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# Clifford Group and Unitary Designs under Symmetry

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# Motivation for our work

What emerges from the interplay of randomness and symmetry?

**Randomness**

- ✓ Investigation in fundamental physics
  e.g. Thermalization
- ✓ Useful in quantum algorithms
  e.g. Randomized benchmarking

**Symmetry**

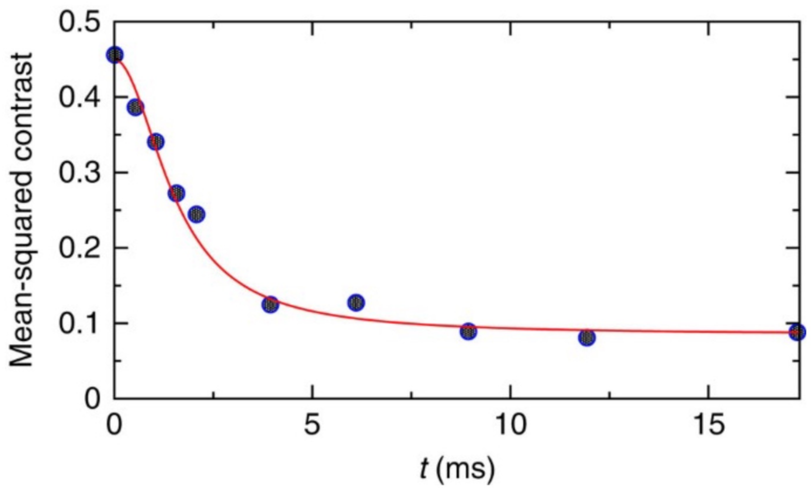
- ✓ Efficient analysis of systems
  e.g. Integrability
- ✓ Opens new physics
  e.g. SPT phase

# Uniform random unitary

Generalized notion of uniform distribution on the unitary group

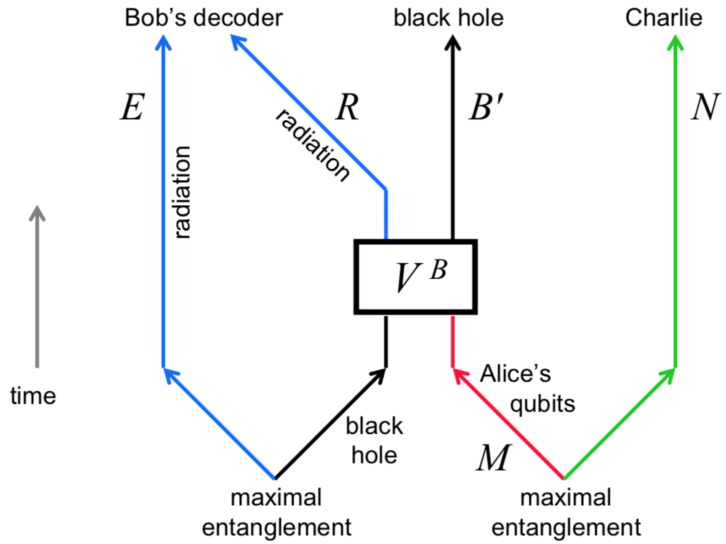
✓ Appears in many fields of fundamental physics.

Thermalization ⟵ Typicality




P. Reimann, Nat. Commun. **7**, 10821 (2016).

Black hole dynamics




P. Hayden and J. Preskill, J. High Energy Phys. 09 (2007) 120.

# Unitary $t$-design

Simulating Haar random unitary is a hard task.

➡ We need a set of unitary operators approximating uniform random unitary.

A compact subgroup $\mathcal{X} \subset \mathcal{U}_N$ is a unitary $t$-design if

Unitary group on $N$ qubits

$$\Phi_{t,\mathcal{X}} = \Phi_{t,\mathcal{U}_N},$$

where $\Phi_{t,\mathcal{X}}$ is the $t$-fold twirling channel over $\mathcal{X}$:

$$\Phi_{t,\mathcal{X}}(L) \coloneqq \int_{U \in \mathcal{U}_N} U^{\otimes t} L U^{\dagger \otimes t} d\mu_{\mathcal{X}}(U) \quad \forall L \in \mathcal{L}_{tN}.$$

Haar measure on $\mathcal{X}$     all linear operators on $tN$ qubits

Unitary $t$-designs for large $t$ are desirable.

# Applications

e.g. Noise twirling

J. Emerson *et al.*, J. Opt. B: Quantum Semiclass. Opt. **7**, S347-S352 (2005).

Randomized benchmarking

C. Dankert *et al.*, Phys. Rev. A **80,** 012304 (2009).

Unitary 2-designs are needed.

Partial quantum tomography $\longleftarrow$ Unitary 3-designs are needed.

H.-Y. Huang, R. Kueng, and J. Preskill, Nat. Phys. **16**, 1050-1057 (2020).

Unitary designs play essential roles in various quantum information processing tasks.

# Clifford group

Pauli group $\mathcal{P}_N := \{\pm 1, \pm i\} \cdot \{\mathrm{I}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}^{\otimes N}$

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathrm{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \mathrm{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Clifford group $\mathcal{C}_N := \left\{ U \in \mathcal{U}_N \middle| U \mathcal{P}_N U^\dagger = \mathcal{P}_N \right\}$

Hadamard, S, CNOT $\in \mathcal{C}_N$

✓ Quantum circuits with Clifford gates can be simulated classically.

D. Gottesman, arXiv:quant-ph/9807006.

✓ The Clifford group is a unitary 3-design.

Z. Webb, Quantum Inf. Comput. **16**, 1379 (2016).

H. Zhu, Phys. Rev. A **96**, 062336 (2017).

# Symmetry in physics

A physical system has symmetry $\mathcal{G}$. $\longleftrightarrow$ $\forall G \in \mathcal{G}\ [H, G] = 0.$

$H$: Hamiltonian

Transverse-field Ising model

$$H = g \sum \mathrm{X}_j \mathrm{X}_k + h \sum \mathrm{Z}_j \qquad \mathcal{G} = \left\{ \mathrm{I}^{\otimes N}, \mathrm{Z}^{\otimes N} \right\} \cong \mathbb{Z}_2$$

Heisenberg XXZ model

$$H = \sum \left[ g \left( \mathrm{X}_j \mathrm{X}_k + \mathrm{Y}_j \mathrm{Y}_k \right) + g' \mathrm{Z}_j \mathrm{Z}_k \right] \qquad \mathcal{G} = \left\{ \left( e^{i\theta \mathrm{Z}} \right)^{\otimes N} \middle| \theta \in \mathbb{R} \right\} \cong \mathrm{U}(1)$$

Heisenberg XXX model

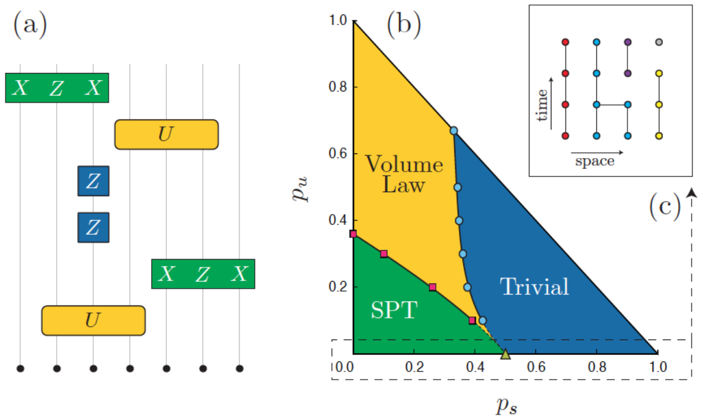
$$H = g \sum \left( \mathrm{X}_j \mathrm{X}_k + \mathrm{Y}_j \mathrm{Y}_k + \mathrm{Z}_j \mathrm{Z}_k \right) \qquad \mathcal{G} = \left\{ \left( e^{i(\theta_{\mathrm{X}} \mathrm{X} + \theta_{Y} \mathrm{Y} + \theta_{\mathrm{Z}} \mathrm{Z})} \right)^{\otimes N} \middle| \theta_\alpha \in \mathbb{R} \right\} \cong \mathrm{SU}(2)$$

✓ Useful for efficient analysis of the systems

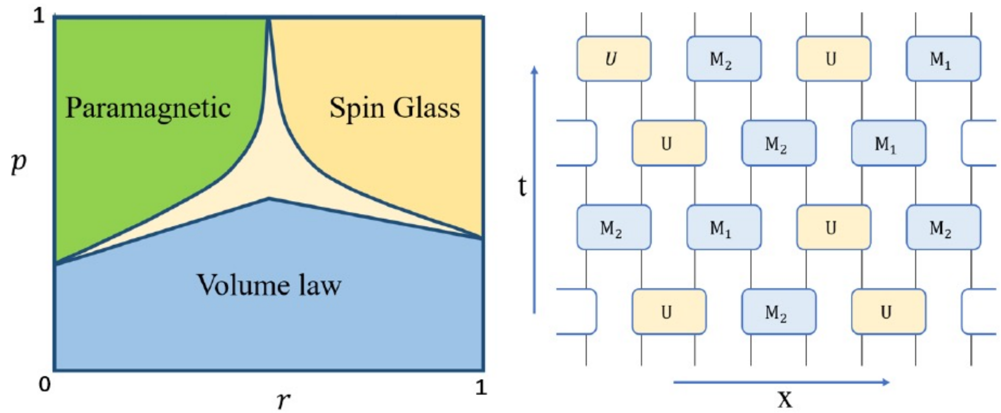
# Symmetry in physics

Symmetry also enriches physics.

e.g. In measurement-induced phase transitions,
interesting features emerge by introducing symmetry constraints.




A. Lavasani, Y. Alavirad, and M. Barkeshli,
Nat. Phys. **17**, 342-347 (2021).




S. Sang and T. H. Hsieh,
Phys. Rev. Research **3**, 023200 (2021).

# Core questions in our work

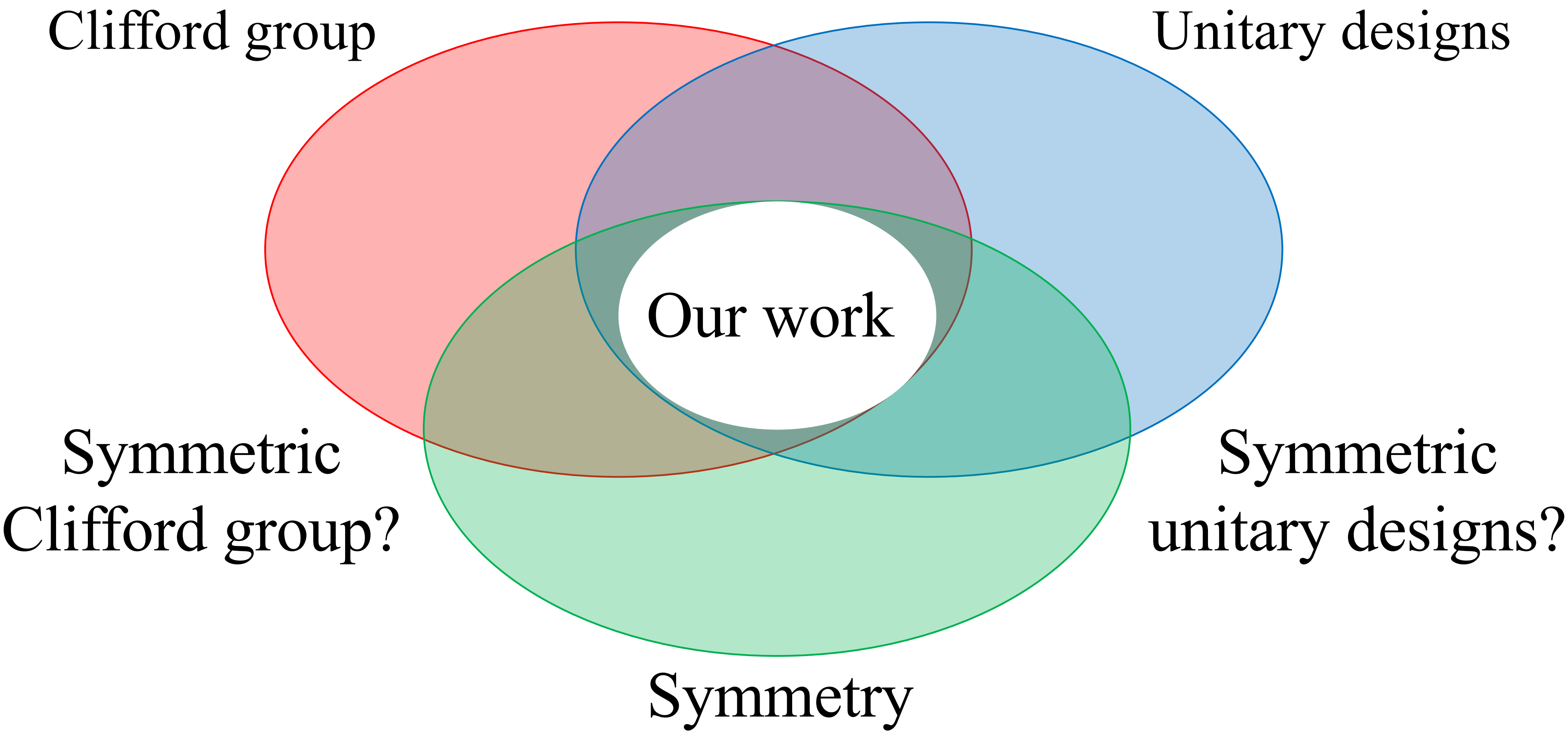
1. Can we characterize the pseudo-randomness of the Clifford group under symmetry?

    ➡ First result

2. What is the explicit construction?

    ➡ Second result

# Setup

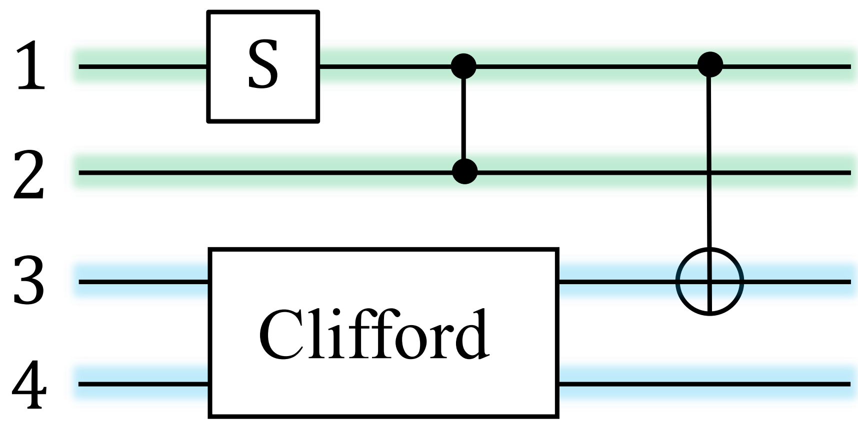

Clifford group
Unitary designs
Our work
Symmetric
Clifford group?
Symmetric
unitary designs?
Symmetry

# Symmetric Clifford group

$\mathcal{G}$-symmetric unitary group $\mathcal{U}_{N,\mathcal{G}} := \{U \in \mathcal{U}_N | \forall G \in \mathcal{G}\ [U, G] = 0\}$

$\mathcal{G}$-symmetric Clifford group $\mathcal{C}_{N,\mathcal{G}} := \mathcal{C}_N \cap \mathcal{U}_{N,\mathcal{G}}$

e.g. $\mathcal{G}$-symmetric Clifford gates for $\mathcal{G} = \{\mathrm{I}, \mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_1\mathrm{Z}_2\}$



# Symmetric unitary $t$-design

Unitary designs approximate uniform distribution on $\mathcal{U}_N$.

$\mathcal{G}$-symmetric unitary designs approximate uniform distribution on $\mathcal{U}_{N,\mathcal{G}}$.

$$\mathcal{U}_{N,\mathcal{G}} \coloneqq \{U \in \mathcal{U}_N | \forall G \in \mathcal{G}\ [U, G] = 0\}$$

A compact subgroup $\mathcal{X} \subset \mathcal{U}_N$ is a $\mathcal{G}$-symmetric unitary $t$-design if

$$\Phi_{t,\mathcal{X}} = \Phi_{t,\mathcal{U}_{N,\mathcal{G}}}. \qquad \Phi_{t,\mathcal{X}}(L) \coloneqq \int_{U \in \mathcal{U}_N} U^{\otimes t} L U^{\dagger \otimes t} d\mu_{\mathcal{X}}(U)$$

The standard unitary designs are included as the case of $\mathcal{G} = \{I\}$.

# Problem

The Clifford group is a unitary 3-design.

**Is the $\mathcal{G}$-symmetric Clifford group a $\mathcal{G}$-symmetric unitary 3-design?**

Does it always hold?

If not, for what symmetry $\mathcal{G}$ does it hold?

# 1st result: randomness of symmetric Clifford group

$\mathcal{G}$: subgroup of $\mathcal{U}_N$

$\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design
if and only if $\exists$subgroup $\underbrace{\mathcal{Q} \subset \mathcal{P}_N \text{ s.t. } \mathcal{U}_{N,\mathcal{G}} = \mathcal{U}_{N,\mathcal{Q}}}$

Symmetry constraint is described by some Pauli subgroup.

$\mathcal{C}_{N,\mathcal{G}} \coloneqq \mathcal{C}_N \cap \mathcal{U}_{N,\mathcal{G}}$
$\mathcal{C}_N$: the Clifford group
$\mathcal{U}_{N,\mathcal{G}} \coloneqq \{U \in \mathcal{U}_N | \forall G \in \mathcal{G}\ [U,G] = 0\}$

✓ Characterized Pauli-symmetric Clifford group in terms of unitary designs

# Examples

Pauli subgroup symmetry

$$\mathcal{G} = \left\{I^{\otimes N}, \mathrm{Z}^{\otimes N}\right\} \cong \mathbb{Z}_2$$

➡ $\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design.

(but not a 4-design)

Non-Pauli subgroup symmetry

$$\begin{cases} \mathcal{G} = \left\{\left(e^{i\theta \mathrm{Z}}\right)^{\otimes N} \middle| \theta \in \mathbb{R}\right\} & \cong U(1) \\ \mathcal{G} = \left\{\left(e^{i(\theta_{\mathrm{X}}\mathrm{X}+\theta_{Y}\mathrm{Y}+\theta_{\mathrm{Z}}\mathrm{Z})}\right)^{\otimes N} \middle| \theta_{\mathrm{X}}, \theta_{Y}, \theta_{\mathrm{Z}} \in \mathbb{R}\right\} & \cong SU(2) \end{cases} \quad \text{for } N \geq 2$$

➡ $\mathcal{C}_{N,\mathcal{G}}$ is not a $\mathcal{G}$-symmetric unitary 3-design.

(a 1-design but not a 2-design)

# Proof sketch

**Theorem S1.** $\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design iff $\exists \mathcal{Q} \subset \mathcal{P}_N$ s.t. $\mathcal{U}_{N,\mathcal{G}} = \mathcal{U}_{N,\mathcal{Q}}$.

"if" part

**Proposition S1.** $\mathcal{C}_{N,\mathcal{Q}}$ is a $\mathcal{Q}$-symmetric unitary 3-design for $\mathcal{Q} \subset \mathcal{P}_N$.

**Lemma S1.** $\forall \mathcal{D} \in \mathfrak{C}_{t,\mathcal{Q}}$, $\Phi_{3,\mathcal{X}} \circ \mathcal{D} = \Phi_{3,\mathcal{X}}$ for $\mathcal{Q} \subset \mathcal{P}_N$, $\mathcal{X} = \mathcal{C}_{N,\mathcal{Q}}$ or $\mathcal{U}_{N,\mathcal{Q}}$.

**Lemma S2.** $\forall L \in \mathcal{M}_{\mathcal{Q}}$, $\Phi_{3,\mathcal{X}}(L) = L$ for $\mathcal{Q} \subset \mathcal{P}_N$.

**Lemma S3.** $\exists \mathcal{D} \in \mathfrak{C}_{t,\mathcal{Q}}$ s.t. $\forall L \in \mathcal{L}(\mathcal{H}^{\otimes 3})$, $\mathcal{D}(L) \in \mathcal{M}_{\mathcal{Q}}$ for $\mathcal{Q} \subset \mathcal{P}_N$.

$\leftarrow$ **Lemmas S4, S16**

$\leftarrow$ **Lemma S14** $\leftarrow$ **Lemmas S12, S13**

**Lemma S15**

| **Lemma S5.** S gates | **Lemma S6.** CZ gates | **Lemma S7.** Clifford gates | **Lemma S8.** Controlled-Pauli gates |
|---|---|---|---|

"only if" part

**Proposition S2.** If $\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design, then $\mathcal{U}_{N,\mathcal{G}} = \mathcal{U}_{N,\mathcal{Q}}$, where $\mathcal{Q} = \langle \{Q \in \mathcal{P}_N^+ \mid \exists G \in \mathcal{G} \text{ s.t. } \mathrm{tr}(GQ) \neq 0\} \rangle$.

**Lemma S9.** If $L = \sum_{P \in \mathcal{P}_N^+} \gamma_P(G) P^{\otimes 3}$, then $\forall U \in \mathcal{C}_{N,\mathcal{G}}$, $U^{\otimes 3} L U^{\dagger \otimes 3} = L$.

**Lemma S17**

**Lemma S10.** If $\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design and $\forall U \in \mathcal{C}_{N,\mathcal{G}}$, $U^{\otimes 3} L U^{\dagger \otimes 3} = L$, then $\forall U \in \mathcal{U}_{N,\mathcal{G}}$, $U^{\otimes 3} L U^{\dagger \otimes 3} = L$.

**Lemma S11.** If $L = \sum_{P \in \mathcal{P}_N^+} \gamma'_P P^{\otimes 3}$ and $\forall U \in \mathcal{U}_{N,\mathcal{G}}$, $U^{\otimes 3} L U^{\dagger \otimes 3} = L$, then $\forall U \in \mathcal{U}_{N,\mathcal{G}}$ and $P \in \mathcal{P}_N^+$ satisfying $\gamma'_P \neq 0$, $UPU^\dagger = P$.

**Lemma S18**

"if" part

Suppose $\mathcal{U}_{N,\mathcal{G}} = \mathcal{U}_{N,\mathcal{Q}}$.

We explicitly construct a channel $\mathcal{D}$ s.t.

$$\mathcal{D} = \Phi_{3,\mathcal{X}} \text{ and } \mathcal{D} = \Phi_{3,\mathcal{U}_{N,\mathcal{G}}}$$

$$\Rightarrow \Phi_{3,\mathcal{X}} = \Phi_{3,\mathcal{U}_{N,\mathcal{G}}}$$

"only if" part

Suppose $\mathcal{C}_{N,\mathcal{G}}$ is a $\mathcal{G}$-symmetric unitary 3-design.

$$\text{e.g. } \mathcal{G} = \left\{ \mathrm{I}^{\otimes 3}, \frac{\mathrm{X}^{\otimes 3} + \mathrm{Z}^{\otimes 3}}{\sqrt{2}} \right\}.$$

$$\Rightarrow \mathcal{U}_{N,\mathcal{G}} = \mathcal{U}_{N,\mathcal{Q}}.$$

with $\mathcal{Q} = \{\pm 1, \pm i\} \cdot \{\mathrm{I}^{\otimes 3}, \mathrm{X}^{\otimes 3}, \mathrm{Y}^{\otimes 3}, \mathrm{Z}^{\otimes 3}\}$

# 2nd result: construction of symmetric Clifford gates

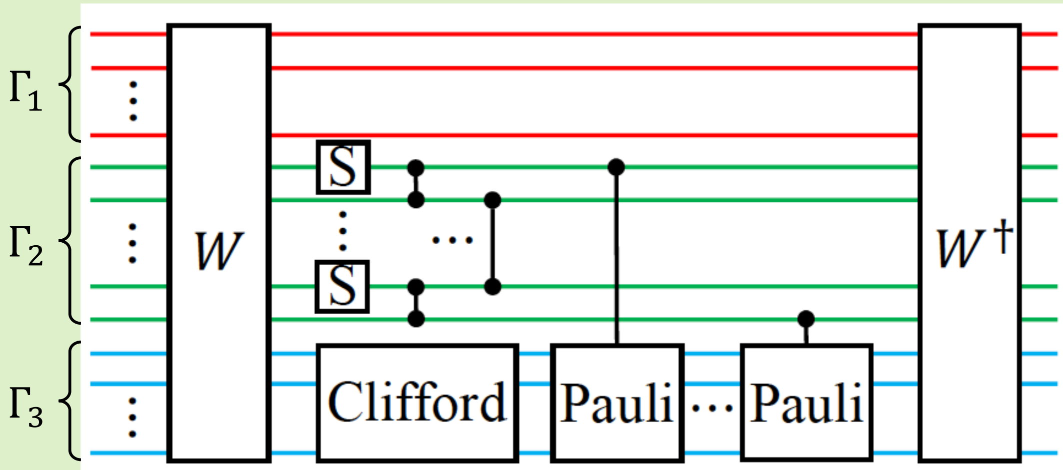
$\mathcal{Q}$: Pauli subgroup

$$\exists W \in \mathcal{C}_N \text{ s.t. } W\mathcal{Q}W^\dagger = \overset{\Gamma_1}{\{\mathrm{I},\mathrm{X},\mathrm{Y},\mathrm{Z}\}^{\otimes N_1}} \otimes \overset{\Gamma_2}{\{\mathrm{I},\mathrm{Z}\}^{\otimes N_2}} \otimes \overset{\Gamma_3}{\{\mathrm{I}\}^{\otimes N_3}}$$

indices for qubits

(up to phase)

Every $\mathcal{Q}$-symmetric Clifford gate can be uniquely expressed as:



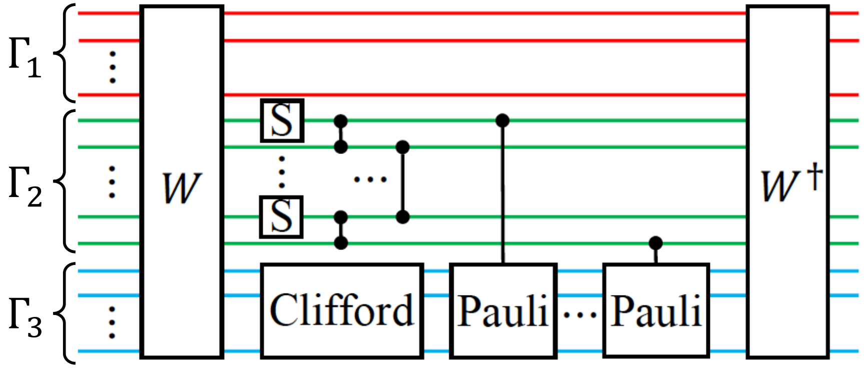

$$W^\dagger\left(\mathrm{T}\prod_{j\in\Gamma_2}\prod_{k\in\Gamma_3}\mathrm{C}(P_{j,k})_{j,k}\right)V$$
$$\times\left(\prod_{j,k\in\Gamma_2, j<k}\mathrm{CZ}_{j,k}^{v_{j,k}}\right)\left(\prod_{j\in\Gamma_2}\mathrm{S}_j^{\mu_j}\right)W$$

with $\mu_j \in \{0,1,2,3\}$, $v_{j,k} \in \{0,1\}$,

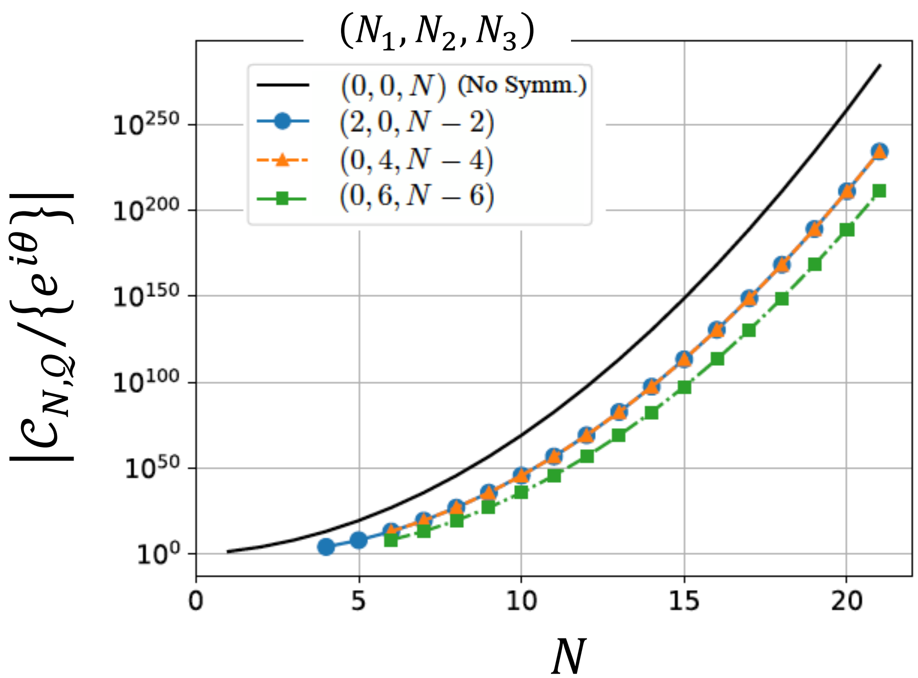
$V \in \mathcal{C}_{N_3}$, $P_{j,k} \in \{\mathrm{I},\mathrm{X},\mathrm{Y},\mathrm{Z}\}$.

✔ Efficient method for generating all the symmetric Clifford gates

# Size of symmetric Clifford gates




$$\left|\mathcal{C}_{N,\mathcal{Q}}/\{e^{i\theta}\}\right| \sim 2^{2(N_3+N_2/2)^2+3(N_3+N_2/2)}$$
$$\sim \left|\mathcal{C}_{N_3+N_2/2}/\{e^{i\theta}\}\right|$$

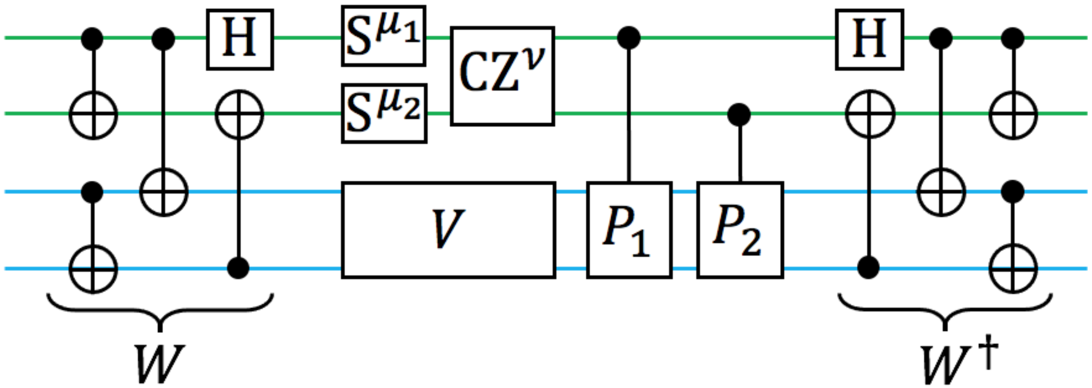



$\left|\mathcal{C}_{N,\mathcal{Q}}/\{e^{i\theta}\}\right|$ is super-exponential,

but it is exponentially smaller than $\left|\mathcal{C}_{N}/\{e^{i\theta}\}\right|$.

# Example

For $Q = \{\mathrm{I}^{\otimes 4}, \mathrm{X}^{\otimes 4}, \mathrm{Y}^{\otimes 4}, \mathrm{Z}^{\otimes 4}\}$, every $U \in \mathcal{C}_{4,Q}$ is uniquely expressed as:




with $\mu_j \in \{0, 1, 2, 3\}$, $\upsilon_{j,k} \in \{0, 1\}$, $V \in \mathcal{C}_2$, $P_j \in \{\mathrm{I}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}^{\otimes 2}$.

$$\left|\mathcal{C}_{4,Q}/\{e^{i\theta}\}\right| \sim 10^8, \ \left|\mathcal{C}_4/\{e^{i\theta}\}\right| \sim 10^{13}$$

➡ Generating $\mathcal{C}_{4,Q}$ by this method is $10^5$ times more efficient than generating $\mathcal{C}_4$ and choosing symmetric gates.

# Proof

Sufficient to consider $\mathcal{Q} = \mathcal{P}_0\{\mathrm{I},\mathrm{X},\mathrm{Y},\mathrm{Z}\}^{\otimes N_1} \otimes \{\mathrm{I},\mathrm{Z}\}^{\otimes N_2} \otimes \{\mathrm{I}\}^{\otimes N_3}$.

<u>Completeness</u> Take the Heisenberg picture.

| | | $j \in \Gamma_1$ | $j \in \Gamma_2$ |
|---|---|---|---|
| $U$ is $\mathcal{Q}$-symmetric | $U \cdot U^\dagger$ | $\mathrm{Z}_j \longmapsto \mathrm{Z}_j$<br>$\mathrm{X}_j \longmapsto \mathrm{X}_j$ | $\mathrm{Z}_j \longmapsto \mathrm{Z}_j$<br>$\mathrm{X}_j \longmapsto$ ? |
| inductively construct $U'$ with the 4 types of gates s.t. | $U'U \cdot U^\dagger U'^\dagger$ | $\mathrm{Z}_j \longmapsto \mathrm{Z}_j$<br>$\mathrm{X}_j \longmapsto \mathrm{X}_j$ | $\mathrm{Z}_j \longmapsto \mathrm{Z}_j$<br>$\mathrm{X}_j \longmapsto \mathrm{X}_j$ |

➡ $U'U$ is a Clifford gate on the third subsystem.

➡ $U$ can be constructed by the 4 types of gates.

<u>Uniqueness</u> Apply $U$ to states $|\Psi\rangle \otimes \left(\otimes_j |x_j\rangle\right) \otimes |\Xi\rangle$ s.t. the number of nonzero $x_j$ is at most 2.

$|x_j\rangle$: Z-basis

# Summary and future perspectives

Summary

arXiv:2306.17559

Proved the $\mathcal{G}$-symmetric Clifford group is a $\mathcal{G}$-symmetric unitary 3-design if and only if the symmetry constraint is described by some Pauli subgroup.

Found a complete and unique construction of the symmetric Clifford groups.

Perspectives

New direction in quantum information processing tasks under symmetry?

Foundation for understanding to many-body systems?