Glue-on AdS holography for $T\bar{T}$-deformed CFTs *JHEP 06 (2023) 117*

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$T\bar{T}$-deformed CFTs with $\mu < 0$ have been proposed by *McGough, Mezei & Verlinde*, 1611.03470 [1] to be holographically dual to Einstein gravity with a finite Dirichlet cutoff.

We generalize the proposal for $\mu > 0$ by *gluing-on* another patch of AdS$_3$, and provide various evidence for this extended holography, now valid for $\mu \in \mathbb{R}$.

Pure Einstein gravity without matter in AdS$_3$:

$$ds^2 = \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{\big(du + \rho\,\bar{\mathcal{L}}(v)\,dv\big)\big(dv + \rho\,\mathcal{L}(u)\,du\big)}{\rho}\right)$$
$$= n_\mu n_\nu dx^\mu dx^\nu + \frac{1}{\zeta}\gamma_{ij}dx^i dx^j, \qquad (1)$$

- Transverse coordinates: $x^i \sim \varphi, t$
  Light-cone coordinates: $u, v = \varphi \pm t$

  $\mathcal{L}(u), \bar{\mathcal{L}}(v)$: arbitrary periodic functions
- Radial coordinate: $\ell^{-2}g_{\varphi\varphi} \equiv r^2 \equiv \zeta^{-1}$
  r: the "proper radius"

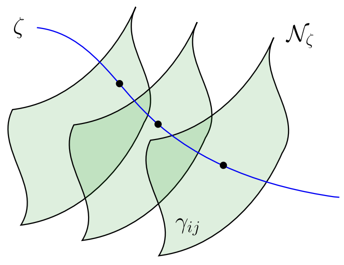


Image courtesy: *R. Szalai*, DOI:10.1007/s11071-020-05891-1

- Structure: foliated by constant ζ surfaces $\mathcal{N}_\zeta$
  Asymptotic boundary: $\mathcal{N}_0$ is at $\zeta \to 0$

Fefferman & Graham, 0710.0919 [2]
Banados, Teitelboim & Zanelli, hep-th/9204099 [3]

"$T\bar{T}$" deformation as the flow of action:

$$\partial_\mu I = 8\pi\int d^2x\, T\bar{T}_{(\mu)} = \pi\int d^2x\,\big(T^{ij}T_{ij} - (T^i_i)^2\big)_{(\mu)},$$
$$x, \bar{x} = \varphi' \pm t' \qquad (2)$$

- **Irrelevant:** initially CFT$_2$: $T\bar{T}_{(\mu=0)} = T_{xx}T_{\bar{x}\bar{x}}$, but conformal symmetry will be broken for $\mu \neq 0$.
- **Solvable:** the deformed spectrum of $\hat{H}(\mu)$ and $\hat{J}(\mu)$ on a cylinder of radius R is a simple function:

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu}{R}E(0) + \frac{4\mu^2}{R^4}J(0)^2}\right), \qquad (3)$$
$$J(\mu) = J(0)$$

of the undeformed spectrum $E(0)$, $J(0)$.
(*under reasonable assumptions*)

Zamolodchikov, hep-th/0401146 [4]
Dubovsky, Flauger & Gorbenko, 1205.6805 [5]
Dubovsky, Gorbenko & Mirbabayi, 1305.6939 [6]
Smirnov & Zamolodchikov, 1608.05499 [7]
Cavaglià, Negro, Szécsényi & Tateo, 1608.05534 [8] *et al*

Cutoff AdS$_3$ duality: Holography within a finite Dirichlet wall

McGough, Mezei & Verlinde, 1611.03470 [1]
Kraus, Liu & Marolf, 1801.02714 [9] *et al*

Dictionary: radial location ζ_c of the cutoff surface $\mathcal{N}_{\zeta_c}$ gets mapped to the deformation parameter μ:

$$\zeta_c = -\frac{c\mu}{3\ell^2} \qquad (4)$$

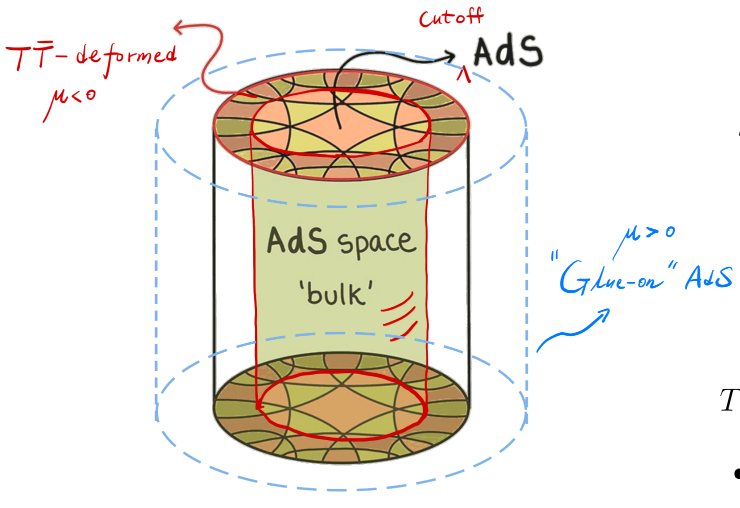
$T\bar{T}$ flow recast geometrically as the i,j components of the Einstein equations.

This is a "non-AdS" / non-CFT duality.
A step towards quantum gravity in reality^{TM}*!*

Caveat: the duality only admits $\zeta_c > 0$ so $\mu < 0$.
But $T\bar{T}$ itself admits $\mu > 0$ with nice properties.

The related proposal of *Guica & Monten*, 1906.11251 [10] admits both signs of μ.

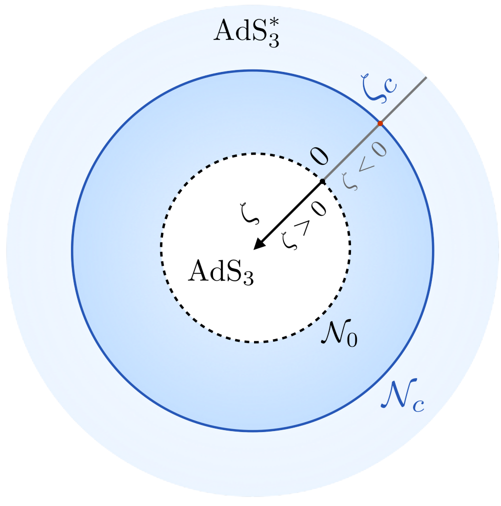
What is the other side of the duality?



Cutoff AdS$_3$ / $T\bar{T}$-deformed CFT$_2$
Image courtesy: *Aldegunde*, 2022 [11]

Proposal: Glue-on AdS$_3$ – beyond the infinity

Cutoff / ***glue-on*** AdS$_3$ **Gravity** $\equiv$ $T\bar{T}$ **deformed** CFT$_2$ **at** $\mathcal{N}_{\zeta_c}$
$\zeta_c > 0$ / $\zeta_c < 0$ $\qquad\qquad \mu \in \mathbb{R}$



Top-down view of a constant t slice

- The metric (1) has a simple pole, thus admits a well-defined **analytic continuation:**

$$\frac{1}{\zeta} \equiv \ell^{-2}g_{\varphi\varphi} \in \mathbb{R}, \qquad \zeta_c = -\frac{c\mu}{3\ell^2} \in \mathbb{R} \qquad (5)$$

What are we doing other than copy-pasting?
"Renormalize" the divergences as $\zeta \to 0^\pm$:

  – introduce $\mathcal{N}_{\zeta=\pm\epsilon}$ and **"glue"** them together;
  – i.e. exclude the $-\epsilon < \zeta < \epsilon$ region, until finally sending $\epsilon \to 0$.

- **Extending the flow:** matching energy momentum (Brown-York) & the $T\bar{T}$ flow equations:

$$T_{ij} = \frac{\sigma_{\zeta_c}}{8\pi G}\Big(K_{ij} - K\gamma_{ij} + \frac{1}{\ell|\zeta_c|}\gamma_{ij}\Big),$$
$$\sigma_{\zeta_c} = \tfrac{|\zeta_c|}{\zeta_c} = \pm 1, \qquad \gamma_{ij} = \zeta_c h_{ij}, \qquad (6)$$

h_{ij}: the induced metric.

The field theory metric γ_{ij} is always positive-definite, while h_{ij} becomes negative-definite for the glue-on region $\rho, \zeta < 0$.

This discrepancy is the origin of $|\zeta_c|$.

- Demanding a **non-singular extended geometry** reproduces bounds on the $T\bar{T}$ deformed theory, e.g.

$$\zeta_c \geq -1 \quad \Leftrightarrow \quad \mu \leq \frac{3\ell^2}{c} \qquad (7)$$

This is a Killing horizon in the glue-on region of the extended geometry; $(\det g_{\mu\nu})_{\zeta=-1} = 0$.

- **Spectrum from the extended geometry:** they are conserved charges $\mathcal{Q}$ of the Killing symmetries, and can be computed with the *covariant formalism.*

Iyer & Wald, Barnich & Brandt et al
With $T\bar{T}$: *Kraus, Monten & Myers*, 2103.13398 [14]

$$\delta E(\mu) \equiv \ell^{-1}\delta\mathcal{Q}_{\partial_{t'}}, \quad \delta J(\mu) \equiv \delta\mathcal{Q}_{\partial_{\varphi'}}.$$

This reproduces $E(\mu)$, $J(\mu)$ with $\mu \in \mathbb{R}$ in (3).
(t', φ'): normalized boundary coordinates:

$$ds_c^2 = \ell^2(-dt'^2 + d\varphi'^2), \qquad \varphi' \sim \varphi' + 2\pi. \qquad (8)$$

- ***State-dependent*** **maps of coordinates:**

$$dt' = \sqrt{\big(1 - \zeta_c(T_u + T_v)^2\big)\big(1 - \zeta_c(T_u - T_v)^2\big)}\,dt,$$
$$d\varphi' = d\varphi + \zeta_c(T_u^2 - T_v^2)\,dt. \qquad (9)$$

$\rightsquigarrow$ correct charges, the modified signal propagation speed $v'_\pm \equiv \pm d\varphi'/dt'$, and $T\bar{T}$ **thermodynamics** upon Wick rotation. In particular,

$$\mu > 0, \quad T_L(\mu)\,T_R(\mu) \leq -\frac{1}{4\pi^2\ell^2\zeta_c} = \frac{3}{4\pi^2 c\mu} = T_H(\mu)^2.$$

$T_{L,R}$: temperatures associated with $u', v' = \varphi' \pm t'$.
T_H: the *Hagedorn temperature*: exceeding T_H leads to a complex entropy.

Giveon, Itzhaki & Kutasov, 1701.05576 [15]
Apolo, Detournay & Song, 1911.12359 [16]

$T\bar{T}$ partition functions from the bulk

- Via bulk on-shell action of the dominant saddle:

$$Z_{T\bar{T}}(\mu) = \mathcal{Z}(\zeta_c) \approx e^{-I(\zeta_c)}. \qquad (10)$$

$I(\zeta_c)$ diverges at $\zeta \to 0^\pm$: we need *holographic renormalization*, with the boundary action:

$$I_{\mathcal{N}_\zeta} := -\frac{1}{8\pi G}\int_{\mathcal{N}_\zeta} d^2x\sqrt{h}\,h^{ij}K_{ij} \qquad (11)$$
$$+ \frac{\sigma_\zeta}{8\pi G}\int_{\mathcal{N}_\zeta} d^2x\sqrt{h}\left(\ell^{-1} - \frac{\ell\mathcal{R}[h]}{4}\log|\zeta|\right).$$

Henningson & Skenderis, hep-th/9806087 [17] *et al*

$\log|\zeta|$: not diff-invariant, due to the Weyl anomaly.
$I_{\mathcal{N}_\zeta}$: consistent with the B-Y stress tensor (6).

$Z_{T\bar{T}}$ agrees with the field theory analysis, using (9):

- **Torus:** modular invariance & sparseness of the "light" spectrum at large $c \rightsquigarrow$ universal form:

$$\log Z_{T\bar{T}}(\mu) \approx \begin{cases} -\frac{1}{2}(\beta_L + \beta_R)\,RE_{\text{vac}}(\mu), & \beta_L\beta_R > 1, \\ -2\pi^2\Big(\frac{1}{\beta_L} + \frac{1}{\beta_R}\Big)RE_{\text{vac}}\Big(\frac{4\pi^2}{\beta_L\beta_R}\mu\Big), & \beta_L\beta_R < 1. \end{cases}$$

Datta & Jiang, 1806.07426 [18]
Apolo, Song & Yu, 2301.04153 [19]
cf. *Hartman, Keller & Stoica*, 1405.5137 [20]

- **Sphere:** maximally symmetric,

Donnelly & Shyam, 1806.07444 [21]

$$\langle T_{ij}\rangle = -\frac{1}{4\pi\mu}\left(1 - \sqrt{1 - \frac{c\mu}{3R^2}}\right)\gamma_{ij}. \qquad (12)$$

Trace relation: $(-R)\,\partial_R \log Z_{T\bar{T}}(\mu) = \int d^2x\sqrt{\gamma}\,\langle T^i_i\rangle$ and the flow equation (2) admit the general solution with a μ-*independent* integration constant a:

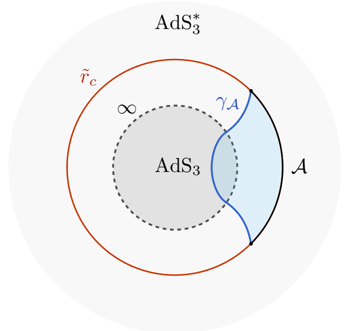
$$\log Z_{T\bar{T}}(\mu, a) = \tfrac{c}{3}\log\Big[\tfrac{R}{a}\Big(1 + \sqrt{1 - \tfrac{c\mu}{3R^2}}\Big)\Big] - \tfrac{R^2}{\mu}\sqrt{1 - \tfrac{c\mu}{3R^2}} + \tfrac{R^2}{\mu}.$$

  – $a = \sqrt{c|\mu|/3}$: recover *Donnelly & Shyam* [21]
    $a = \epsilon$: also a valid choice, where the RG length scale ϵ is decoupled from μ.
  – Enlarge the **space of $T\bar{T}$ deformed theories:** with independent parameters (μ, a).
  – The $\log|\zeta|$ in (11) guarantees that $I = -\log Z_{T\bar{T}}$ satisfies the $T\bar{T}$ flow (2); not the case for [21].

Caputa, Datta, Jiang & Kraus, 2011.04664 [22]
Li, 2012.14414 [23]

- Future: understand the **entanglement structure** of $T\bar{T}$ deformation with the help of bulk geometry.

Lewkowycz, Liu, Silverstein & Torroba, 1909.13808 [24]



RT surface for the extended AdS$_3$

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Inspire: inspirehep.net/authors/2640135
GitHub: github.com/bryango