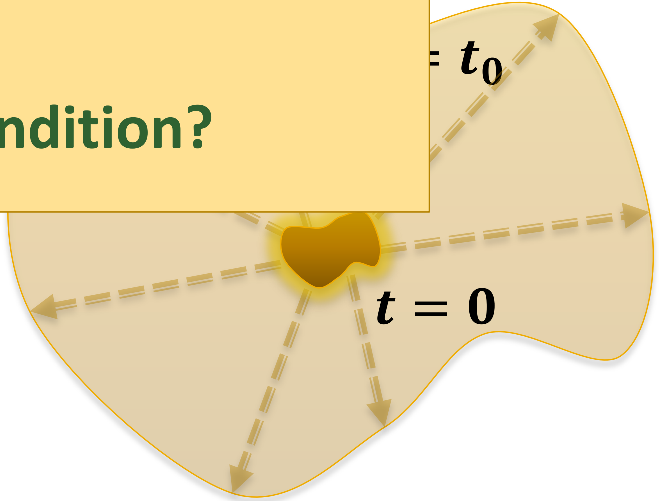
($N_{\rm tot}$: total boson number、$d_{j,X}$: distance from X

The speed of boson diffusion is finite!

➡ Concentration on the vacuum is crucial.

(No generalization had been known…)

➡ **Can one prove it in a more realistic condition?**



Generalization to arbitrary initial states

- Faupin, Lemm and Sigal

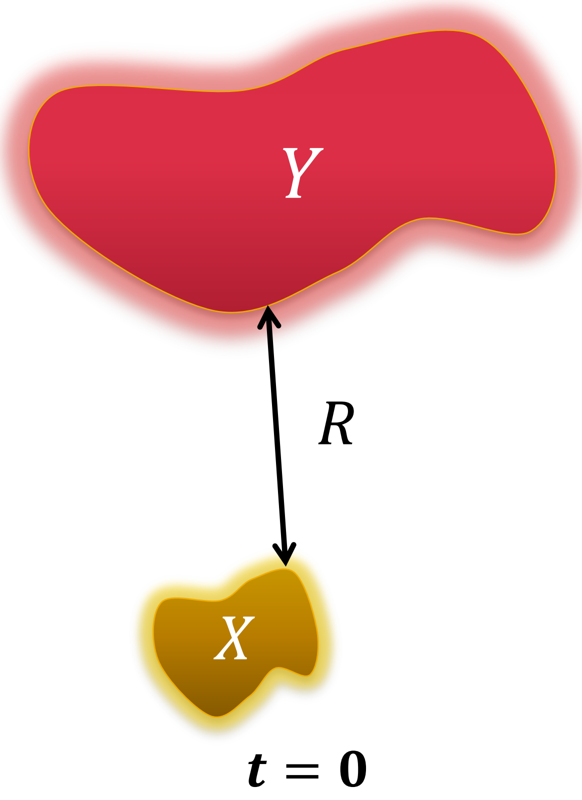
J. Faupin, M. Lemm, and I. M. Sigal, Phys. Rev. Lett. **128**, 150602 (2022).

- Macroscopic transport

$$\langle\psi|\hat{n}_X|\psi\rangle \geq (1-\eta)N_{\mathrm{tot}} \xrightarrow[e^{-iHt}]{} \mathrm{Prob}(\hat{n}_Y \geq \xi N_{\mathrm{tot}})\ (\xi > \eta)$$

- **Transport of a macroscopic number of bosons has finite speed!**

→ Adiabatic spacetime localization observables (ASTLO) was utilized



Lieb-Robinson bound in Bose-Hubbard model

- Wang and Hazzard $f(\hat{n}_i, \hat{n}_{i_1}, \hat{n}_{i_2}, \dots, \hat{n}_{i_k}) = \frac{U}{2}\hat{n}_i(\hat{n}_i - 1) - \mu\hat{n}_i$

➡ $R \propto \sqrt{N_{\mathrm{tot}}}t, \quad v = \mathcal{O}(\sqrt{N_{\mathrm{tot}}})$ Light cone (N_{tot} : total boson number)

Z. Wang and K. R. Hazzard, PRX Quantum **1**, 010303 (2020)

- Initial state: steady state with low-boson density

- Lieb-Robinson bound Thm. 1 in **Kuwahara and Saito, PRL, 127, 070403 (2021)**

$$\left\| \left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right) \right] \rho_0 \right\|_1 \le \left\| O_{X_0} \right\| \exp\left(-C_1 \frac{R}{t \log R} + C_2 \log R \right)$$

C_1, C_2 : constants of $\mathcal{O}(1)$

➡ It gives $O_{X_0}(t) \approx O_{X_0}\left(H_{X_0[R]}, t\right)$ for $R \gtrsim t \log^2(t)$

  - Almost linear light cone up to logarithmic corrections

Lieb-Robinson bound in Bose-Hubbard model

- Yin and Lucas   Yin and Lucas, PRX, **12**, 021039 (2022)

→ Improvement to the linear Light cone

- Initial state: steady state with low-boson density

- Lieb-Robinson bound for average of commutator

$$\left|\mathrm{tr}\left\{\left[O_{X_0}(t), O_Y\right]\rho_0\right\}\right| \leq \left\|O_{X_0}\right\| \cdot \|O_Y\| e^{-c(d_{X_0,Y}-vt)}$$

→ Weaker than the trace norm $\left\|\left[O_{X_0}(t), O_Y\right]\rho_0\right\|_1$

- But, in 1D case, the condition of the steady state can be removed!

Lieb-Robinson bound in Bose-Hubbard model

- Yin and Lucas Yin and Lucas, PRX, **12**, 021039 (2022)

➡ Improvement to the linear Light cone

- Initial state: steady state

  - Lieb-Robinson boun

$$\left|\mathrm{tr}\{[O_{X_0}(t), O$$

This point is critical in developing efficiency guaranteed algorithm for simulating quantum dynamics

e.g., using Haah-Hastings-Kohtari-Low algorithm

J. Haah, M. Hastings, R. Kothari, and G. H. Low, 2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS) (2018) pp. 350–360.

➡ Weaker than the trace norm $\left\|[O_{X_0}(t), O_Y]\,\rho_0\right\|_1$

- But, in 1D case, the condition of the steady state can be removed!

Our new results

Kuwahara, Vu and Saito, arXiv:2206.14736

- Optimal forms of the effective light cone

➡ **(1) Speed of boson transport (Result 1 in arXiv:2206.14736)**

$$[\hat{n}_X(t)]^s \preccurlyeq \left[\hat{n}_{X[R]} + e^{-\mathcal{O}(R/t)} + \mathcal{O}(st)\right]^s$$

➡ **(2) Speed of total information propagation (Result 2 in arXiv:2206.14736)**

$$\left\|\left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right)\right] \rho_0\right\|_1 \le \left\|O_{X_0}\right\| e^{-C(R/t^D)^{1/(\kappa D)}}$$

Initial state ρ_0 : low-boson density $\max_i[\mathrm{tr}(\hat{n}_i^s \rho)] \le \frac{1}{e}\left(\frac{b_\rho}{e} s^\kappa\right)^s$

Our new results

Kuwahara, Vu and Saito, arXiv:2206.14736

- Optimal forms of the effective light cone

➡ **(1) Speed of boson transport (Result 1 in arXiv:2206.14736)**

$$[\hat{n}_X(t)]^s \preccurlyeq \left[\hat{n}_{X[R]} + e^{-\mathcal{O}(R/t)} + \mathcal{O}(st)\right]^s$$

➡ **(2) Speed of total information propagation (Result 2**

$$\left\|\left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right)\right]\rho_0\right\|_1 \le \left\|O_{X_0}\right\| e^{-C(R/t^D}$$

The probability of the boson number at each site decays (sub)exponentially

Initial state ρ_0 : low-boson density $\max_i[\mathrm{tr}(\hat{n}_i^s\rho)] \le \frac{1}{e}\left(\frac{b_\rho}{e}s^\kappa\right)^s$

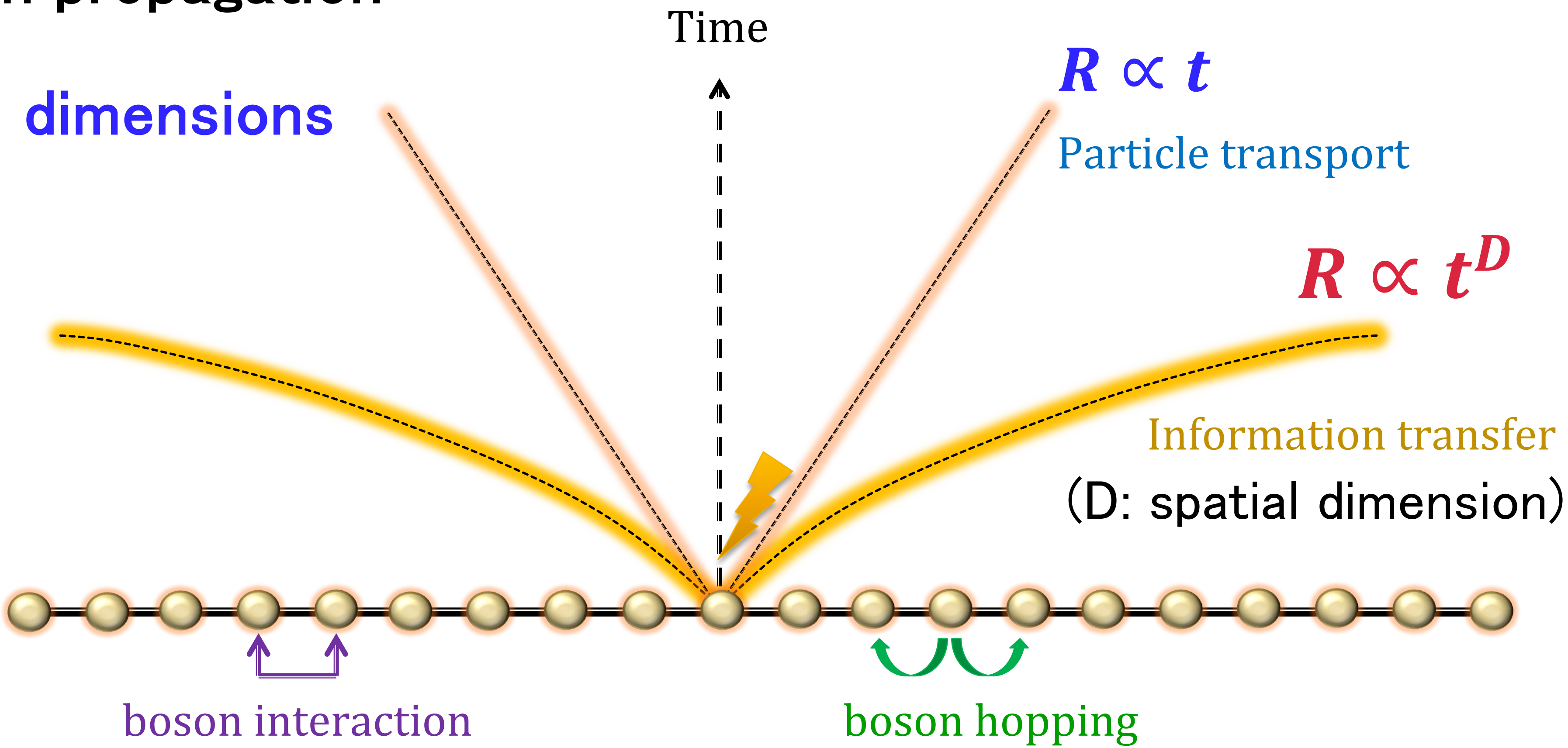
Our new results

Kuwahara, Vu and Saito, arXiv:2206.14736

- Schematic pictures of the effective light cones

  - Speed of information propagation

→ Accelerating in high dimensions



Our new results

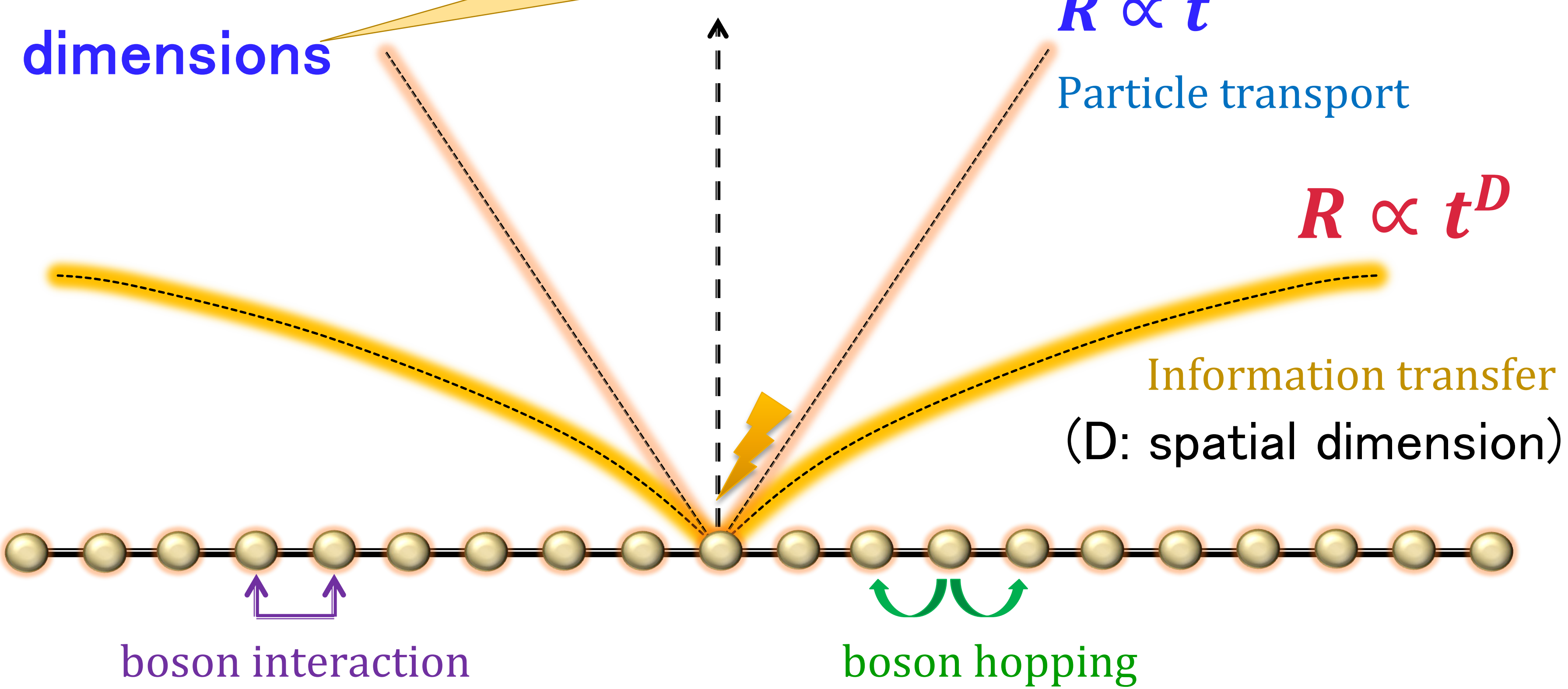
ito, arXiv:2206.14736

- Schematic pictures of the effective

▪ Speed of information propagation

➡ Accelerating in high dimensions

There exist quantum dynamics that saturate the bound.



$R \propto t$

Particle transport

$R \propto t^D$

Information transfer

(D: spatial dimension)

boson interaction

boson hopping

Our new results

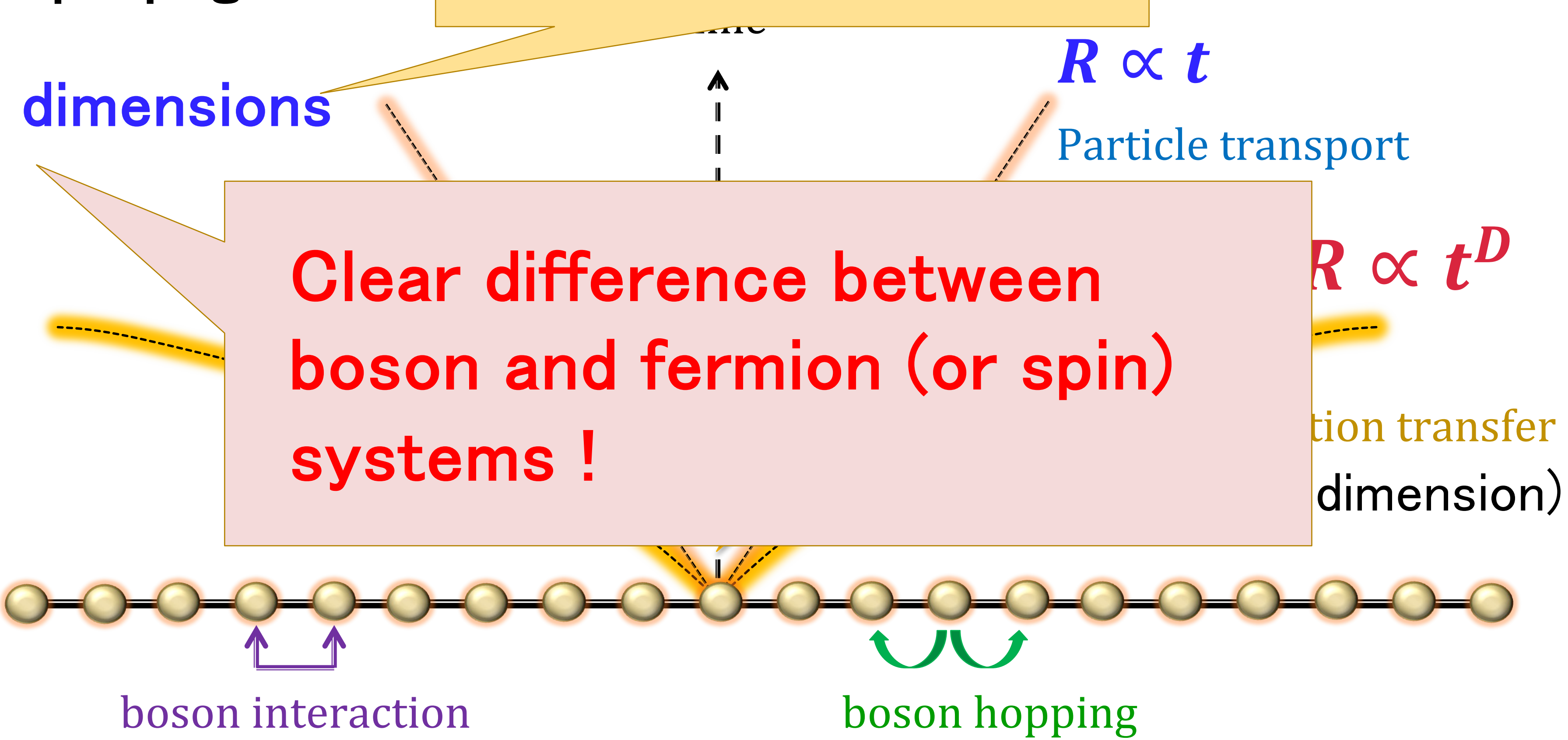
ito, arXiv:2206.14736

- Schematic pictures of the effective
  - Speed of information propagation

→ Accelerating in high dimensions

There exist quantum dynamics that saturate the bound.

Clear difference between boson and fermion (or spin) systems !

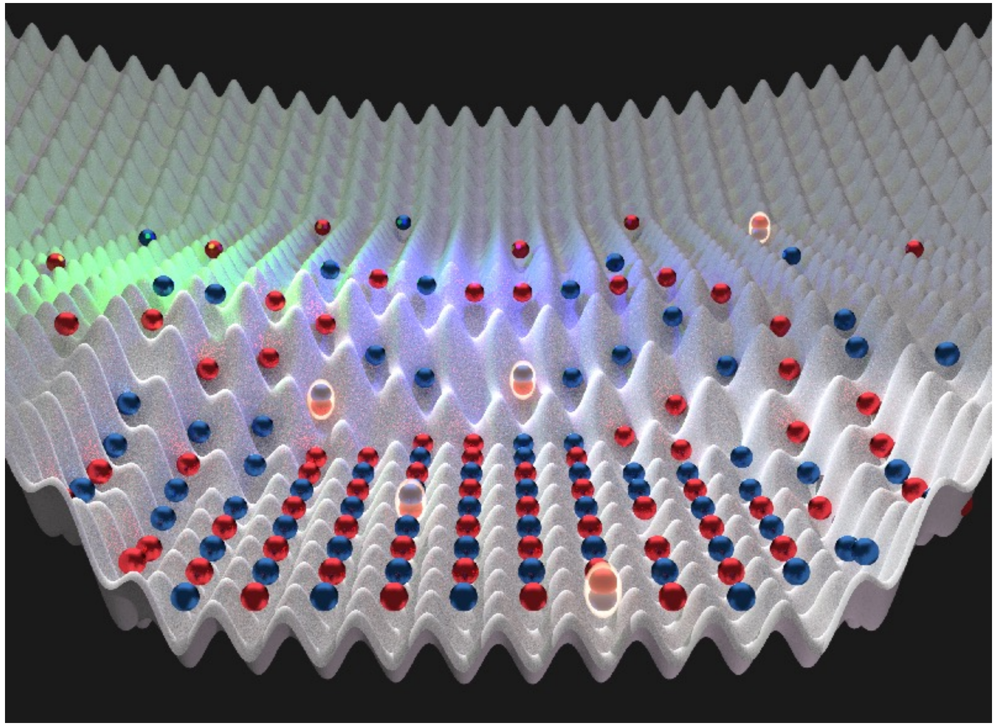


Application

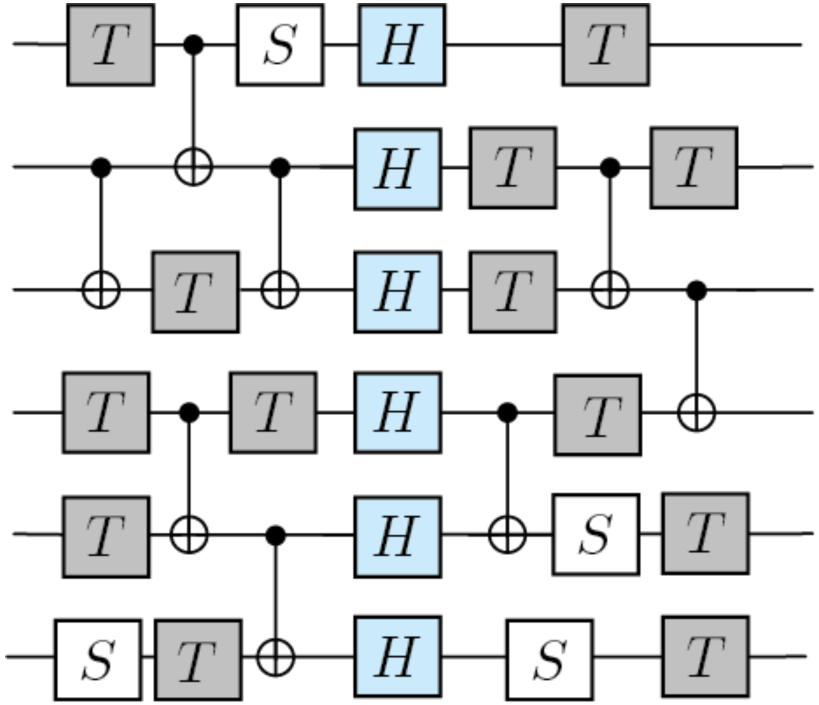
- **Gate complexity of quantum simulation**

Kuwahara, Vu and Saito, arXiv:2206.14736

How many elementary quantum gates (e.g., CNOT, Hadamard, Phase shift gates) are sufficient to implement the quantum dynamics



? ≈



Quantum dynamics by the Bose–Hubbard model e^{-iHt} (N: system size, D: spatial dimension)

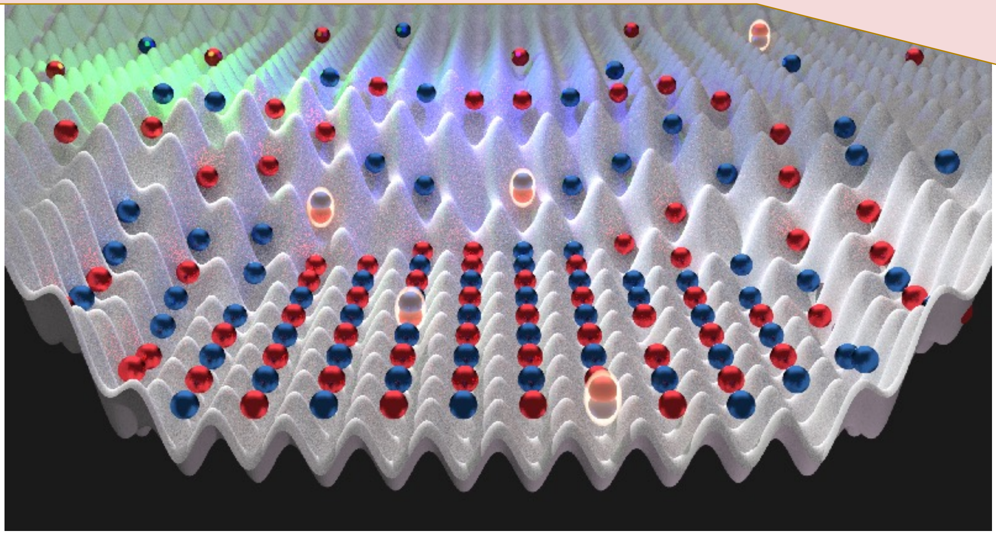
Sufficient number of quantum gates up to an error ε

$$Nt^{D+1}\mathbf{polylog}(Nt/\varepsilon)$$

Result 3 in arXiv:2206.14736

Kuwahara, Vu and Saito, arXiv:2206.14736

- .g., CNOT, Hadamard, Phase
  ne quantum dynamics



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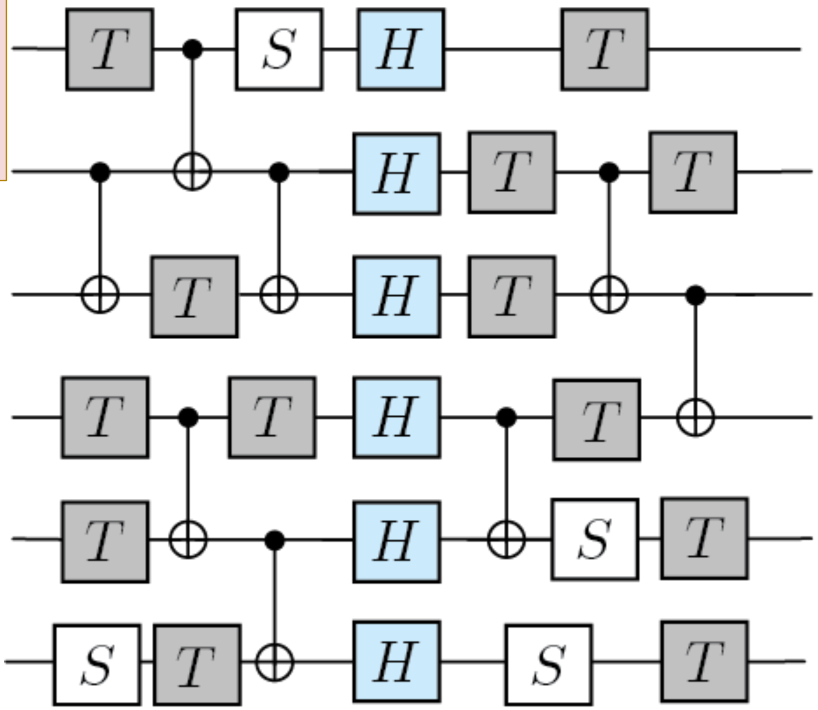


4: Proofs of main results

Summary:

Derivation of the finite speed of boson transport

$$[\hat{n}_X(t)]^s \preccurlyeq \left[\hat{n}_{X[R]} + e^{-\mathcal{O}(R/t)} + \mathcal{O}(st)\right]^s$$

Proof idea 1:
Extending the result by Schuch-Harrison-Osborne-Eisert

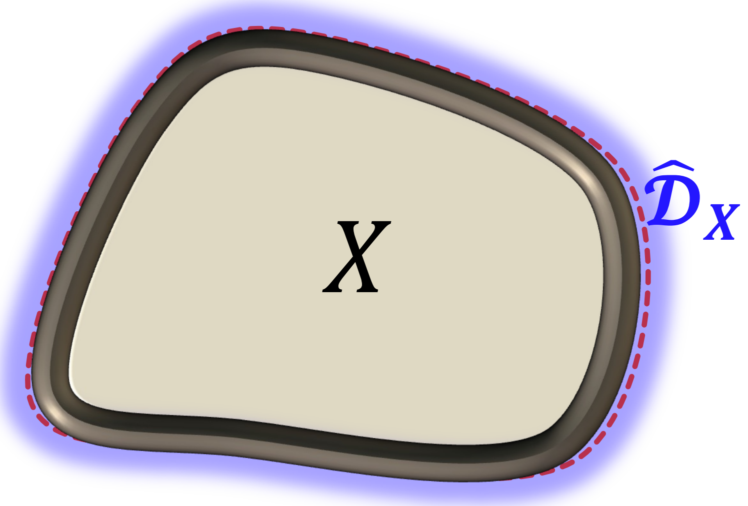
N. Schuch, S. K. Harrison, T. J. Osborne, and J. Eisert, Phys. Rev. A **84**, 032309 (2011).

- Original result implies

$$\hat{n}_X(\tau) \preccurlyeq \hat{n}_X + c_{\tau,1}\hat{\mathcal{D}}_X + c_{\tau,2}$$

$$\hat{\mathcal{D}}_X = \sum_{i\in\partial X}\sum_{j\in\Lambda} e^{-d_{i,j}}\hat{n}_j$$ ➡ Boson number on the surface of X

$$c_{\tau,1}, c_{\tau,2} = e^{\mathcal{O}(\tau)}$$



- Generalization to $\hat{n}_X^s(\tau)$

[Subtheorem 1 in arXiv:2206.14736]

$$[\hat{n}_X(\tau)]^s \preccurlyeq \left[\hat{n}_X + c_{\tau,1}\hat{\mathcal{D}}_X + c_{\tau,2}s\right]^s$$

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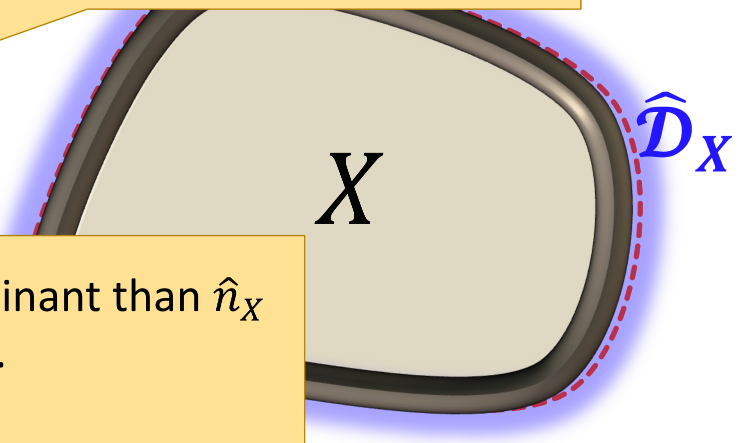
$$\hat{\mathcal{D}}_X = \sum_{i\in\partial X}\sum_{j\in\Lambda} e^{-d_{i,j}}\hat{n}_j$$

➡ Boson number on the surface

$$c_{\tau,1}, c_{\tau,2} = e^{\mathcal{O}(\tau)}$$

Meaningful only for a short time $\boldsymbol{\tau = \mathcal{O}(1)}$ …



- Generalization to $\hat{n}_X^s(\tau)$

[Subtheorem 1 in arXiv:2206.14736]

$$[\hat{n}_X(\tau)]^s \preccurlyeq [\hat{n}_X + c_{\tau,1}\hat{\mathcal{D}}_X + c_{\tau,2}s]$$

$\hat{\mathcal{D}}_X$ should be less dominant than $\hat{n}_X$ if X is sufficiently large.

↕

Most of the boson may concentrate on ∂X.

Proof idea 2: Appropriate choice of X

- Optimization problem as $\max_{\ell} \langle\psi| \hat{n}_{X[\ell]} + c_{\tau,1}\widehat{\mathcal{D}}_{X[\ell]} + c_{\tau,2} |\psi\rangle$ for $\forall|\psi\rangle$

$$\Leftarrow \langle\psi|\hat{n}_X(\tau)|\psi\rangle \le \langle\psi|\hat{n}_{X[\ell]}(\tau)|\psi\rangle \le \langle\psi| \hat{n}_{X[\ell]} + c_{\tau,1}\widehat{\mathcal{D}}_{X[\ell]} + c_{\tau,2} |\psi\rangle$$

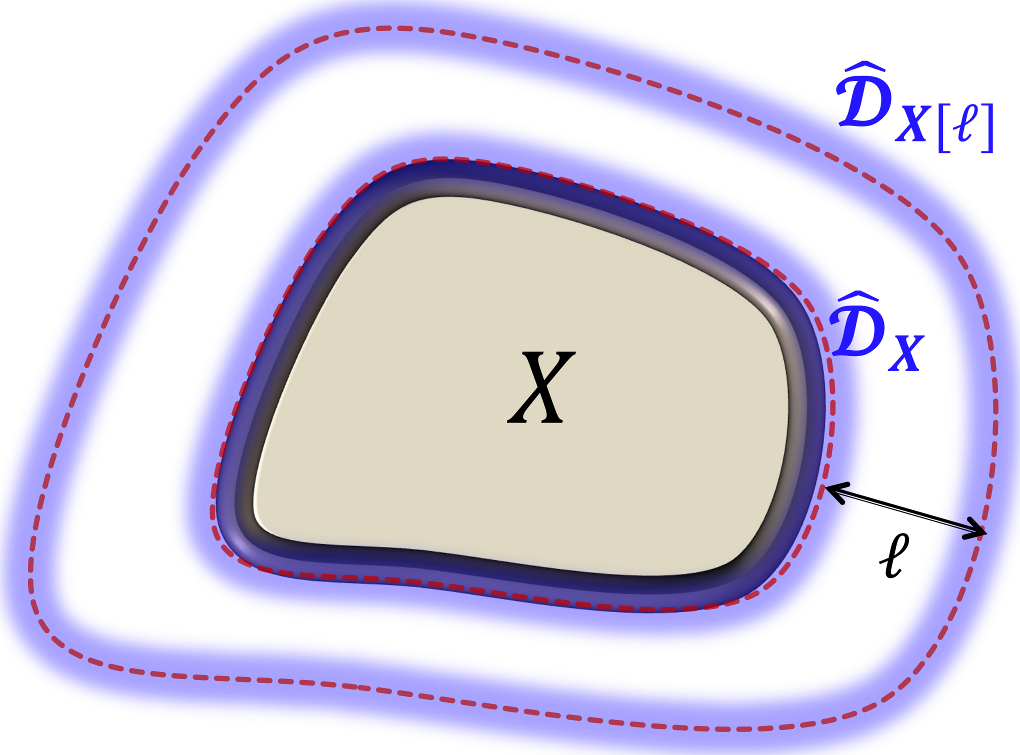
- Optimal choice gives

[Propositions 16,18 in arXiv:2206.14736]

$$\hat{n}_X(\tau) \preccurlyeq \hat{n}_{X[\ell]} + e^{-\mathcal{O}(\ell)} + c_{\tau,2}$$

⇓ General order *s*

$$[\hat{n}_X(\tau)]^s \preccurlyeq \left[\hat{n}_{X[\ell]} + e^{-\mathcal{O}(\ell)} + c_{\tau,2}s\right]^s$$



Proof idea 3:
Connecting short-time evolutions

- From time 0 to τ, we have

$$\hat{n}_X(\tau) \lesssim \hat{n}_{X[\ell]} + e^{-\mathcal{O}(\ell)} + c_{\tau,2}$$

↓

- From time τ to 2τ, we have

$$\hat{n}_X(2\tau) \lesssim \hat{n}_{X[\ell]}(\tau) + e^{-\mathcal{O}(\ell)} + c_{\tau,2} \lesssim \hat{n}_{X[2\ell]} + 2\left(e^{-\mathcal{O}(\ell)} + c_{\tau,2}\right)$$

↓ Repeating the same process

- From time $(m-1)\tau$ to $m\tau$, we have

$$\hat{n}_X(m\tau) \lesssim \hat{n}_{X[m\ell]} + m\left(e^{-\mathcal{O}(\ell)} + c_{\tau,2}\right)$$

Proof idea 3: Connecting short-time evolutions

- From time 0 to τ, we have

$$\hat{n}_X(\tau) \preccurlyeq \hat{n}_{X[\ell]} + e^{-\mathcal{O}(\ell)} + c_{\tau,2}$$

$$\Downarrow$$

- From time τ to 2τ, we have

$$\hat{n}_X(2\tau) \preccurlyeq \hat{n}_{X[\ell]}(\tau) + e^{-\mathcal{O}(\ell)} + c_{\tau,2}$$

$$\Downarrow$$ Repeating the same process

- From time $(m-1)\tau$ to $m\tau$, we have

$$\hat{n}_X(m\tau) \preccurlyeq \hat{n}_{X[m\ell]} + m\left(e^{-\mathcal{O}(\ell)} + c_{\tau,2}\right)$$

Choosing m, ℓ, τ as

$$\boldsymbol{t = m\tau\,,\ \ m\ell = R} \Rightarrow \boldsymbol{m = t/\tau\,,\ \ell = \tau R/t\,,\ \tau = \mathcal{O}(1)}$$

$$\boldsymbol{\hat{n}_X(t) \preccurlyeq \hat{n}_{X[R]} + \frac{t}{\tau}\left(e^{-\mathcal{O}(R/t)} + c_{\tau,2}\right)}$$

4: Proofs of main results

Summary:

Derivation of the Lieb-Robinson light cone

$$\left\| \left[O_{X_0}(t) - O_{X_0}\left(H_{X_0[R]}, t\right) \right] \rho_0 \right\|_1 \leq \left\| O_{X_0} \right\| e^{-C\left(R/t^D\right)^{1/(\kappa D)}}$$

Proof idea 1: Effective Hamiltonian theory

[Corollary 26 in arXiv:2206.14736]

- $\Pi_{i,\le q}$: projection for boson truncation up to q at the site i
- Effective Hamiltonian $\widetilde{H} := \overline{\Pi}_{X,q} H \overline{\Pi}_{X,q}$ $\qquad \overline{\Pi}_{X,q} = \prod_{i\in X} \Pi_{i,\le q}$
- Initial state ρ_0 : low-boson density $\qquad \max_i[\mathrm{tr}(\hat{n}_i^s \rho)] \le \frac{1}{e}\left(\frac{b_\rho}{e} s^\kappa\right)^s$
- Error estimation for arbitrary projection $\overline{\Pi}$ ($\overline{\Pi}^c = 1 - \overline{\Pi}$)

Frobenius norm

$$\left\|O(H,\tau)\rho - O(\bar{\Pi}H\bar{\Pi},\tau)\rho\right\|_1 \le \left\|\bar{\Pi}^{\mathrm{c}} O(\tau)\sqrt{\rho}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}} O\sqrt{\rho(\tau)}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}}\sqrt{\rho}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}}\sqrt{\rho(\tau)}\right\|_F$$

$$+ \int_0^\tau \left\|\bar{\Pi} H_0 \bar{\Pi}^{\mathrm{c}} O(\tau-\tau_1)\sqrt{\rho(\tau_1)}\right\|_F d\tau_1 + \int_0^\tau \left\|\bar{\Pi} H_0 \bar{\Pi}^{\mathrm{c}}\sqrt{\rho(\tau_1)}\right\|_F d\tau_1,$$

Free boson term

Proof idea 1:
Effective Hamiltonian theory

[Corollary 26 in arXiv:2206.14736]

- $\Pi_{i,\leq q}$: projection for boson truncation up to q at the site i
- Effective Hamilt
- Initial state ρ_0 :
- Error estimation for arbi rojection $\bar{\Pi}$ $(\bar{\Pi}^c = 1 - \bar{\Pi})$

It roughly gives

$$\|[\boldsymbol{O}(\boldsymbol{H},\boldsymbol{t}) - \boldsymbol{O}(\bar{\boldsymbol{\Pi}}\boldsymbol{H}\bar{\boldsymbol{\Pi}},\boldsymbol{t})]\,\boldsymbol{\rho}_0\|_1 \lesssim \mathbf{tr}\,[\bar{\boldsymbol{\Pi}}^c\,\boldsymbol{\rho}_0(\mathbf{t})\,\bar{\boldsymbol{\Pi}}^c]$$

It is critical to estimate the boson number distribution after the time evolution e^{-iHt}

Frobenius norm

$$\left\|O(H,\tau)\rho - O(\bar{\Pi}H\bar{\Pi},\tau)\rho\right\|_1 \leq \left\|\bar{\Pi}^{\mathrm{c}}O(\tau)\sqrt{\rho}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}}O\sqrt{\rho(\tau)}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}}\sqrt{\rho}\right\|_F + \left\|\bar{\Pi}^{\mathrm{c}}\sqrt{\rho(\tau)}\right\|_F$$

$$+\int_0^\tau \left\|\bar{\Pi}H_0\bar{\Pi}^{\mathrm{c}}O(\tau-\tau_1)\sqrt{\rho(\tau_1)}\right\|_F d\tau_1 + \int_0^\tau \left\|\bar{\Pi}H_0\bar{\Pi}^{\mathrm{c}}\sqrt{\rho(\tau_1)}\right\|_F d\tau_1,$$

Free boson term

Proof idea 2: Boson number distribution after time evolution

- $\Pi_{i,\leq q}$: projection for boson truncation up to q at the site i
- Initial state ρ_0 : low-boson density
- Upper bound for the boson number distribution after time evolution

Lemma 28 in arXiv:2206.14736]

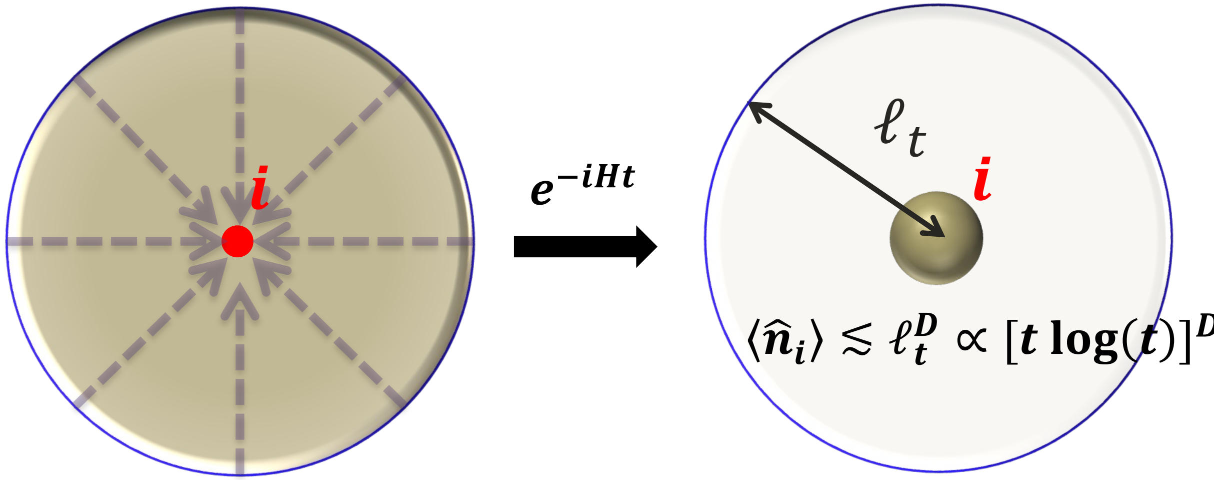
Result 1 (finite speed of boson transport).

$$[\hat{n}_X(t)]^s \preccurlyeq \left[\hat{n}_{X[R]} + e^{-\mathcal{O}(R/t)} + \mathcal{O}(st)\right]^s$$

$$\Downarrow$$

The number of bosons at each site is at most of $\mathcal{O}(\ell_t^D)$

$\ell_t \propto t\log(t)$

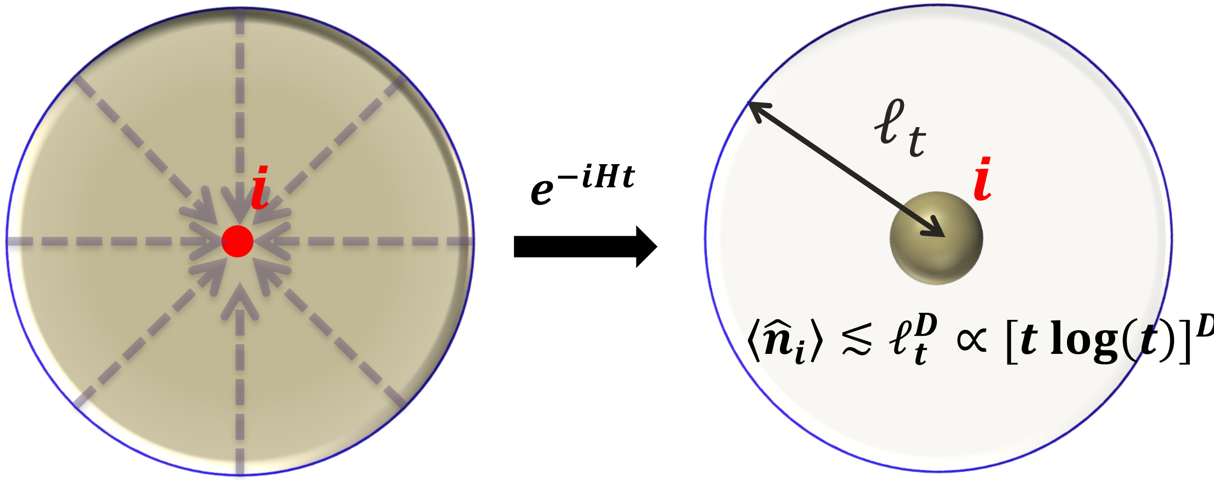


Proof idea 3:
Lieb-Robinson velocity（looser bound）

- The number of boson truncation should be taken as $\boldsymbol{\mathcal{O}(\ell_t^D) \propto [t \log(t)]^D}$
- One-site energy ∝ boson number truncation $= \tilde{\mathcal{O}}(t^D)$
- Lieb-Robinson velocity $= \tilde{\mathcal{O}}(t^D)$

Theorem 2 in arXiv:2206.14736]

$$\left\|\left(O_{X_0}(t) - O_{X_0}(H_{X[R]}, t)\right)\rho_0\right\|_1 \le \exp\left[-c\left(\frac{R}{t(t\log t)^D}\right)^{1/\kappa} + \log(|X_0[R]|)\right]$$



Proof idea 3:
Lieb-Robinson velocity (looser bound)

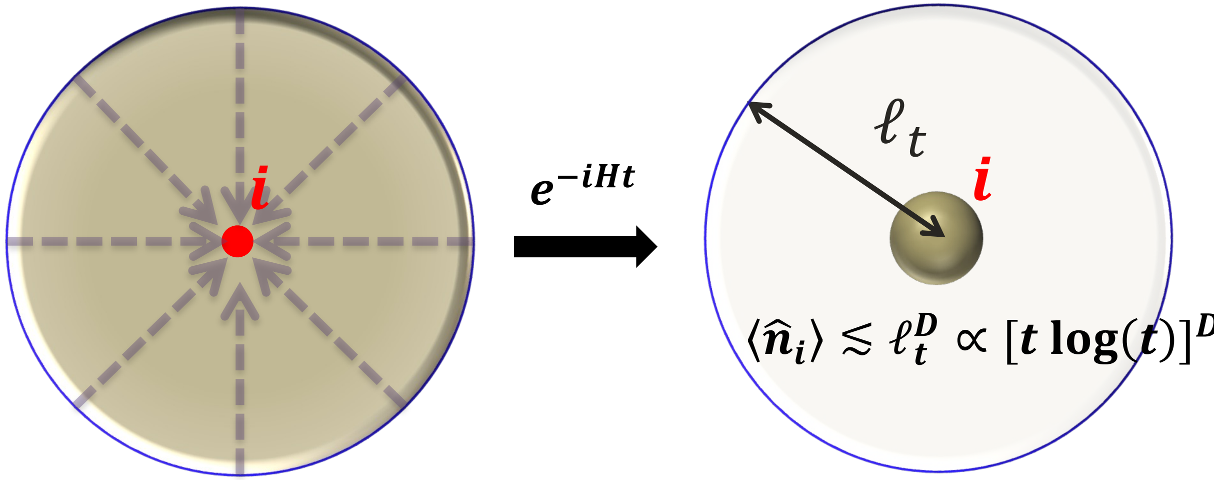
- The number of boson truncati $\mathbf{g(t)]^D}$
- One-site energy ∝ boson numl
- Lieb-Robinson velocity $= \tilde{\mathcal{O}}(t^D)$

Worse than the expected speed $\tilde{\mathcal{O}}(t^{D-1})$

How to improve it?

Theorem 2 in arXiv:2206.14736]

$$\left\| \left(O_{X_0}(t) - O_{X_0}(H_{X[R]}, t)\right) \rho_0 \right\|_1$$
$$\le \exp\left[-c\left(\frac{R}{t(t\log t)^D}\right)^{1/\kappa} + \log(|X_0[R]|)\right]$$



Proof idea 4:
Average boson number

- Point: **Boson concentration cannot occur simultaneously**

- One-dimensional region:

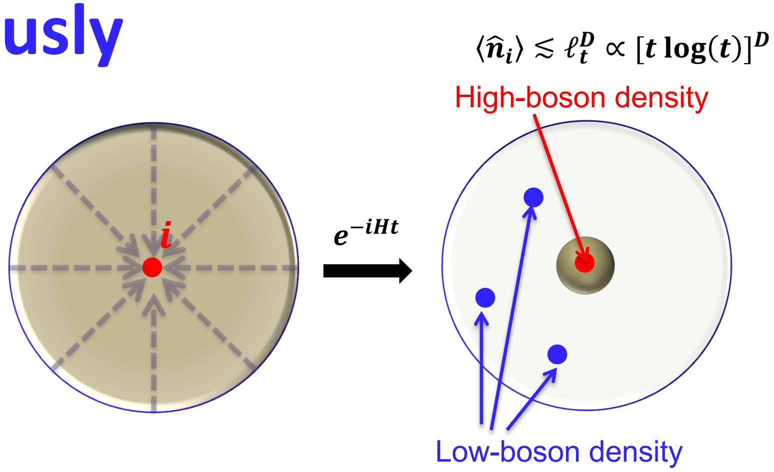
Average boson number $\le \tilde{\mathcal{O}}(t^{D-1})$

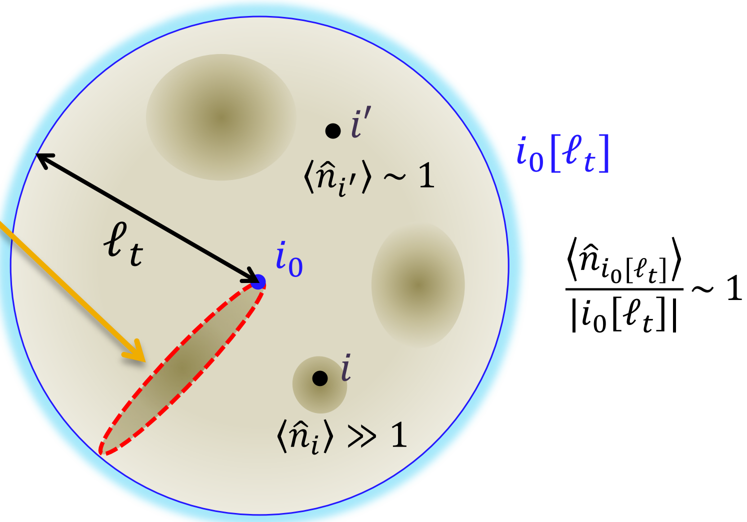
↓

- Lieb-Robinson velocity $= \tilde{\mathcal{O}}(t^{D-1})$

- Difficulty:

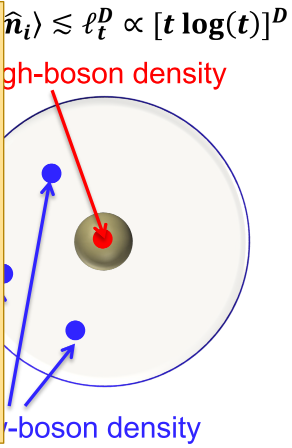
The configuration of the boson densities is not fixed. A quantum state includes various configurations as superposition.





Proof idea 4:
Average boson number

- Point: **Boson concentration cannot occur simultaneously**



Over half of the proof is devoted to this improvement !

Can we get a simpler proof?

- One-dimensional region:

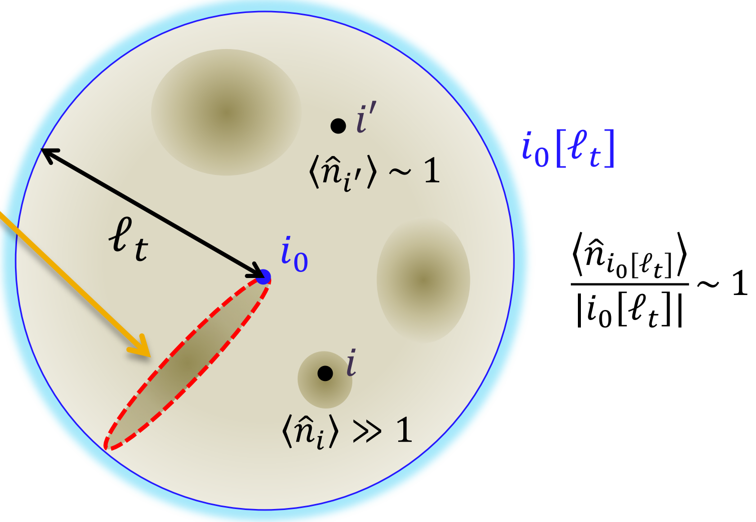
Average boson number $\leq \tilde{\mathcal{O}}(t^{D-1}$

⬇

- Lieb-Robinson velocity $= \tilde{\mathcal{O}}(t^{D-1})$

- Difficulty:

The configuration of the boson densities is not fixed. A quantum state includes various configurations as superposition.



5: Optimal transfer protocol in bosonic systems

Summary:

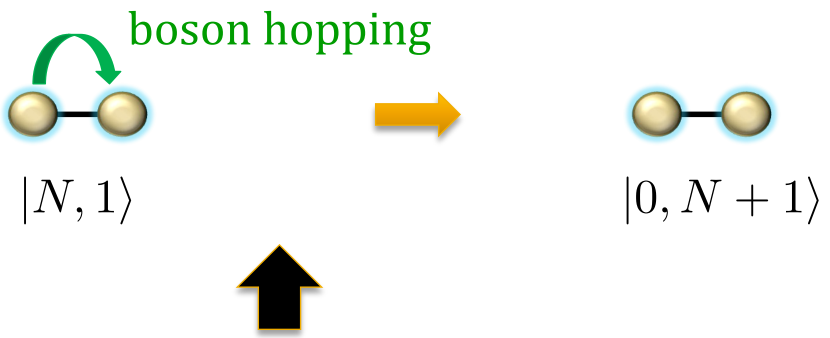
Optimality of the **Effective light cone as** $R \propto t^D$

Step 1: Collection of bosons onto the information path
Step 2: CNOT operation on the information path

Step 1:
Collection of bosons onto the information path

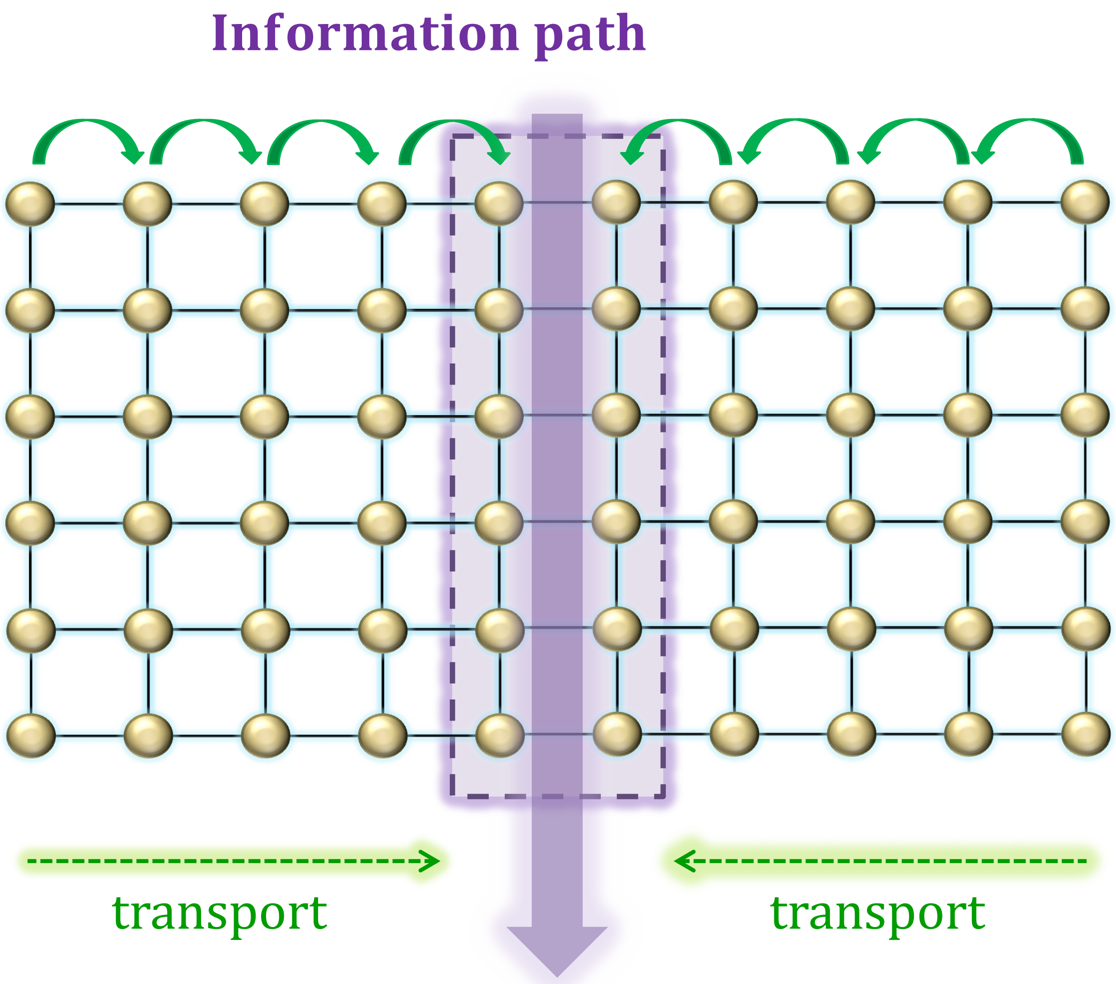
- Initial state: Mott state with one boson

- Boson hopping



- It takes a time of $\mathcal{O}(1)$

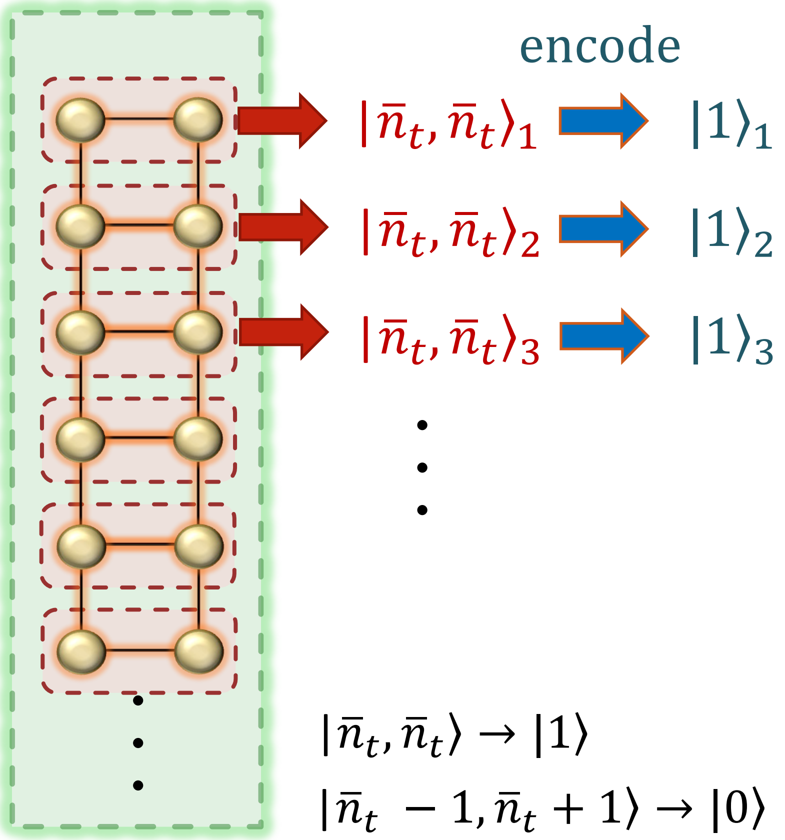
Using half of the time, $\bar{n}_t = \mathcal{O}(t^{D-1})$ bosons concentrate on the information path.



$v \propto \bar{n}_t = \mathcal{O}(t^{D-1})$

Step 2:
CNOT operation on the information path

- Encoding to qubit



encode

$|\bar{n}_t,\bar{n}_t\rangle_1 \rightarrow |1\rangle_1$

$|\bar{n}_t,\bar{n}_t\rangle_2 \rightarrow |1\rangle_2$

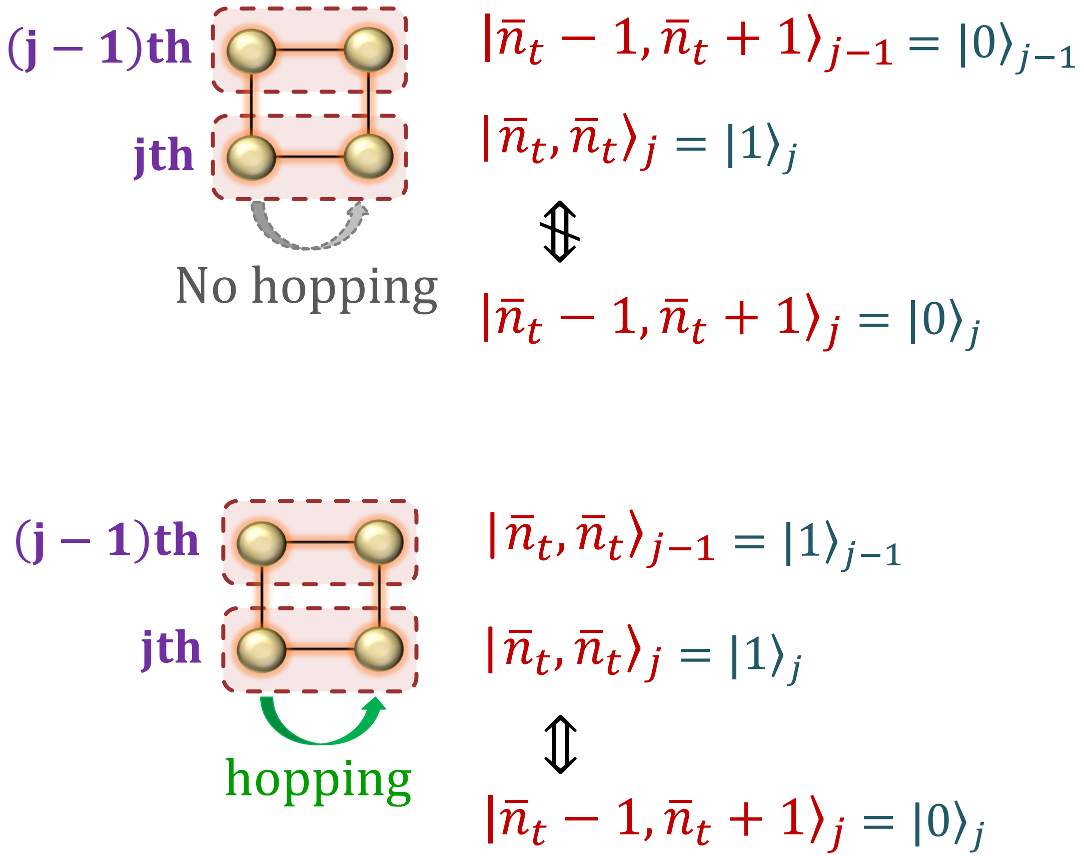
$|\bar{n}_t,\bar{n}_t\rangle_3 \rightarrow |1\rangle_3$

⋮

$|\bar{n}_t,\bar{n}_t\rangle \rightarrow |1\rangle$

$|\bar{n}_t - 1,\bar{n}_t + 1\rangle \rightarrow |0\rangle$

- CNOT operation



(j – 1)th

jth

No hopping

$|\bar{n}_t - 1,\bar{n}_t + 1\rangle_{j-1} = |0\rangle_{j-1}$

$|\bar{n}_t,\bar{n}_t\rangle_j = |1\rangle_j$

$\Updownarrow$

$|\bar{n}_t - 1,\bar{n}_t + 1\rangle_j = |0\rangle_j$

(j – 1)th

jth

hopping

$|\bar{n}_t,\bar{n}_t\rangle_{j-1} = |1\rangle_{j-1}$

$|\bar{n}_t,\bar{n}_t\rangle_j = |1\rangle_j$

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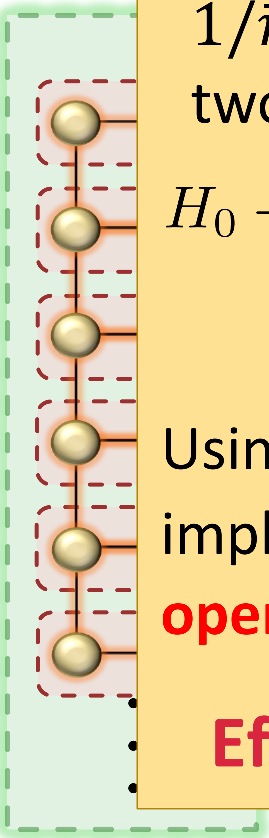
$|\bar{n}_t - 1,\bar{n}_t + 1\rangle_j = |0\rangle_j$

Step 2:
CNOT operation on the information path

- Encoding to qubit



$|\bar{n}_t - 1, \bar{n}_t + 1\rangle \to |0\rangle$

The operation can be implemented by $1/\bar{n}_t$ time using the Hamiltonian with two-body interactions

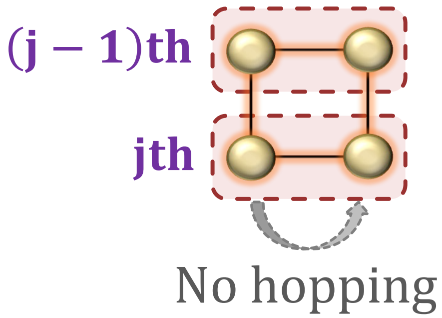
$$H_0 + h(\hat{n}_2 - \hat{n}_1)\hat{n}_3 + U(\hat{n}_3\hat{n}_4 + \hat{n}_4 - \bar{n}_t)$$

$$h, U \gg 1$$

Using the latter half of the time, one can implement **$(t/2)\bar{n}_t = \mathcal{O}(t^D)$ CNOT operations** on the information path.

Effective light cone as $R \propto t^D$

- CNOT operation

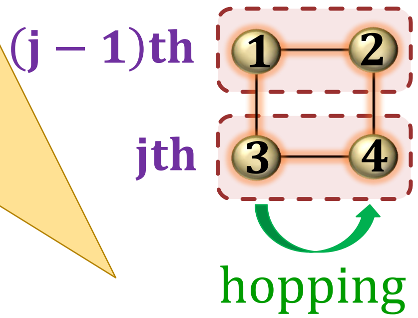


$|\bar{n}_t - 1, \bar{n}_t + 1\rangle_{j-1} = |0\rangle_{j-1}$

$|\bar{n}_t, \bar{n}_t\rangle_j = |1\rangle_j$

$\Updownarrow$

$|\bar{n}_t - 1, \bar{n}_t + 1\rangle_j = |0\rangle_j$



$|\bar{n}_t, \bar{n}_t\rangle_{j-1} = |1\rangle_{j-1}$

$|\bar{n}_t, \bar{n}_t\rangle_j = |1\rangle_j$

$\Updownarrow$

$|\bar{n}_t - 1, \bar{n}_t + 1\rangle_j = |0\rangle_j$

Summary

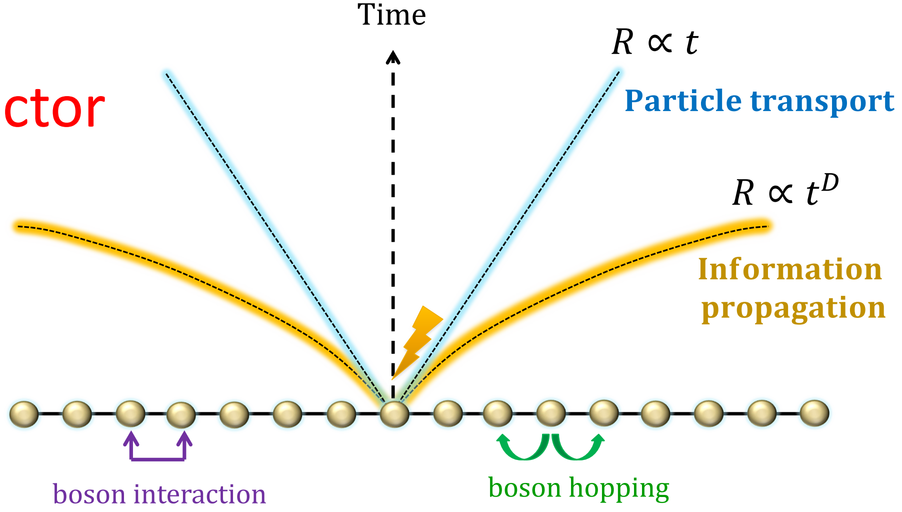
- We have identified the effective light cones for the Bose-Hubbard type model.

$$H = \sum_{\langle i,j \rangle} J_{i,j}(b_i b_j^\dagger + \mathrm{h.c.}) + \sum_i f(\hat{n}_i, \hat{n}_{i_1}, \hat{n}_{i_2}, \ldots, \hat{n}_{i_k})$$

- The obtained light cone is optimal up to a logarithmic factor

(1) Speed of boson transport $R \propto t$
(Result 1 in arXiv:2206.14736)

(2) Speed of total information propagation $R \propto t^D$
(Result 2 in arXiv:2206.14736)



- Gate complexity of the quantum simulation

$$Nt^{D+1}\mathbf{polylog}(Nt/\varepsilon)$$

Future direction:

- Prove the linear light cone in other natural setups (e.g., translation invariance)
- Integrating long-range interactions and boson interactions

Summary

- We have identified the effective light cones for the Bose-Hubbard type model.

$$H=\sum_{\langle i,j\rangle} J_{i,j}(b_i b_j^\dagger+\mathrm{h.c.})+\sum_i f(\hat{n}_i,\hat{n}_{i_1},\hat{n}_{i_2},\dots,\hat{n}_{i_k})$$

- The obtained light cone is optimal up to a logarithmic factor

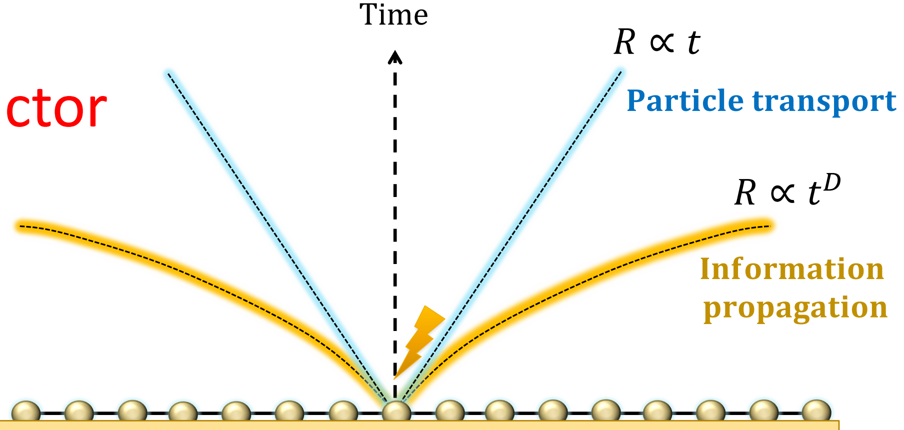
(1) Speed of boson transport $R \propto t$
(Result 1 in arXiv:2206.14736)

(2) Speed of tota
(Result 2 in arXi

Partial success
Lemm and Kuwahara, in preparation

- Gate complexity quantum simulation

$Nt^{D+1}\mathrm{polylog}(Nt/\varepsilon)$



Partial success (macroscopic boson transport)
Vu-Kuwahara-Saito, arXiv:2307.01059

Future direction:

- Prove the linear light cone in other natural setups (e.g., translation invariance)
- Integrating long–range interactions and boson interactions