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Local operator quench and inhomogeneous time evolution

Coming soon

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Supplements

Deformed hamiltonians:

$$H_v = \int_0^L \frac{dx}{2\pi}\left(v(x)T(x) + \bar{v}(x)\bar{T}(x)\right)$$

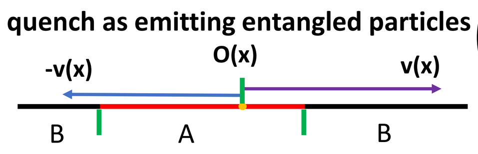
where $v(x) = v(x+L)$ and $\bar{v}(x) = \bar{v}(x+L)$ are two independent smooth real functions, dubbed deformation functions.
Here $T(x)$ and $\bar{T}(x)$ are the chiral and anti-chiral energy-momentum density, namely, $T+\bar{T}$ is the energy density and $T-\bar{T}$ the momentum density.

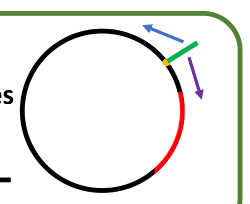
$$v_{\text{Möbius}}(x) = 1 - \tanh 2\theta \cos\left(\frac{2\pi x}{L}\right), v_{\text{SSD}}(x) = 2\sin^2\left(\frac{\pi x}{L}\right), v_{\text{CSD}}(x) = 2\cos^2\left(\frac{\pi x}{L}\right),$$

For $\theta = 0, H_v = H$. In SSD limit, $\theta \to \infty, H_v \to H_{\text{SSD}}$

Quasi-particle picture

Local quench as emitting entangled particles





Topological surface operators in general dimension $d > 2$

$$Q_\xi(\Sigma) = -\int_\Sigma d\sigma^\mu \xi^\nu T_{\mu\nu}$$

In $d = 2$

$$Q_\xi(\Sigma) = \oint_\Sigma \frac{1}{2\pi i}\left(dz\xi(z)T(z) - d\bar{z}\bar{\xi}(z)\bar{T}(z)\right)$$

Use heisenberg equation and OPE

$$O_t(z) := e^{itQ_\xi(\Sigma)}O(z)e^{-itQ_\xi(\Sigma)} \qquad \frac{d}{dt}O_t(z) = i[Q_\xi(\Sigma), O_t(z)]$$

$$[Q_\xi(\Sigma), O(z)] = \oint_C \frac{dw}{2\pi i}\xi(w)T(w)O(z) = \left(h\xi'(z)O(z) + \xi(z)\partial O(z)\right)$$

↓

$$\frac{dz_t}{dt} = i\xi(z_t), \quad \frac{d\bar{z}_t}{dt} = i\bar{\xi}(\bar{z}_t).$$

↓

Deformation contain velocity (Wick rotation to Lorentzian time)

$$\frac{dx_t^1}{dt} = v(x_t^1), \quad \text{with} \quad v(x^1) := \xi(z)|_{z=ix^1}$$

$$\frac{dx_t^1}{dt} = -\bar{v}(x_t^1), \quad \text{with} \quad \bar{v}(x^1) := \bar{\xi}(\bar{z})|_{\bar{z}=-ix^1}$$

Gravity dual

Consider three dimensional Euclidean gravity . Use the metric given by [4] we get the dual metric

$$ds^2 = 4GR\left(L(W)dW^2 + \bar{L}(\bar{W})d\bar{W}^2\right) + \left(R^2e^{2\rho} + 16G^2L(W)\bar{L}(\bar{W})e^{-2\rho}\right)dWd\bar{W} + R^2d\rho^2$$

$$W = x + i\tau = i\bar{w}, \bar{W} = x - i\tau = -iw$$

↓ In terms of $(w^{\text{New},i}, \bar{w}^{\text{New},i})$ coordinate

$$ds^2 = -4GR\left(\bar{L}(\bar{w})\left(\frac{\partial\bar{w}}{\partial\bar{w}_i^{New}}\right)^2(d\bar{w}_i^{New})^2 + L(w)\left(\frac{\partial w}{\partial w_i^{New}}\right)^2(dw_i^{New})^2\right) + \left(R^2e^{2\rho} + 16G^2L(\bar{w})L(w)e^{-2\rho}\right)\left(\frac{\partial\bar{w}}{\partial\bar{w}_i^{New}}\right)\left(\frac{\partial w}{\partial w_i^{New}}\right)dw_i^{New}d\bar{w}_i^{New} + R^2d\rho^2$$

Here we further do a coordinate transformation and then find the metrics look like AdS_3 space

$$L(w) = \langle T_{ww}(w)\rangle_i$$

$$X = \frac{w^{\text{New},i} - \bar{w}^{\text{New},i}}{2i}, \tau = \frac{w^{\text{New},i} + \bar{w}^{\text{New},i}}{2}$$

Analytic continuation $\tau \to it$

Definition of r [;2 means the coeffient coupled with e^(2 ρ)]

$$e^{-\rho} = \frac{-2}{\sqrt{\frac{f^i_{XX;2}(X,t)}{R^2}}}\left(r - \sqrt{r^2 + \frac{f^i_{XX;2}(X,t)}{R^2}}\right)$$

$$ds^2 = R^2d\rho^2 + f_{XX}(\rho,X,t)dX^2 + f_{Xt}(\rho,X,t)dXdt + f_{tt}(\rho,X,t)dtdt$$

Metric near the boundary (r -> Inf)

For i=1,2

$$ds^2 = R^2\left[\frac{dr^2}{r^2} + r^2\left(dX^2 - \tilde{C}_\alpha^2dt^2\right)\right]$$

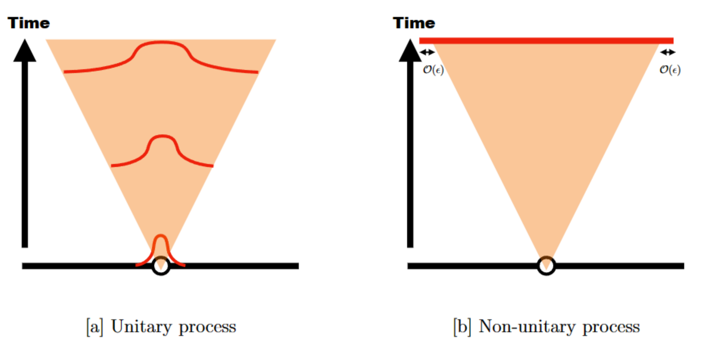
$$\tilde{C}^\alpha(X) = \begin{cases}\left[1 - \cos\left(\frac{2\pi X}{L}\right)\tanh 2\theta\right] & \text{for } \alpha = \text{Möbius} \\ 2\sin^2\left(\frac{\pi X}{L}\right)\end{cases}$$

For i=3,4

$$ds^2 = R^2\left[\frac{dr^2}{r^2} + r^2\left(dX^2 - dt^2\right)\right]$$

Setup & Motivation

$$\left|\Phi_a(\tilde{t})\right\rangle = \mathcal{N}_a e^{-iH_1\tilde{t}}e^{-\epsilon H_2}\mathcal{O}(x)|0\rangle$$
$$\left|\Phi_b(\tilde{t})\right\rangle = \mathcal{N}_b e^{-\epsilon H_2}e^{-iH_1\tilde{t}}\mathcal{O}(x)|0\rangle$$



[a] Unitary process   [b] Non-unitary process

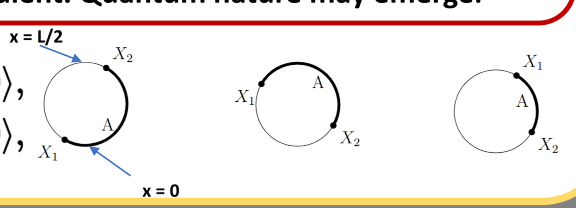
In the case of [a], "information" introduced by the smearing is delocalized during the time evolution. In the case of [b], the information introduced by the smearing is spatially localized.

The growth of operator considered is induced by the Euclidean and Lorentzian time evolution. Our undeformed, sine-square deformed, and Mobius Hamiltonians in 2d CFTs on the circle with the circumstance L. The system considered are defined on the spatial circle.

Motivation :

1. In this setup, entanglement entropy, two point function and so on can be analytically computable.
2. It is possible to study non-equilibrium process in the larger system than that numerically-computable.
3. We study inhomogeneous Lorentzian time evolution as an extend excited state respect to uniform case. SSD/Mobius quenches may make the system have the temperature gradient. Quantum nature may emerge.

$$|\Psi_1\rangle = \mathcal{N}_1 e^{-iH_{\text{Möbius}}t_1}e^{-\epsilon H_0}\mathcal{O}(x)|0\rangle, |\Psi_2\rangle = \mathcal{N}_2 e^{-\epsilon H_0}e^{-iH_{\text{Möbius}}t_1}\mathcal{O}(x)|0\rangle,$$
$$|\Psi_3\rangle = \mathcal{N}_3 e^{-iH_0t_0}e^{-\epsilon H_{\text{Möbius}}}\mathcal{O}(x)|0\rangle, |\Psi_4\rangle = \mathcal{N}_4 e^{-\epsilon H_{\text{Möbius}}}e^{-iH_0t_0}\mathcal{O}(x)|0\rangle,$$



Main Result

Entanglement entropy

Now, by calculating the entanglement entropy where the local operator is inserted at any point in circle, we found that the smear order indeed influence the result.

Take x=0 in case a as an example

$$S_{A,1} \approx \begin{cases} \frac{c}{3}\log\left[\frac{L}{\pi}\sin\left[\frac{\pi(X_1-X_2)}{L}\right]\right] & t_{1,-} > t_1 > 0 \\ \frac{c}{3}\log\left[\frac{L}{\pi}\sin\left[\frac{\pi(X_1-X_2)}{L}\right]\right] + \frac{c}{6}\log\left[\frac{\sin[\pi\alpha_O]}{\pi\alpha_O}\cdot\frac{(4\pi^2)\sin\left(\frac{\pi X_1}{L}\right)\sin\left(\frac{\pi X_2}{L}\right)(t_1-\tilde{t}_{1,-})(t_1-\tilde{t}_{2,-})}{L\epsilon\sin\left[\frac{\pi(X_1-X_2)}{L}\right]}\right] & t_{2,+} > t_1 > t_{1,-} \\ \frac{c}{3}\log\left[\frac{\sin[\pi\alpha_O]}{\pi\alpha_O}\right] + \frac{c}{6}\log\left[\frac{16\pi^6\sin^2\left(\frac{\pi X_1}{L}\right)\sin^2\left(\frac{\pi X_2}{L}\right)}{L^4\epsilon^2}\left(t_1^2 + \tilde{t}_{1,+}\tilde{t}_{t_1,-}\right)\left(t_1^2 + \tilde{t}_{2,+}\tilde{t}_{t_2,-}\right)\right] & t_1 > t_{2,+} \end{cases}$$

$$S_{A,2} \approx \begin{cases} \frac{c}{3}\log\left[\frac{L}{\pi}\sin\left[\frac{\pi(X_1-X_2)}{L}\right]\right] & t_{2,+} > t_1 > 0 \\ \frac{c}{3}\log\left[\frac{\sin[\pi\alpha_O]}{\pi\alpha_O}\right] + \frac{c}{6}\log\left[\frac{16\pi^6L^2\sin^2\left(\frac{\pi X_1}{L}\right)\sin^2\left(\frac{\pi X_2}{L}\right)}{(L^2+4\pi^2t_1^2)^2\epsilon^2}\left(t_1^2 - \tilde{t}_{1,+}\tilde{t}_{t_1,-}\right)\left(t_1^2 - \tilde{t}_{2,+}\tilde{t}_{t_2,-}\right)\right] & t_1 > t_{2,+} \end{cases}$$

Large t behavior

$$S_{A,1} \approx \frac{c}{3}\log\left(\frac{t}{\epsilon}\right) + \frac{c}{3}\log\left(\frac{t}{L}\right),$$

$$S_{A,2} \approx \frac{c}{3}\log\left[\frac{\sin[\pi\alpha_O]}{\pi\alpha_O}\right] + \frac{c}{3}\log\left[\frac{\pi L\sin\left(\frac{\pi X_1}{L}\right)\sin\left(\frac{\pi X_2}{L}\right)}{\epsilon}\right]$$

$$\langle T_{ww}(w_X)\rangle_1 \approx \left(\frac{2\pi}{L}\right)^2\left[-\frac{c}{24} + \frac{4h_O\pi^2L^2\epsilon^2}{\left[2\pi t_1\sin\left[\frac{\pi X}{L}\right]\sin\left[\frac{\pi x}{L}\right] + L\sin\left[\frac{\pi(X-x)}{L}\right]\right]^4}\right],$$

$$\langle T_{ww}(w_X)\rangle_2 \approx \left(\frac{2\pi}{L}\right)^2\left[-\frac{c}{24} + \frac{h_O\pi^2\epsilon^2\left[L^2 - 4\pi^2t_1^2\sin^2\left[\frac{\pi x}{L}\right] + 4\pi t_1L\sin\left[\frac{\pi x}{L}\right]\cos\left[\frac{\pi x}{L}\right]\right]^2}{L^2\left[2\pi t_1\sin\left[\frac{\pi X}{L}\right]\sin\left[\frac{\pi x}{L}\right] + L\sin\left[\frac{\pi(X-x)}{L}\right]\right]^4}\right],$$

$$\langle T_{ww}(w_X)\rangle_3 \approx \left(\frac{2\pi}{L}\right)^2\left[-\frac{c}{24} + \frac{4\pi\epsilon^2h_O\sin^4\left[\frac{\pi x}{L}\right]}{L^2\sin^4\left[\frac{\pi(t_0-x+X)}{L}\right]}\right],$$

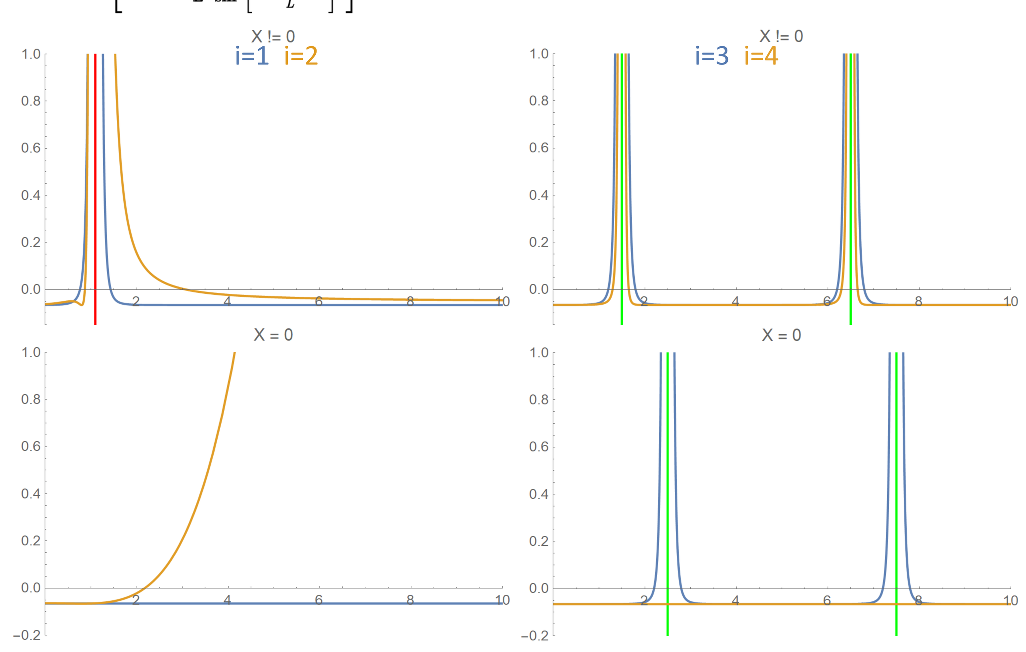
$$\langle T_{ww}(w_X)\rangle_4 \approx \left(\frac{2\pi}{L}\right)^2\left[-\frac{c}{24} + \frac{4\pi\epsilon^2h_O\sin^4\left[\frac{\pi(t_0-x)}{L}\right]}{L^2\sin^4\left[\frac{\pi(L_0-x+X)}{L}\right]}\right],$$

SSD limit

For $i = 1, 2$, the chiral (anti-chiral) energy density is localized around $t_1 = \frac{L}{2\pi}\left[\cot\left(\frac{\pi X}{L}\right) - \cot\left(\frac{\pi x}{L}\right)\right]$

For $i = 3, 4$, the peaks of it at $t_0 = \pm(x - X) + nL$.

One possible interpretation of this energy localization is that the local operator at the spatial location x creates two local excitations as called quasi particles, and then they propagate to left and right with the velocities $v = \pm 2\sin^2\left[\frac{\pi x}{L}\right]$ for i = 1,2 and $v = \pm 1$ for i = 3,4.



While there are still something interesting beyond the description by the propagation of quasiparticles.

For i=2 this accumulation of energy around X = 0 can not be because the time for quasi-particles to arrive at the origin is divergent.

For i = 4, the t-dependence of chiral and anti-chiral energy densities exhibits the dynamical behavior that is not explained by this quasiparticle picture.

Conclusion

The growth of smearing hides the information of the local operator.

Survival of information about the local operator:

If the local operator is smeared by the Euclidean evolution induced by the homogeneous Hamiltonian, and subsequently we evolve the system with the SSD Hamiltonian, then the entanglement entropy logarithmically grows with time. This behavior is independent of the local operator inserted into the state.

In the different time-ordered non-equilibrium process where we evolve the system and subsequently, the Euclidean evolution smears the local operator, the entanglement entropy saturates to the constant value that depends on the local operator inserted.

Details

In [a], we smear the local operator, then evolve the system. In [b], we evolve the system, and then smear the local operator. The orange shadow region illustrates the space-time where the local operator is delocalized by the dynamics.

Delocalize **Consider 1d Ising model** without external magnetic field

$$H(\sigma) = -J\sum_{i=1,\ldots,L-1}\sigma_i^z\sigma_{i+1}^z$$

Use Baker–Campbell–Hausdorff formula

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + \frac{1}{1!}[\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \cdots$$

$$e^{ItH}\sigma_j^xe^{-ItH} = \sigma_j^x + It[H,\sigma_j^x] - \frac{t^2}{2}[H,[H,\sigma_j^x]] + \cdots = \sigma_j^x + 2J\sigma_j^y(\sigma_{j+1}^z + \sigma_{j-1}^z) + f(j, j-1, j+1, j-2, j+2) + \cdots$$

We can see here the higher order of time will depend on wider distance, so called delocalize

Calculation of entanglement entropy

$$H_{\text{Möb}} = H_0 - \frac{\tanh(2\theta)}{2}(H_+ + H_-),$$

$$H_\pm = \int_0^L \frac{d\sigma}{2\pi}\left(e^{\pm 2\pi w/L}T(w) + e^{\mp 2\pi\bar{w}/L}\bar{T}(\bar{w})\right)$$

$$H_0 = \int_0^L \frac{d\sigma}{2\pi}T_{\tau\tau}(\sigma) = \int_0^L \frac{d\sigma}{2\pi}\left(T(w) + \bar{T}(\bar{w})\right)$$

$$z = e^{\frac{2\pi}{L}w} \quad \tilde{z} = f(z) = -\frac{\cosh\theta z - \sinh\theta}{\sinh\theta z - \cosh\theta}$$

$$H_{\text{Möb}}^{(\tilde{z})} = \frac{1}{iL\cosh(2\theta)}\oint \tilde{z}T(\tilde{z})d\tilde{z} - \frac{\pi c}{12L} = \frac{2\pi}{L\cosh(2\theta)}L_0^{(\tilde{z})} - \frac{\pi c}{12L}.$$

$$e^{H_{\text{Mob}}\tau}\mathcal{O}^{(\tilde{z})}(\tilde{z},\bar{\tilde{z}})e^{-H_{\text{Mob}}\tau} = \lambda^{2\alpha_O}\mathcal{O}^{(\tilde{z})}(\lambda\tilde{z},\lambda\bar{\tilde{z}})$$

$$\lambda := \exp\left(\frac{2\pi\tau}{L\cosh(2\theta)}\right) \qquad f(z_{\text{new}}) = \lambda f(z)$$

$$z_{\text{new}} = \frac{[(1-\lambda)\cosh 2\theta - (\lambda+1)]z + (\lambda-1)\sinh 2\theta}{(1-\lambda)\sinh 2\theta\cdot z + [(\lambda-1)\cosh 2\theta - (\lambda+1)]}$$

Thus, entanglement entropy for i=1 logarithmically grows with t while i=2 depends on the details of local operator, not t. For i=2 the information about the local operator remains even for large t.

While the uniform evolution i=3,4 both do not loss the information of the local operator, and the behavior are periodical.

$$S_{A,i,E}^{(n)} = \frac{1}{1-n}\log\left[\frac{\left\langle\mathcal{O}_n^\dagger\left(z_\epsilon^{\text{New},i},\bar{z}_\epsilon^{\text{New},i}\right)\sigma_n(z_{X_1},\bar{z}_{X_1})\bar{\sigma}_n(z_{X_2},\bar{z}_{X_2})\mathcal{O}_n\left(z_{-\epsilon}^{\text{New},i},\bar{z}_{-\epsilon}^{\text{New},i}\right)\right\rangle}{\left\langle\mathcal{O}^\dagger\left(z_\epsilon^{\text{New},i},\bar{z}_\epsilon^{\text{New},i}\right)\mathcal{O}\left(z_{-\epsilon}^{\text{New},i},\bar{z}_{-\epsilon}^{\text{New},i}\right)\right\rangle^n}\right] - \frac{c(1+n)}{24n}\log\left[\prod_{i=1,2}\left|\frac{dz_{X_i}}{dw_{X_i}}\right|\right],$$

By mapping from the cylinder to the complex plane

$$(z,\bar{z}) = \left(e^{\frac{2\pi w}{L}}, e^{\frac{2\pi\bar{w}}{L}}\right)$$ **and define cross ratio**

$$(z_{c,i},\bar{z}_{c,i}) = \left(\frac{\left(z_\epsilon^{\text{New},i} - z_{X_1}\right)\left(z_{X_2} - z_{-\epsilon}^{\text{New},i}\right)}{\left(z_{X_1} - z_{-\epsilon}^{\text{New},i}\right)\left(z_\epsilon^{\text{New},i} - z_{X_2}\right)}, \frac{\left(\bar{z}_\epsilon^{\text{New},i} - \bar{z}_{X_1}\right)\left(\bar{z}_{X_2} - \bar{z}_{-\epsilon}^{\text{New},i}\right)}{\left(\bar{z}_{X_1} - \bar{z}_{-\epsilon}^{\text{New},i}\right)\left(\bar{z}_\epsilon^{\text{New},i} - \bar{z}_{X_2}\right)}\right)$$

$$S_{A,i,E}^{(n)} = \frac{1}{1-n}\log\left[|z_{X_1} - z_{X_2}|^{-4nh_n}|1 - z_c|^{4nh_n}G_n(z_c,\bar{z}_c)\right] - \frac{c(n+1)}{6n}\log\left(\frac{2\pi}{L}\right)$$

In 2d holographic CFT, take the Von Neumann limit [2]

$$S_{A,i,E} = \frac{c}{6}\log\left[\frac{z_{c,i}^{\frac{1-\alpha_O}{2}}\bar{z}_{c,i}^{\frac{1-\bar{\alpha}_O}{2}}\left(1 - z_{c,i}^{\alpha_O}\right)\left(1 - \bar{z}_{c,i}^{\bar{\alpha}_O}\right)}{\alpha_O\bar{\alpha}_O}\right] + \frac{c}{6}\log\left[|z_{X_1} - z_{X_2}|^2\right] - \frac{c}{6}\log\left[|1 - z_{c,i}|^2\right] - \frac{c}{3}\log\left(\frac{2\pi}{L}\right)$$

Calculation of energy density

The three dimensional gravity corresponds to 2d holographic CFT can be obtained by computing the expectation value of energy density.

Definition of chiral part energy density expectation

$$\langle T_{ww}(w_X,\bar{w}_X)\rangle_{i,E} = \text{tr}(\rho_{i,E}T_{ww}(w_X,\bar{w}_X)) = \frac{\left\langle\mathcal{O}^\dagger\left(w_\epsilon^{\text{New},i},\bar{w}_\epsilon^{\text{New},i}\right)T_{ww}(w_X,\bar{w}_X)\mathcal{O}\left(w_{-\epsilon}^{\text{New},i},\bar{w}_{-\epsilon}^{\text{New},i}\right)\right\rangle}{\left\langle\mathcal{O}^\dagger\left(w_\epsilon^{\text{New},i},\bar{w}_\epsilon^{\text{New},i}\right)\mathcal{O}\left(w_{-\epsilon}^{\text{New},i},\bar{w}_{-\epsilon}^{\text{New},i}\right)\right\rangle}$$

By performing the conformal map from cylinder to the flat space and using the Ward-Takahashi indentity

$$\langle T_{zz}(z)\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\rangle = \sum_{i=1}^2\left[\frac{h_O}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\right]\langle\mathcal{O}(z_1,\bar{z}_1)\mathcal{O}(z_2,\bar{z}_2)\rangle$$

$$\langle T_{ww}(w_X,\bar{w}_X)\rangle_{i,E} = -\frac{c}{24}\left(\frac{2\pi}{L}\right)^2 + h_O\left(\frac{dz_X}{dw_X}\right)^2\left[\frac{1}{z_X - z_\epsilon^{\text{New},i}} - \frac{1}{z_X - z_{-\epsilon}^{\text{New},i}}\right]^2$$

Main References

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