Recovery maps/protocols in Quantum black holes and their SYK realizations

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QIMG 2023 @ YITP

September 6th, 2023

Based on a joint work (to appear soon) with

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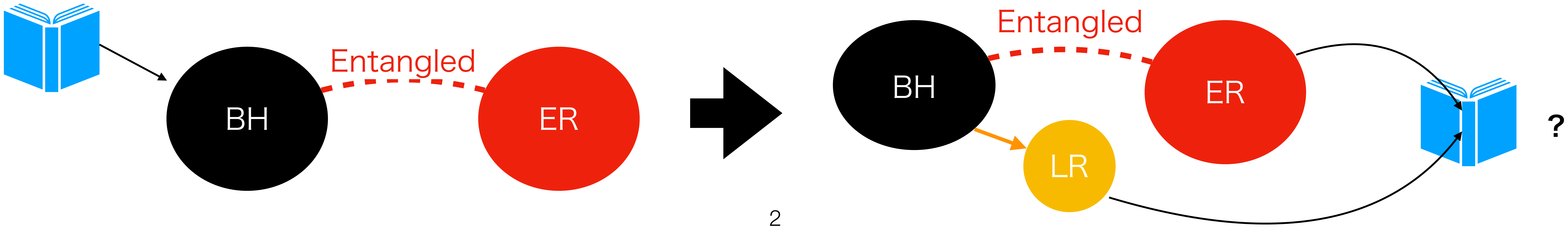
Hayden-Preskill (HP) protocol

★ **Information in black hole**: Unitarity of quantum gravity means that information thrown into a black hole is not destroyed under an black hole evaporation process, and the information is emitted as Hawking radiation, and we can recover the information from the Hawking radiation

- Question

When can we recover information thrown into a black hole from Hawking Radiation (HR)?

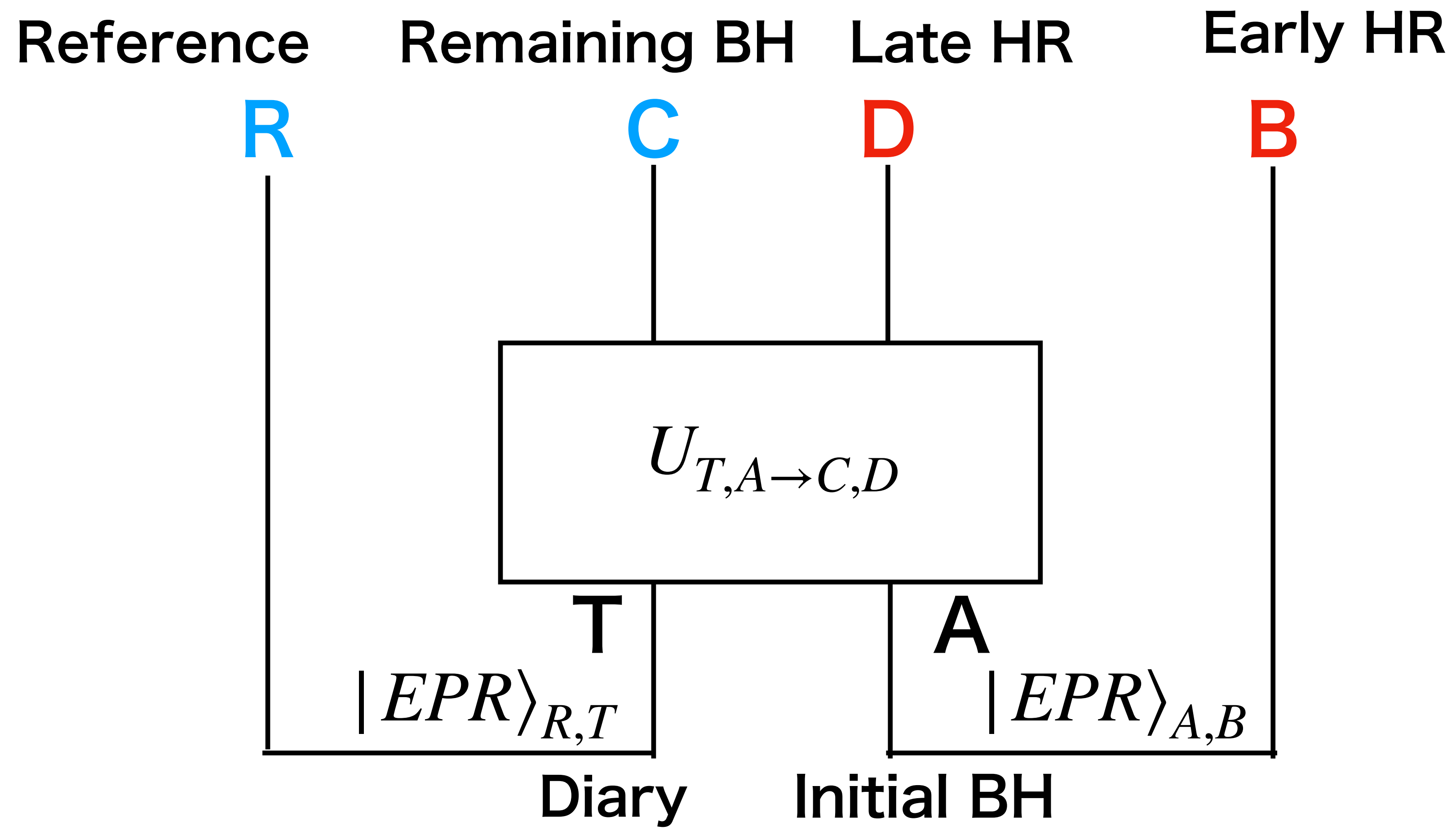
- Hayden-Preskill setup (protocol) [Hayden-Preskill '07]
  - An initial BH is (maximally) entangled with an Early Hawking Radiation (ER) → After the Page time
  - How much Late Hawking radiation (LR) do we need to recover the information by using ER and LR?



Q. How much Late Hawking radiation do we need to recover the information by using HR?

- If a reference system (R) (entangled with a diary (T)) is not correlated with the remaining BH (C), $\rho_{R,C} \overset{?}{\approx} \rho_R \otimes \rho_C$, then we can recover the diary information from Hawking radiation (D+B).

→Decoupling condition [Hayden-Preskill '07]



For Haar random unitaries (simplified BH time evolution)

$$\int dU \left|\left| \rho_{R,C} - \rho_R \otimes \rho_C \right|\right|_1 \leq \frac{d_T}{d_D}$$

($||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$, d_T = dim of the diary system, d_D = dim of Late HR)

→ If we have **sufficient large Late Hawking radiation compared with the diary, that is,** $d_T \ll d_D$, then we can recover the diary from Hawking radiation (D+B) [Hayden-Preskill '07].

✱But, how we can recover the diary from the HR?

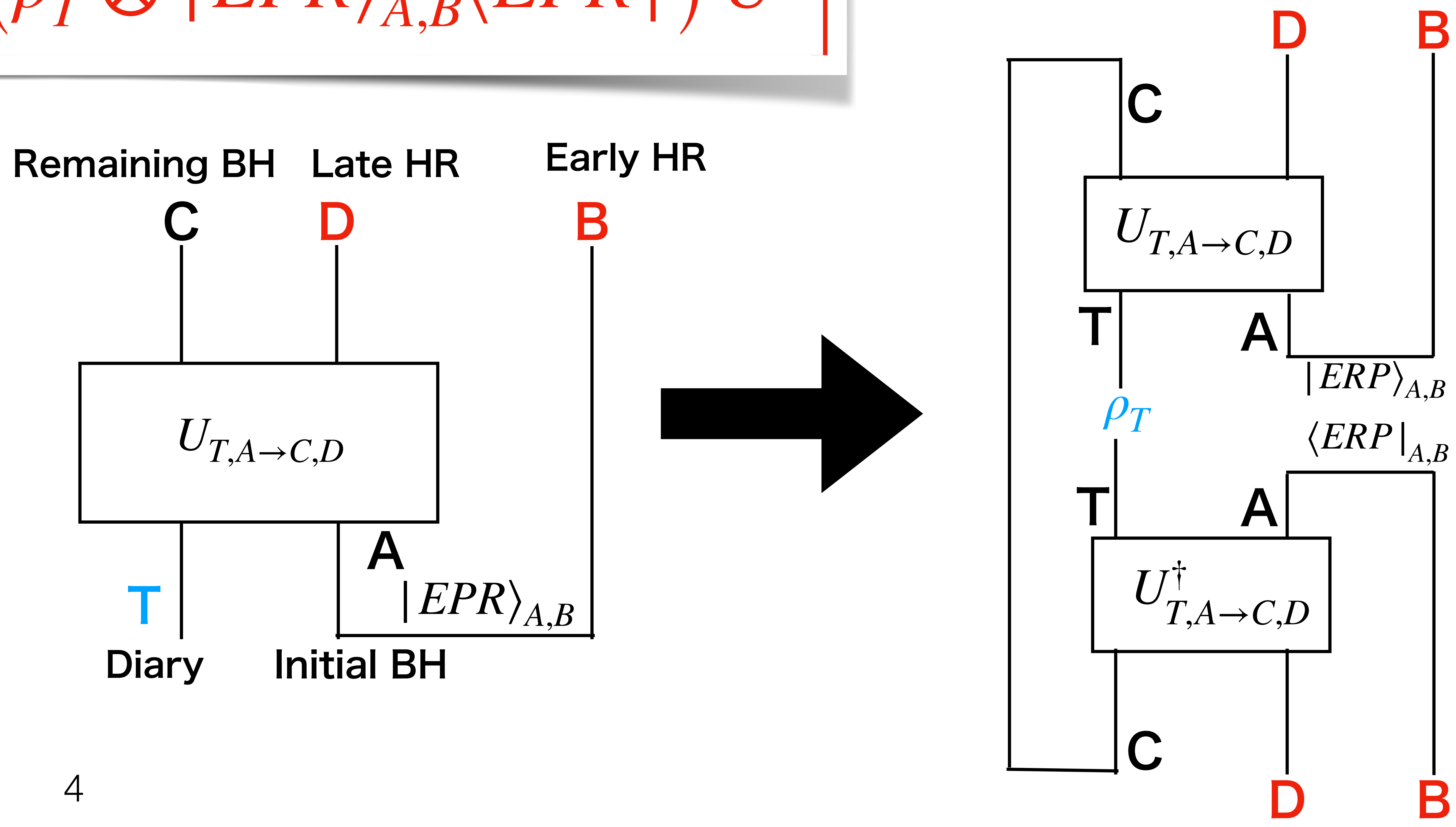
How we can recover?

- We would like to find a method or protocol to recover the diary from the HR

■ Basic strategy → Construct a recovery map for a quantum channel defined through the HP protocol

  - Quantum channel (map from a density matrix to a density matrix) for the HP protocol

$$\boxed{\mathcal{N}_{T\to D,B}[\rho_T] = \mathrm{Tr}_C \left[U \left(\rho_T \otimes |EPR\rangle_{A,B}\langle EPR| \right) U^\dagger \right]}$$



- Quantum channel for the HP protocol: $\mathcal{N}_{T\to D,B}[\rho_T] = \mathrm{Tr}_C\left[U\left(\rho_T \otimes |EPR\rangle_{A,B}\langle EPR|\right)U^\dagger\right]$

- Recovery channel (map) for the quantum channel $\mathcal{R}_{D,B\to T}$ satisfying $\mathcal{R}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\rho_T]\right] = \rho_T$

  ◆ The quantum channel is not a simple unitary map

  → it is non-trivial whether such a recovery map exists.

- However, we can have such a recovery map as long as the decoupling condition (approximately) holds

$$\boxed{\int dU \left|\left|\rho_{R,C} - \rho_R \otimes \rho_C\right|\right|_1 \leq \frac{d_T}{d_D} \ll 1}$$

- Of course, a recovered result is also approximate one; $\mathcal{R}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\rho_T]\right] \approx \rho_T$.

- Quantum channel for the HP protocol

$$\mathcal{N}_{T\to D,B}[\rho_T] = \mathrm{Tr}_C\left[U\left(\rho_T \otimes |EPR\rangle_{A,B}\langle EPR|\right)U^\dagger\right]$$

- Recovery channel for the quantum channel $\mathscr{R}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\rho_T]\right] = \rho_T$

Generally, we can consider the Petz recovery map [Petz' 88, Petz '02] as such a recovery map

$$\mathscr{R}_{D,B\to T}[\mathcal{O}_{D,B}] = \sigma_T^{1/2}\mathcal{N}^\dagger_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\sigma_T]^{-1/2}\mathcal{O}_{D,B}\ \mathcal{N}_{T\to D,B}[\sigma_T]^{-1/2}\right]\sigma_T^{1/2}$$

σ_T : Some reference state with full rank. We can choose this by hand.

$\mathcal{N}^\dagger$: Adjoint channel of $\mathcal{N}$ defined by $\mathrm{Tr}_{D,B}\left[\mathcal{N}_{T\to D,B}[\rho_T]\,O_{D,B}\right] = \mathrm{Tr}_T[\rho_T\,\mathcal{N}^\dagger_{D,B\to T}[O_{D,B}]]$ → Next slide

- However, it is difficult to understand the Petz map intuitively.

→ **Is there a simpler recovery map for the HP quantum channel?**

*Our main proposal:

In a highly chaotic system, we can use a simplified Petz map, that is, the "Petz-lite"[Penington-Shenker-et al., '19];

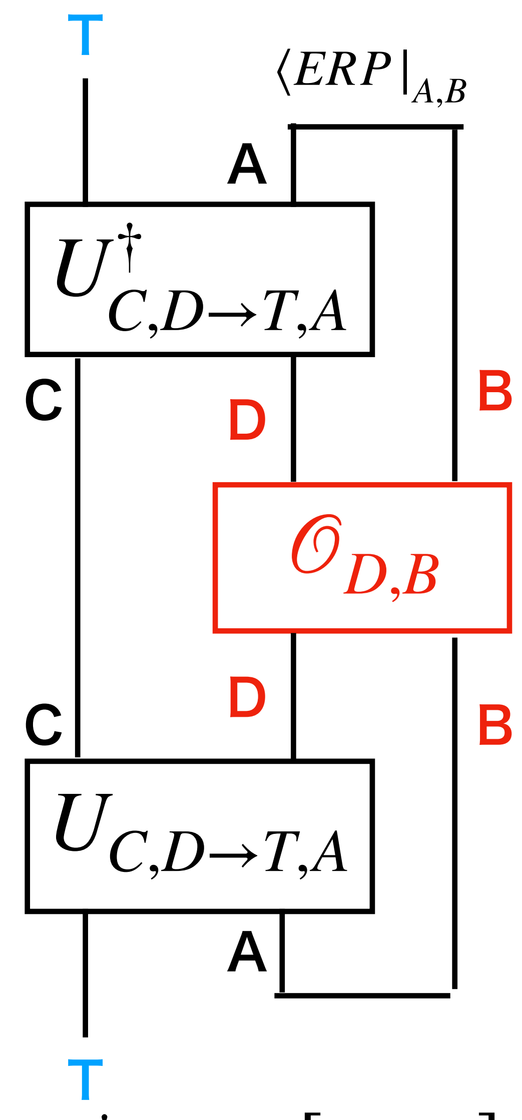
$$\mathscr{R}_{D,B\to T}[\mathcal{O}_{D,B}] = \sigma_T^{1/2}\mathcal{N}^{\dagger}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\sigma_T]^{-1/2}\mathcal{O}_{D,B}\ \mathcal{N}_{T\to D,B}[\sigma_T]^{-1/2}\right]\sigma_T^{1/2} \quad \text{(Original Petz map)}$$

→"**Petz-lite**" for the Haar random HP channel (Normalized s.t. $\mathrm{Tr}\left[\mathscr{R}^{Lite}_{D,B\to T}[\mathcal{N}_{T\to D,B}[\rho_T]]\right] \approx \mathrm{Tr}\left[\rho_T\right] = 1$)

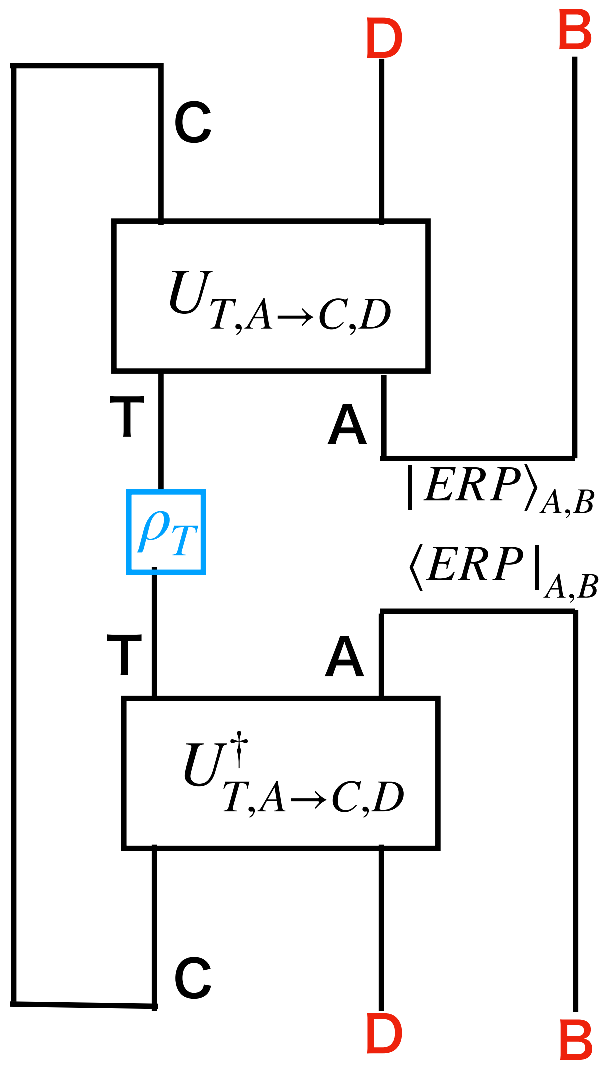
$$\boxed{\mathscr{R}^{Lite}_{D,B\to T}[\mathcal{O}_{D,B}] = \frac{d_C}{1+\left(\frac{d_D}{d_T}\right)^2}\mathcal{N}^{\dagger}_{D,B\to T}\left[\mathcal{O}_{D,B}\right]}$$

$$\mathcal{N}^{\dagger}_{D,B\to T}\left[\mathcal{O}_{D,B}\right] = {}_{A,B}\langle TFD|\,U^{\dagger}_{C,D\to T,A}\left(\mathcal{O}_{D,B}\right)U_{C,D\to T,A}\,|TFD\rangle_{A,B}$$

$$\mathcal{N}_{T\to D,B}[\rho_T] = \mathrm{Tr}_C\left[U_{T,A\to C,D}\left(\rho_T\otimes|EPR\rangle_{A,B}\langle EPR|\right)U^{\dagger}_{T,A\to C,D}\right]$$



$\mathcal{N}^{\dagger}_{D,B\to T}\left[\mathcal{O}_{D,B}\right]$



$\mathcal{N}_{T\to D,B}[\rho_T]$

This kind of simplification, more precisely, simplification to the Yoshida-Kitaev recovery protocol [Yoshida-Kitaev '17] in the HP protocol, was pointed out by Yoshida. [Yoshida'21,⋯]

We have checked the validity of the Petz-lite for the Haar random HP protocol, and we have also checked part of the validity of the recovery map for the SYK HP protocol, but the remaining part is still in progress.

$$\mathscr{R}^{Lite}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\rho_T]\right] \overset{?}{\approx} \rho_T$$

Check the Petz lite in (Haar random) HP protocol

- Let us quickly check the validity of the Petz-lite under the Haar average.

To do so, we focus on the Haar averaged result

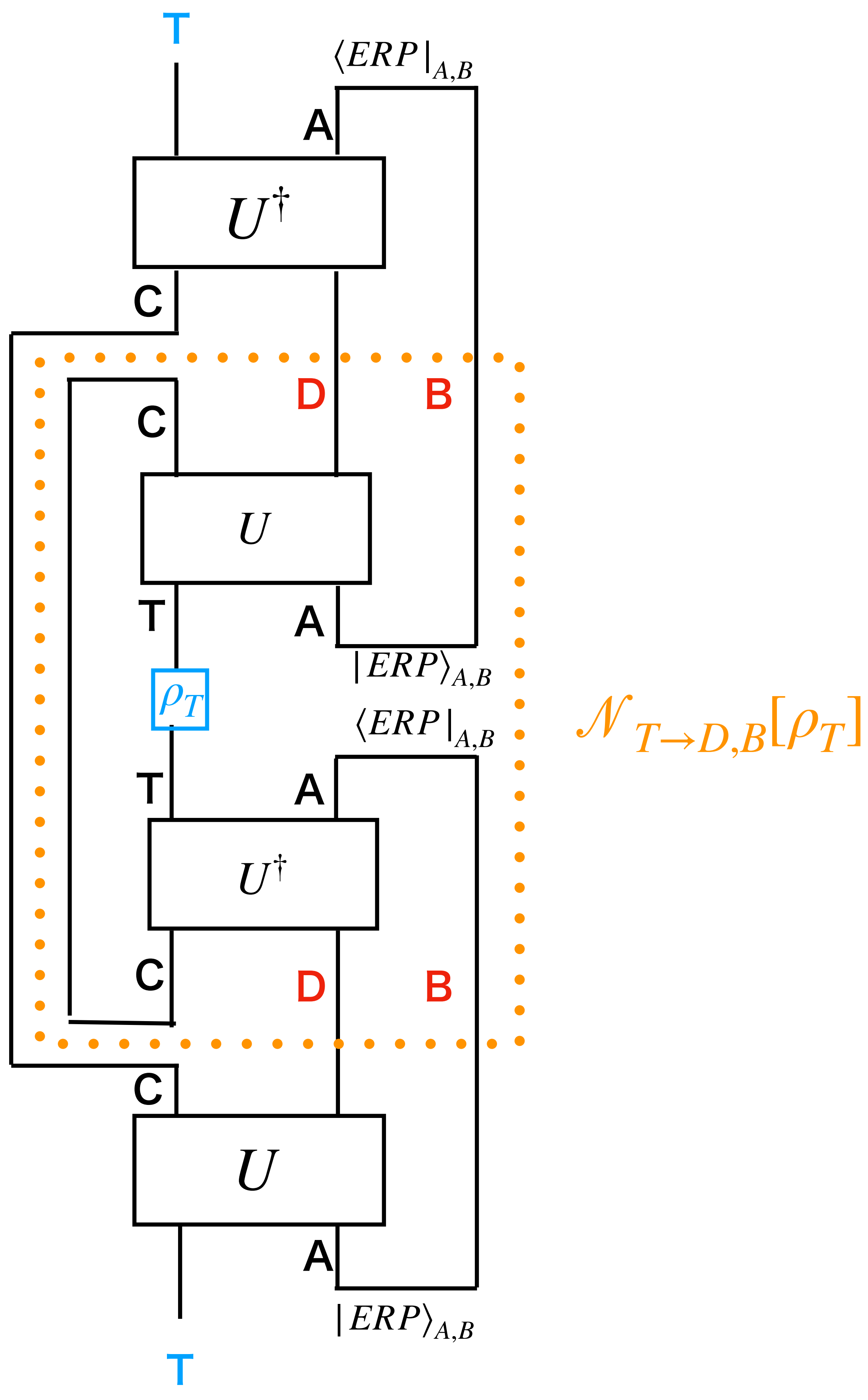
$$\int dU\, \mathcal{R}^{Lite}_{D,B\to T}[\mathcal{N}_{T\to D,B}[\rho_T]] \approx \frac{1}{1+\left(\frac{d_T}{d_D}\right)^2}\left(\rho_T + \left(\frac{d_T}{d_D}\right)^2 \cdot \frac{1}{d_T} I_T\right).$$

- 2nd term→ Disconnected contribution, which is related to a Hawking saddle
- 1st term → Connected contribution between $\mathcal{N}_{T\to D,B}$ and $\mathcal{N}^{\dagger}_{D,B\to T}$, which is related to a Replica wormhole saddle

★When we have sufficiently large late Hawking radiation compared with the diary,

$$\mathcal{R}^{Lite}_{D,B\to T}[\mathcal{N}_{T\to D,B}[\rho_T]] \approx \rho_T \quad \text{for} \quad d_T \ll d_D.$$

We have also checked the relative entropy $\int dU\, S(\mathcal{R}^{Lite}_{D,B\to T}[\mathcal{N}_{T\to D,B}[\rho_T]]\,||\,\rho_T)$ is also consistent with the above result.



$\mathcal{N}^{\dagger}_{D,B\to T}\left[\mathcal{N}_{T\to D,B}[\rho_T]\right]$

Contents of Talk

1. Introduction and Petz-lite for Haar random case ✔
2. Petz-lite for SYK case (SYK HP)
3. Summary

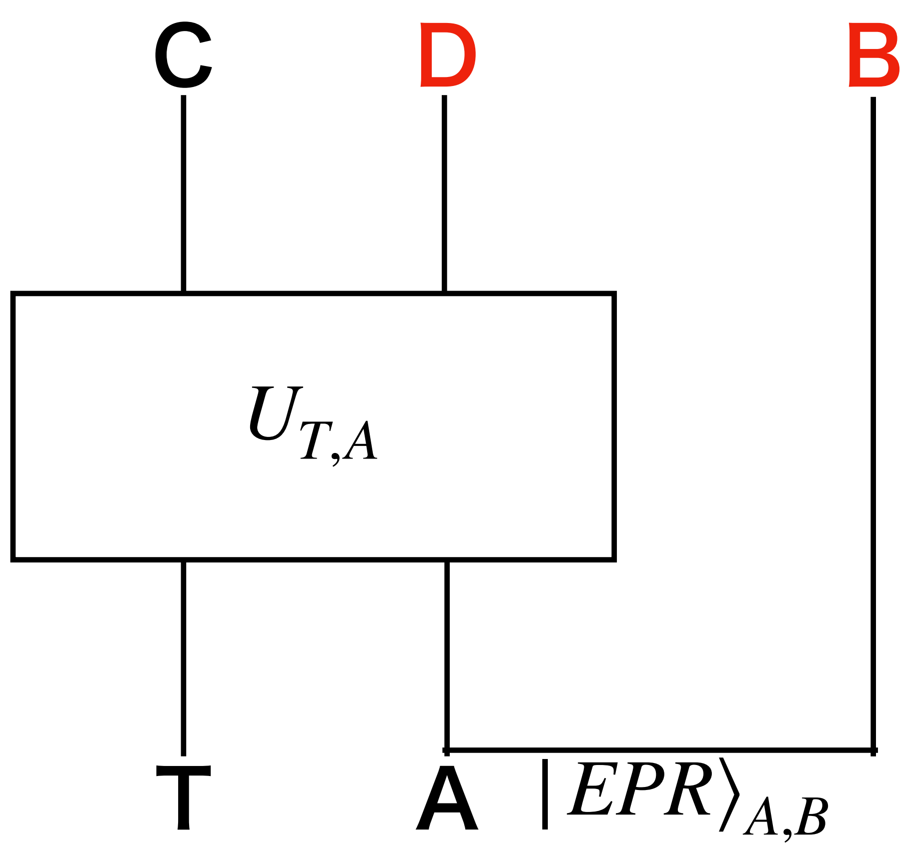
Quantum Channel of the HP protocol in SYK

- Focus on the setup of an HP protocol in the SYK model [Chandrasekaran-Levine '22]

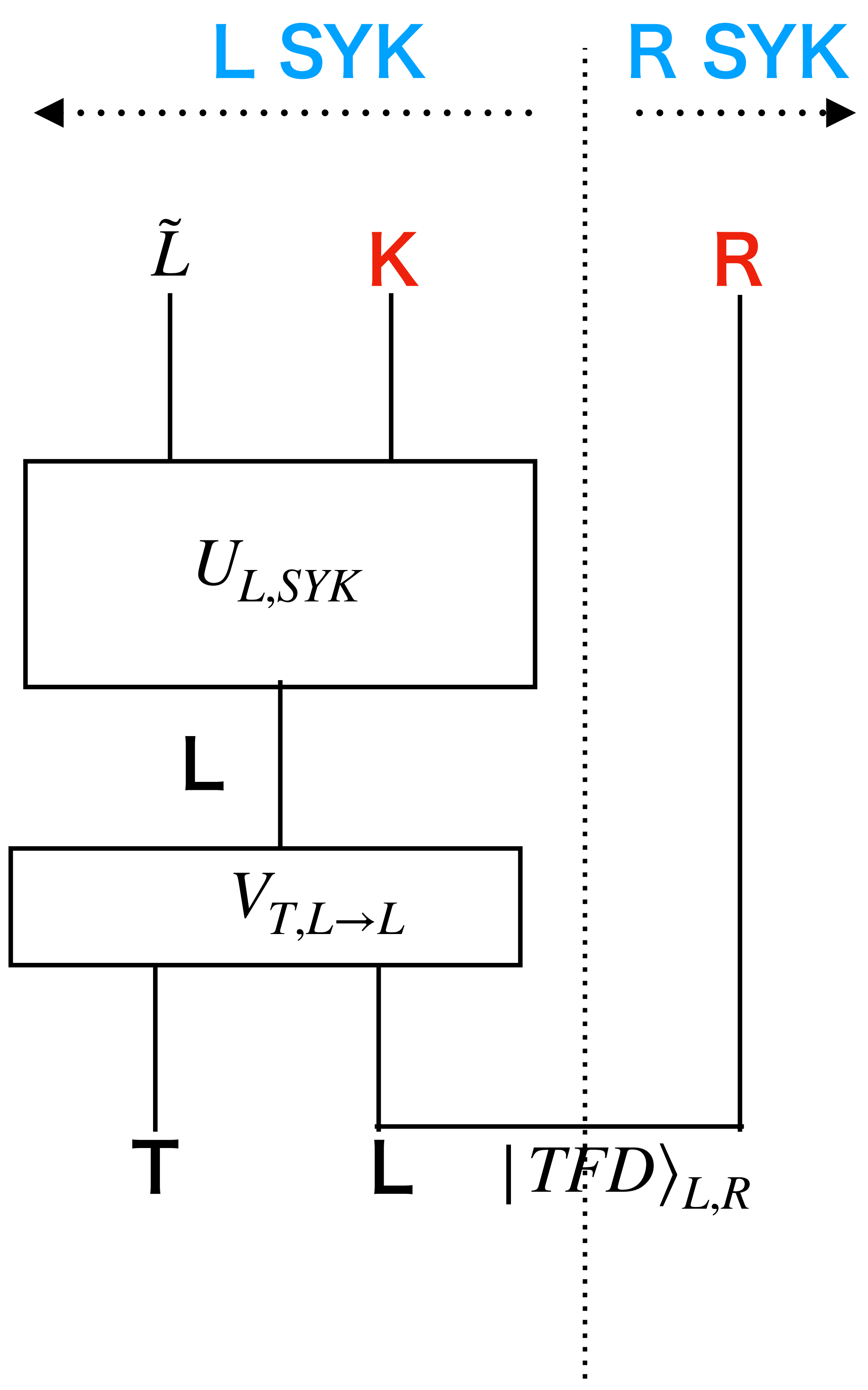
- To use the setup, we assume that a diary consists of only 1-qubit: $d_T = 2$

→ Check the validity of the Petz lite, which recover the 1-qubit diary information from the Hawking radiation.

- In the SYK HP protocol, consider the setup;



Original HP protocol



SYK HP protocol

$\tilde{L}$: sub-system with (N-K) Majorana fermion=Remaining BH (C)

K: sub-system with K Majorana fermion=Late Radiation (D)

$$(N \gg 1, K \gg 1, (N-K) \gg 1)$$

R: R system with N Majorana fermion=Early Radiation (B)

$U_{L,SYK} = e^{-itH_{L,SYK}}$: Unitary time evolution by the left SYK Hamiltonian

$$H_{L,SYK} = (i)^{q/2} \sum_{1 \le i_1 \le i_2 \le \cdots \le i_q \le N} j_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

$V_{T,L\to L}$: Embedding map from $T, L \to L$ with $d_T = 2$

$$V(|T'\rangle_T \otimes |TFD\rangle) = \begin{cases} |TFD\rangle & T' = 0 \\ \psi_{i,L}(0)|TFD\rangle & T' = 1 \end{cases} \qquad (i \in \tilde{L})$$

$$|TFD\rangle_{L,R} = \frac{1}{\sqrt{Z(\beta)}} \sum_i e^{-\beta E_i/2} |E_i\rangle_L \otimes |E_i\rangle_R^*$$

- Quantum channel (QC)

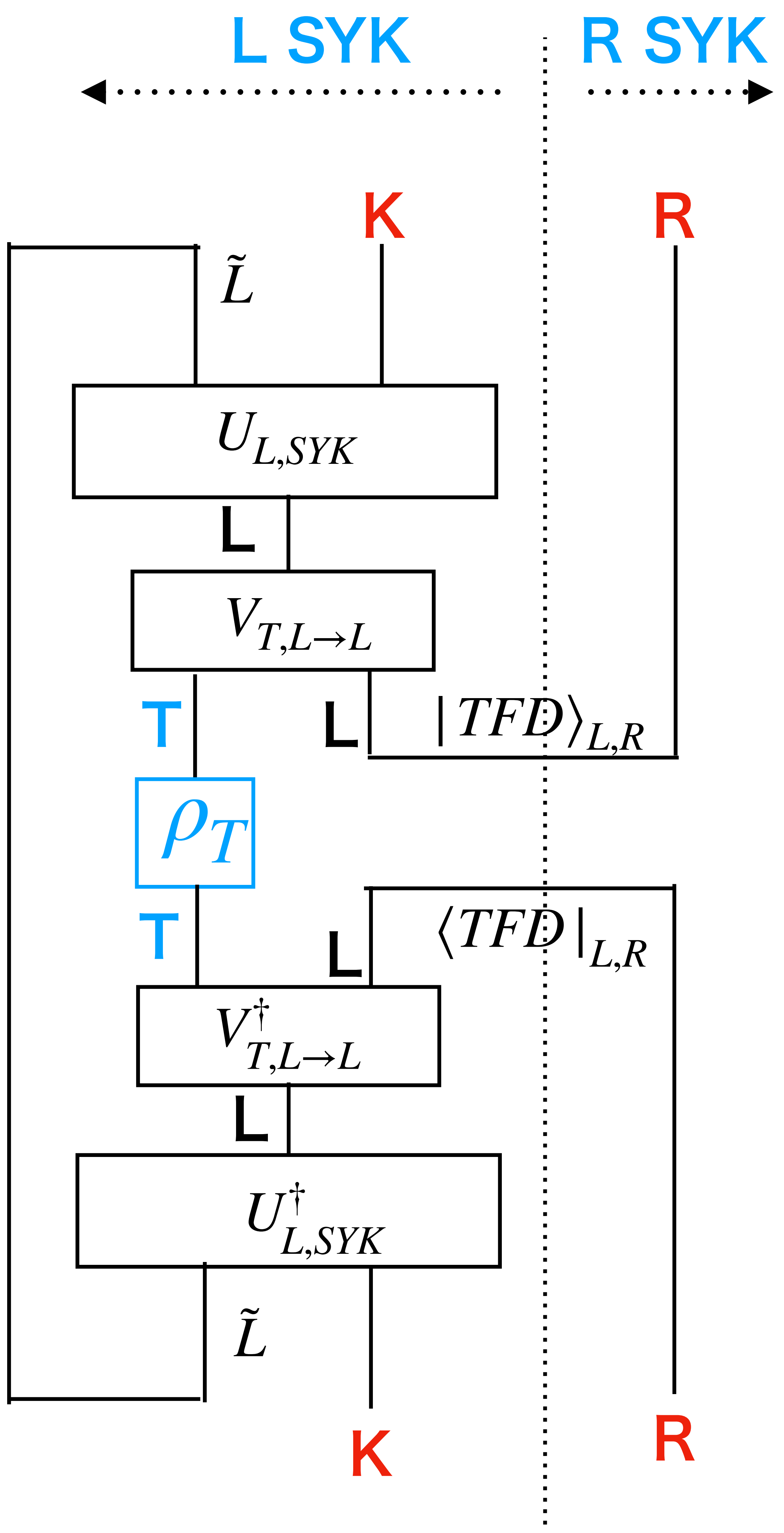
$$\mathcal{N}^{SYK}_{T\to K,R}[\rho_T] = \mathrm{Tr}_{\tilde{L}}\left[U_L V_{T,L\to L}\left(\rho_T \otimes |TFD\rangle_{L,R}\langle TFD|\right) V^\dagger_{T,L\to L} U^\dagger_L\right]$$

- Ex.

$$\mathcal{N}^{SYK}_{T\to K,R}[\,|1\rangle\langle 0|\,] = \mathrm{Tr}_{\tilde{L}}\left[U_L \psi_{i,L}(0)|TFD\rangle_{T,A}\langle TFD|U^\dagger_L\right]$$

- Adjoint quantum channel

$$\mathcal{N}^{SYK\dagger}_{K,R\to T}[\mathcal{O}_{KR}] = {}_{L,R}\langle TFD|\left(V^\dagger_{L\to T,L} U^\dagger_L \mathcal{O}_{KR} U_L V_{L\to T,L}\right)|TFD\rangle_{L,R}$$



SYK HP QC

Recovery channel for the HP protocol in SYK

The Petz lite for the SYK setup is given by (with the normalization Tr $\left[\mathscr{R}^{Lite}_{K,R\to T}[\mathcal{N}_{T\to K,R}[\rho_T]]\right] \approx$ Tr $[\rho_T] = 1)$

$$\mathscr{R}^{Lite}_{K,R\to T} = \frac{\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta}{1 + \left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1 | \mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle} \mathcal{N}^\dagger_{K,R\to T}$$

$\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta = ($**Tr** $\left[(\rho_{\tilde{L}})^2\right])^{-1}$: **"effective dim." of** $\tilde{L}$

$\sim$ **effective dim. of Remaining BH**

$(\rho_{\tilde{L}} =$ **Tr**$_{K,R}\left[|TFD\rangle_{L,R}\langle TFD|\right])$

c.f. $\mathscr{R}^{Lite}_{D,R\to T} = \dfrac{d_C}{1 + \left(\frac{d_D}{d_T}\right)^2} \mathcal{N}^\dagger_{T\to D,R}$ (Haar random HP case, d_C : dim. of Remaining BH)

- In this SYK HP protocol, let us check the matrix elements

$$\langle T_1 | \mathscr{R}^{Lite}_{K,R\to T}[\mathcal{N}_{T\to K,R}[\rho_T]] | T_2\rangle \overset{?}{\approx} \langle T_1 | \rho_T | T_2\rangle \quad \text{for } \forall \rho_T$$

Or, equivalently

$$\langle T_1 | \mathscr{R}^{Lite}_{K,R\to T}[\mathcal{N}_{T\to K,R}[|T_3\rangle\langle T_4|]] | T_2\rangle \overset{?}{\approx} \langle T_1 | T_3\rangle\langle T_4 | T_2\rangle = \delta_{T_1T_3}\delta_{T_2T_4} \quad \textbf{for } T_i = 0,1 \quad (i = 1,\cdots,4)$$

- For example, focus on the following matrix element,

$$0 \overset{?}{\approx} \langle 1 | \mathscr{R}^{Lite}_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle\langle 0|]]|1\rangle = \frac{\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1|\mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle}{1 + \left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1|\mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle}$$

  - To understand the behavior of the element, we need to evaluate

$$\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1|\mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle = \frac{\langle TFD|\psi_{i,L}(t)\left(I_{\tilde{L}} \otimes \gamma_{KR}\right)\psi_{i,L}(t)|TFD\rangle}{\mathrm{Tr}\left[\gamma_{KR}^2\right]} \qquad \left(\gamma_{KR} = \mathrm{Tr}_{\tilde{L}}\left[|TFD\rangle_{L,R}\langle TFD|\right]\right)$$

It is difficult to evaluate the quantity analytically, but basically, we are interested in typical results.

→ Consider the “averaged correlator” over the possible embedding into $\tilde{L}$ (consisting of $(N-K)$ fermions)

$$\frac{\langle TFD|\psi_{i,L}(t)\left(I_{\tilde{L}} \otimes \gamma_{KR}\right)\psi_{i,L}(t)|TFD\rangle}{\mathrm{Tr}\left[\gamma_{KR}^2\right]} \longrightarrow \frac{1}{N-K}\sum_{i=1}^{N-K}\frac{\langle TFD|\psi_{i,L}(t)\left(I_{\tilde{L}} \otimes \gamma_{KR}\right)\psi_{i,L}(t)|TFD\rangle}{\mathrm{Tr}\left[\gamma_{KR}^2\right]} \qquad (i \in \tilde{L})$$

- By using techniques in [Chandrasekaran-Levine '22], we evaluated the "Renyi-two" correlator in the large-βJ limit with $\beta J \ll N/K$ or the large-q (SYK_q) limit with $q \ll N/K$)

$$\frac{1}{N-K}\sum_{i=1}^{N-K}\frac{\langle TFD|\psi_{i,L}(t)\left(I_{\tilde{L}}\otimes\gamma_{KR}\right)\psi_{i,L}(t)|TFD\rangle}{\mathrm{Tr}\left[\gamma_{KR}^{2}\right]}\overset{t\gg1}{\approx}G_{2\beta}(2\beta)\left[1-\frac{K}{N}e^{\frac{\pi}{\beta}t}\cdot\#(q,\beta,J,C)\right]+\mathcal{O}((K/N)^{2})$$

- $G_{2\beta}$: SYK thermal two point function with periodicity 2β.
- We have assumed the condition $K/N \ll 1$.
- $\#(q,\beta,J,C)$: constant given by the SYK model parameters (q,J,C) and β

Therefore, $\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1 | \mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle \sim G_{2\beta}(2\beta)\left[1 - \frac{K}{N}e^{\frac{\pi}{\beta}t}\cdot \#(q,\beta,J,C)\right] + \mathcal{O}((K/N)^2)$

- Around initial times $t \sim 1$, the quantity is approximately $G_{2\beta}(2\beta)$,

- This quantity includes the critical time $t_{Recover}$, where the expansion w.r.t. (K/N) becomes breakdown

$$\frac{K}{N}e^{\frac{\pi}{\beta}t} = \exp\left(\frac{\lambda}{2}(t - t_{Recover})\right) \qquad \left(\lambda = \frac{2\pi}{\beta},\quad t_{Recover} = 2t_{scrambling} = \frac{\beta}{\pi}\log(N/K),\quad t_{scrambling} = \frac{\beta}{2\pi}\log(N/K)\right)$$

Around the critical time $t_{Recovery} = 2t_{Scrambling}$, the coefficient of $G_{2\beta}(2\beta)$ becomes small, so the quantity also becomes small. This gives the our expected result around $t_{Recovery} = 2t_{Scrambling}$

$$0 \overset{?}{\approx} \langle 1 | \mathcal{R}^{Lite}_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle\langle 0|]]|1\rangle = \frac{\left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1 | \mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle}{1 + \left\langle \hat{d}_{\tilde{L}} \right\rangle_\beta \cdot \langle 1 | \mathcal{N}^\dagger_{D,R\to T}[\mathcal{N}_{T\to D,R}[|0\rangle_T\langle 0|]]|1\rangle} \overset{t\sim t_{Recovery}}{\approx} \underbrace{G_{2\beta}(\beta)\cdot(small)}_{\sim 0}$$

- The deviation from the expected result means an error for the recovery map (approximate recovery).

Interpretation of the SYK Result

- The time scale $t_{Recovery} = 2t_{Scrambling}$ is natural since the correlator is evaluated as the Renyi-two quantity. It can be interpreted as follows;

  1. The quantum channel $\mathcal{N}_{T\to K,R}$ scrambles an input state and diffuse the information over the system (in 1-st system=Original system); This takes the scrambling time $t_{Scrambling}$.

  2. The recovery map $\mathcal{N}^{\dagger}_{K,R\to T}$ is the "opposite" process of the above quantum channel (in 2nd system = Recovery system); This also takes the scrambling time $t_{Scrambling}$.

  →In total, it take twice the scrambling time, $t_{Recovery} = 2t_{Scrambling}$!

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1. Introduction and Petz-lite for Haar random case ✔
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Summary

- We propose that in a highly chaotic system, one can use the "Petz-lite" as the recovery map;

$$\mathscr{R}_{K,R\to T}[\mathcal{O}_{K,R}] = \sigma_T^{1/2}\mathcal{N}^{\dagger}_{K,R\to T}\left[\mathcal{N}_{T\to K,R}[\sigma_T]^{-1/2}\mathcal{O}_{K,R}\ \mathcal{N}_{T\to K,R}[\sigma_T]^{-1/2}\right]\sigma_T^{1/2}$$ (Original Petz map).

$$\to \mathscr{R}^{Lite}_{K,R\to T}[\mathcal{O}_{K,R}] = \frac{1}{N}\mathcal{N}^{\dagger}_{K,R\to T}\left[\mathcal{O}_{K,R}\right]$$ (N : suitable normalization factor)

  - In the Petz lite, the recovery time is twice the scrambling time, not the scrambling time.

- We have checked the validity of the Petz lite for the Haar random unitary.

- Also, we have checked some parts (one matrix component in this talk) of the validity of the Petz lite for the SYK model. The remaining parts are still under calculation.

Future work

- Bulk Interpretation?
- Numerical analysis
- Relation to recovery error (upper) bound [Nakata-Tezuka ’23] ?
- Fully analytic results (without using the K/N expansion or with resuming their contributions)
- Variants of the HP setup
- More higher dimensional ($d_T \geq 3$) code subspace case?
- HP setup in complex SYK or 2dCFT with conserved charge?
- Chaotic-Integrable Transition? (In progress)
- Relation to the Gao-Jafferis-Wall traversable wormhole (protocol) [Gao-Jafferis-Wall ’16, Gao-Jafferis ’19,⋯], Size-winding protocol[Brown-Gharibyan-Leichenauer-et al.. ’19,⋯]?
- 2d CFT case, other models (e.g., Chaotic Spin chain)?
- ……

Thank you

for your attention!