Symmetries as Ground States

Sanjay Moudgalya
Caltech $\rightarrow$ TUM

SM, Olexei I. Motrunich, arXiv: 2309.15167

Related Works:
- arXiv: 2108.10824 [PRX 12, 011050 (2022)]
- arXiv: 2209.03370 [Annals of Physics 455, 169384 (2023)]
- arXiv: 2209.03377
- arXiv: 2302.03028 [PRB 107, 224312 (2023)]

YITP Kyoto
28th September 2023

Introduction: Symmetries in Quantum Many-Body Physics

Symmetries in Many-Body Physics

- Symmetries: Organizing principle for physical phenomena.

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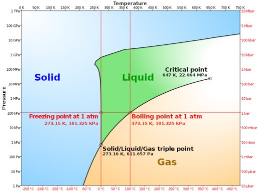
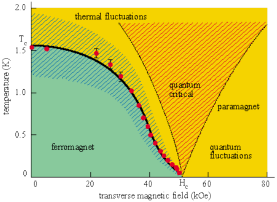
- Conventional phases of matter and transitions understood using "Spontaneous symmetry breaking" (Landau paradigm)





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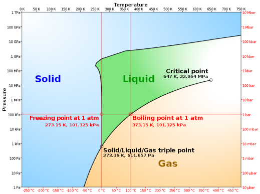
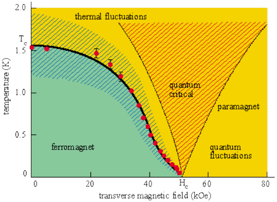
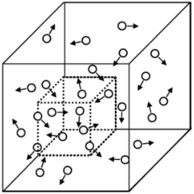
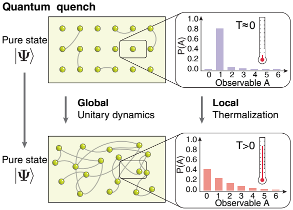




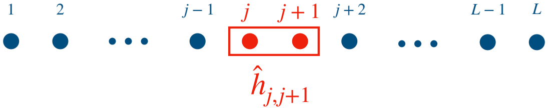
- Extensive conserved quantities required for correct thermodynamics, e.g., in the definition of Gibbs ensembles





Symmetries in Quantum Many-Body Physics

- Physics of quantum many-body systems studied using toy models: Spin chain governed by a local Hamiltonian $H = \sum_{j=1}^{L} \hat{h}_{j,j+1}$.



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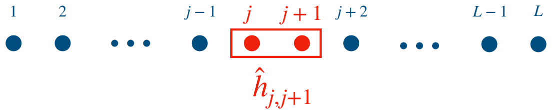

- Example:
  - Hilbert space $\mathcal{H}$: Spanned by spins $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$ on each site.
  - Operators: Magnetization $Z_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, Spin-Flip $X_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
  - Transverse Field Ising Model: $H = \sum_{j=1}^{L} \widehat{h}_{j,j+1}$, $\hat{h}_{j,j+1} = (X_j X_{j+1} + g Z_j)$.

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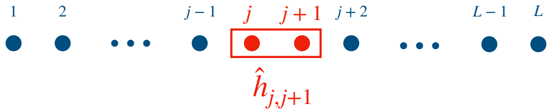

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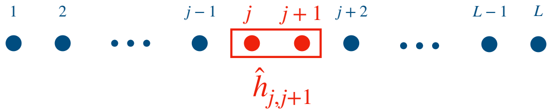

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- Z_2 symmetry of Ising Model: Parity of total spin $Q_\alpha = \prod_{j=1}^{L} Z_j$.

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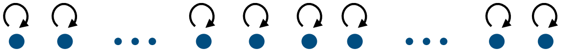

$$Q_\alpha = u_1 \; u_2 \; \cdots \; u_{j-1} \; u_j \; u_{j+1} u_{j+2} \; \cdots \; u_{L-1} \; u_L$$

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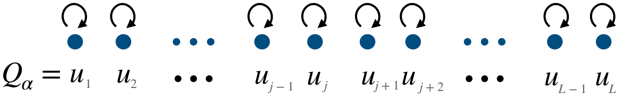

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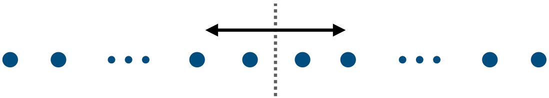

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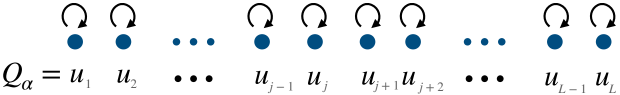

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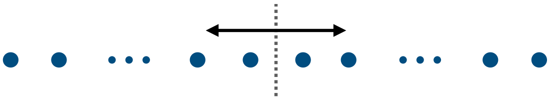

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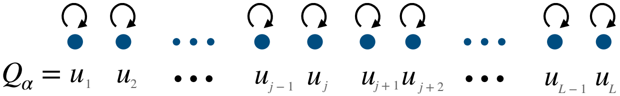

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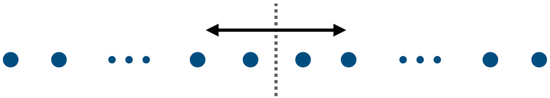

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Are these the most general physical symmetries?

Beyond Conventional Symmetries

- Several recent works: Conventional symmetries are not sufficient!

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Beyond Conventional Symmetries

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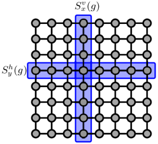
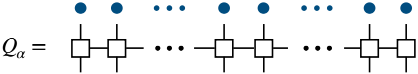
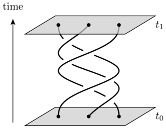
- Equilibrium Physics:[1] Subsystem/Higher-form symmetries, Fractons, Categorical/MPO symmetries, Generalized Landau paradigm, . . .







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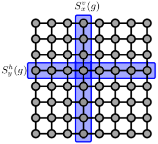
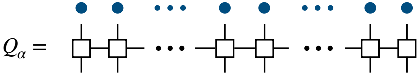
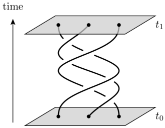
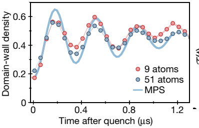
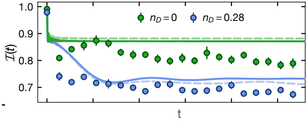


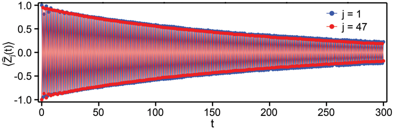




- Non-Equilibrium Physics:[2] Slow thermalization due to quantum scars, Hilbert space fragmentation, strong zero modes, ...







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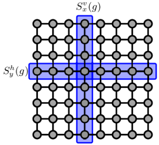

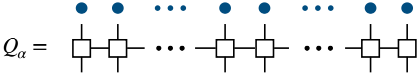

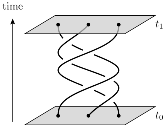

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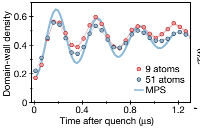

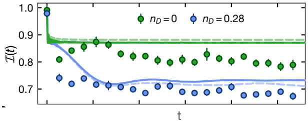

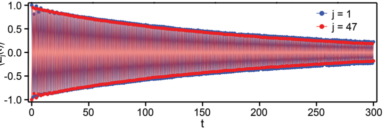

Characterizing Properties of Symmetries?

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Background: Quantum Dynamics and Weak Ergodicity Breaking

Review Articles:

- M.Serbyn, D.A.Abanin, Z.Papić, arXiv: 2011.09486
- Z.Papić, arXiv: 2108.03460
- **SM**, B. Andrei Bernevig, Nicolas Regnault, arXiv: 2109.00548
- A.Chandran, T.Iadecola, V.Khemani, R.Moessner, arXiv: 2206.11528

Ergodicity in Isolated Quantum Systems

[3] J.M.Deutsch (1991), M.Srednicki (1994)

[4] **SM, B.A.Bernevig, N.Regnault (2021)**

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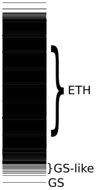
Ergodicity in Isolated Quantum Systems

- Eigenstate Thermalization Hypothesis (ETH)[3]:
  Eigenstates $|E\rangle$ are "thermal"
  - $E \iff \beta$, $\mathrm{Tr}_B(|E\rangle\langle E|) \sim e^{-\beta H|_A}$
  - **Volume law** entanglement: $S = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) \sim V_A$.



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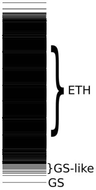
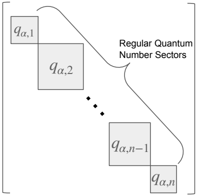


- Symmetric Hamiltonians: Block-diagonalized into symmetry sectors labelled by eigenvalues under $\{Q_\alpha\}$.

- Ergodicity/ETH expected **within** each sector



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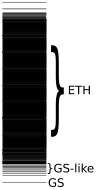

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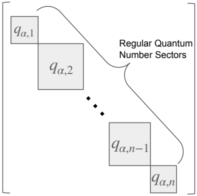

Assumption $\sim$ All "blocks" are explained by symmetries

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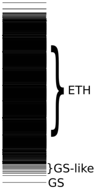

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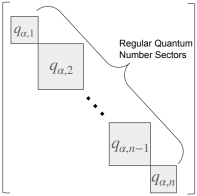

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- Recent analytical[4] and experimental[5] discovery of "weak" ergodicity breaking put this into question

[3] J.M.Deutsch (1991), M.Srednicki (1994)

[4] **SM, B.A.Bernevig, N.Regnault (2021)**

[5] M.Serbyn, D.A.Abanin, Z.Papic (2020)

Weak Ergodicity Breaking

[6] **SM, S.Rachel, B.A.Bernevig, N.Regnault (2017)**

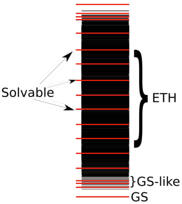
[7] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019); V.Khemani, M.Hermele, R.Nandkishore (2019)

[8] **SM, A.Prem, R.Nandkishore, N.Regnault, B.A.Bernevig (2019)**

Weak Ergodicity Breaking

Quantum Many-Body Scars

- Solvable eigenstates deep in the spectrum[6]
- Mid-spectrum: $S \sim \log L \implies$ ETH violation!
- Equally spaced quasiparticle tower
  $\implies$ Revivals from simple initial states



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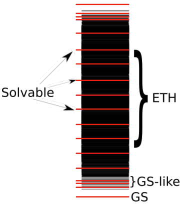
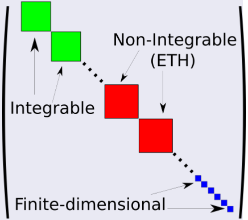


Hilbert Space Fragmentation

- Local Hamiltonians with **exp. many** disconnected blocks:[7] $\mathcal{H} = \bigoplus_{\alpha=1}^{\sim\exp(L)} \mathrm{span}\left\{e^{-iHt}\left|R_\alpha\right\rangle\right\}$
- Blocks **not distinguished** by conventional symmetry quantum numbers, vastly different properties![8]



[6] **SM, S.Rachel, B.A.Bernevig, N.Regnault (2017)**

[7] P.Sala, T.Rakovszky, R.Verresen, M.Knap, F.Pollmann (2019); V.Khemani, M.Hermele, R.Nandkishore (2019)

[8] **SM, A.Prem, R.Nandkishore, N.Regnault, B.A.Bernevig (2019)**

Outstanding Issue for Symmetries

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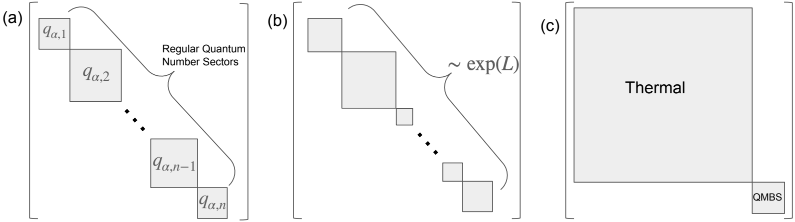

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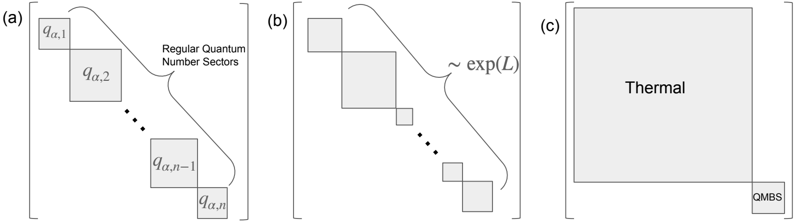

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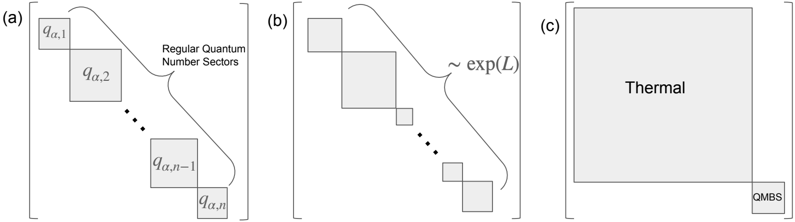

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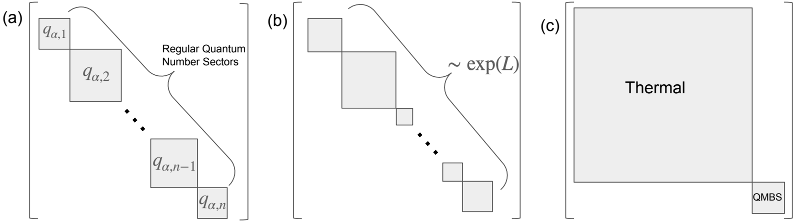

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What is an appropriate definition of a symmetry/conserved quantity?

Our Recent Works: Symmetries and Commutant Algebras

SM, O.I.Motrunich
- arXiv: 2108.10824 [PRX 12, 011050 (2022)]
- arXiv: 2209.03370 [Ann. Phys. 455, 169384 (2023)]
- arXiv: 2209.03377
- arXiv: 2302.03028 [PRB 107, 224312 (2023)]
- arXiv: 2309.15167

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- We want symmetries to commute with H: $[Q_\alpha, \sum_j \widehat{h}_{j,j+1}] = 0$

$$\text{Ising Model:} \quad \left[\prod_j Z_j, \sum_j (X_j X_{j+1} + g Z_j)\right] = 0$$

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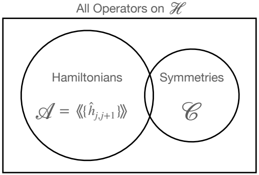
Symmetries in $\mathcal{C}$ $\Longleftrightarrow$ Families of Hamiltonians in $\mathcal{A}$

Commutant Algebras: Block Structures

- $\mathcal{A}$ and $\mathcal{C}$ are von Neumann algebras (closed under $\dagger$), centralizers of each other (Double Commutant Theorem).



[9] **SM, O.I.Motrunich (2021-23)**

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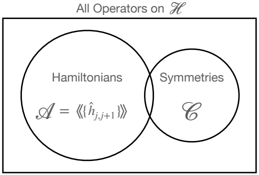
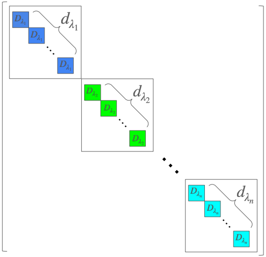
  $\{d_\lambda\}$: Irreps of $\mathcal{C}$ = Degeneracies





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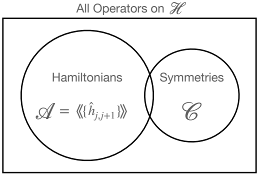

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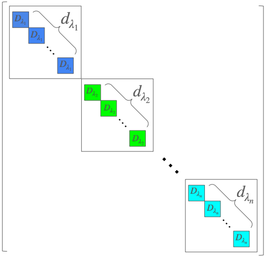

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- Symmetry Sectors: Basis in which **all** local terms $\{\widehat{h}_{j,j+1}\}$ **simultaneously** block diagonal![9]

[9] **SM, O.I.Motrunich (2021-23)**

Symmetries are Ground States!

[10] **SM, O.I.Motrunich (2023)**

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- Operators on $\mathcal{H}$ are states in a doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}$

$$\widehat{O} = \sum_{\mu,\nu} o_{\mu\nu} \left|v_\mu\right\rangle\!\left\langle v_\nu\right| \iff |\widehat{O}) = \sum_{\mu,\nu} o_{\mu\nu} \left|v_\mu\right\rangle \otimes \left|v_\nu\right\rangle$$

- Rewrite commutant condition after converting operators to states

$$[\widehat{h}_{j,j+1}, \widehat{Q}_\alpha] = 0 \quad \iff \quad \overbrace{(\widehat{h}_{j,j+1} \otimes \mathbb{1} - \mathbb{1} \otimes \widehat{h}^T_{j,j+1})}^{\widehat{\mathcal{L}}_{j,j+1} :=} |\widehat{Q}_\alpha) = 0.$$

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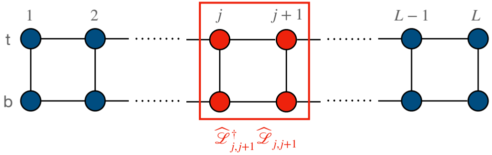

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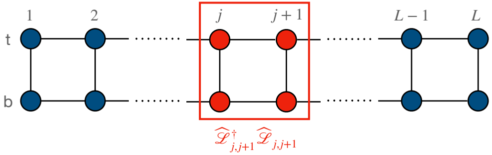

- Number of ground states $= \dim(\mathcal{C})$.

[10] SM, O.I.Motrunich (2023)

Z_2 Symmetry = Ising Ferromagnet

- Bond Algebra: $\mathcal{A}_{Z_2} = \langle\langle \{X_j X_{j+1}\}, \{Z_j\} \rangle\rangle$

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- Generalization: $SU(q)$ symmetry $=$ $SO(q^2)$ ferromagnet.

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Ground States = Exact Symmetries
Low-Energy Excitations = Approximate Symmetries?

[12] X.Chen, T.Zhou (2019); C.Sünderhauf *et al.* (2019); D.Bernard, T.Jin (2019)

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- Made precise using a "Brownian Circuit"[12]

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Autocorrelation Functions

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Gapless Symmetries and Slow-Modes

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[15] **SM**, A.Prem, D.A.Huse, A.Chan (2020)

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- Generalization: Dipole symmetry,[15] low-energy modes $\sim k^4$, autocorrelation decay $\sim t^{-1/4}$ $\implies$ Subdiffusion

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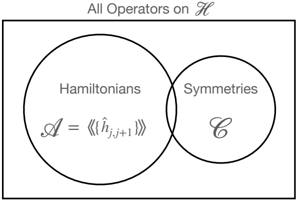
- $\mathcal{C}$ = Ground states of super-Hamiltonian $\widehat{\mathcal{P}}$, which contains info on slow-modes



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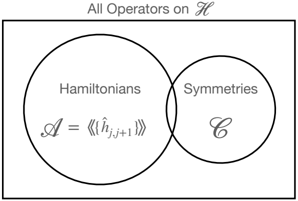

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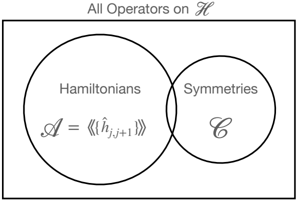

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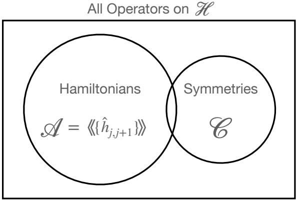

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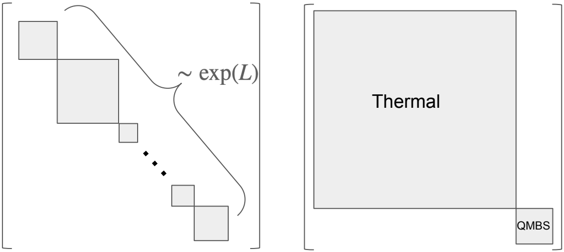
No explicit restriction on the structure of symmetries in $\mathcal{C}$:
Novel unconventional symmetries!

Unconventional Symmetries: Weak Ergodicity Breaking



Hilbert Space Fragmentation

- $t - J_z$ Hamiltonian:[16] hopping with two species of particles,

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Quantum Many-Body Scars

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- Ground states: $\{|\Phi_{n,0}\rangle_t \otimes |\Phi_{m,0}\rangle_b\} \implies \mathcal{C}_{\text{scar}} = \langle\!\langle \{|\Phi_{n,0}\rangle\langle\Phi_{m,0}|\} \rangle\!\rangle.$

[18] **SM**, O.I.Motrunich (2022)

Quantum Many-Body Scars

- QMBS eigenstates: $H\,|\psi_n\rangle = E_n\,|\psi_n\rangle \iff [H, |\psi_n\rangle\langle\psi_n|] = 0.$
- $\mathcal{C}_{\text{scar}} = \langle\!\langle |\psi_n\rangle\langle\psi_n| \rangle\!\rangle$ is a valid symmetry algebra if there exists a corresponding locally-generated $\mathcal{A}_{\text{scar}}$.
- Spin-$\frac{1}{2}$ ferromagnetic multiplet $\{|\Phi_{n,0}\rangle = (S^-_{\text{tot}})^n\,|F\rangle\}$, $|F\rangle = |\uparrow\cdots\uparrow\rangle$

$$\mathcal{A}_{\text{scar}} := \langle\!\langle \{R_{j,j+1}\sigma^\alpha_{j+2}, \sigma^\alpha_{j-1}R_{j,j+1}\} \rangle\!\rangle, \quad R_{j,j+1} = \frac{1}{4} - \vec{S}_j\cdot\vec{S}_{j+1}.$$

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- Projectors onto some **special** states can be viewed as symmetries![18]

[18] **SM**, O.I.Motrunich (2022)

Unconventional Slow-Modes

[19] **SM, O.I.Motrunich (2023)**

[20] J.Feldmeier, W. Witczak-Krempa, M.Knap (2022)

[21] L.Gotta, **SM**, L.Mazza (2023)

[22] M.C.Bañuls, D.A.Huse, J.I.Cirac (2021)

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$$\overline{|\langle\Phi_{n,k}(0)|\Phi_{n,k}(t)\rangle|^2} \geq \left|\overline{(\widehat{A}(0)|\widehat{A}(-t))}\right|^2 = e^{-2\kappa \sin^2(\frac{k}{2})t} \sim e^{-\frac{ct}{L^2}}.$$

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- Asymptotic QMBS:[21] Fidelity decay timescale diverges with L even though state orthogonal to QMBS – generically not possible![22]

[19] **SM, O.I.Motrunich (2023)**

[20] J.Feldmeier, W. Witczak-Krempa, M.Knap (2022)

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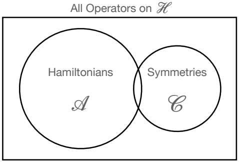
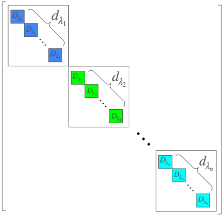
Summary and Open Questions

- Local Building Blocks $\Longleftrightarrow$ Symmetry
  $\mathcal{A}$: Local algebra $\mathcal{C}$: Commutant algebra
- Symmetries are ground states, slow-modes are low-energy excitations
- Locality restrictions on generators of $\mathcal{A}$ instead of $\mathcal{C}$ $\Longrightarrow$ Novel symmetries
- Non-invertible/categorical symmetries in this language? Dualities?[23]
- Symmetry = Shadow of topological order?[24]





- "Classification" of kinds of symmetries and slow-modes with locality?
- Approximate symmetries without exact symmetries? Stability to perturbations?

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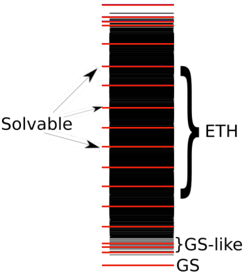
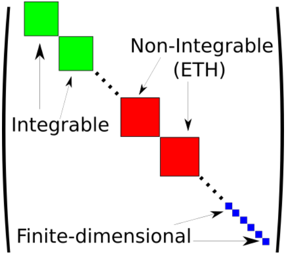
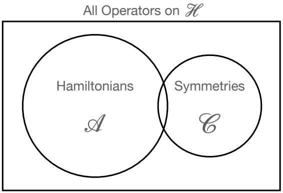
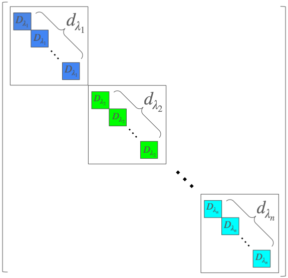
[23] E.Cobanera, G.Ortiz, Z.Nussinov (2011); H.Moradi, Ö.M.Aksoy, J.H.Bardarson, A.Tiwari (2023)

[24] J.McGreevy (2022); A.Chatterjee, X.-G. Wen (2023); H.Moradi, S.F.Moosavian, A.Tiwari (2022)


Solvable
ETH
}GS-like
GS


Non-Integrable
(ETH)
Integrable
Finite-dimensional


All Operators on $\mathscr{H}$
Hamiltonians
$\mathscr{A}$
Symmetries
$\mathscr{C}$


D_{λ_1}
d_{λ_1}
D_{λ_2}
d_{λ_2}
D_{λ_n}
d_{λ_n}

Thank You!

Commutant Algebras: Block Structures (Informal)

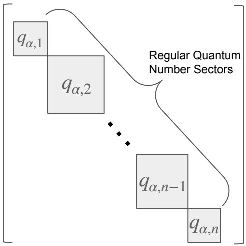
- Symmetry Sectors: Basis in which **all** local terms $\{\widehat{h}_{j,j+1}\}$ **simultaneously** block diagonal!
- Entire families of Hamiltonians built from $\{\widehat{h}_{j,j+1}\}$ block-diagonal in same basis, including $H = \sum_j \widehat{h}_{j,j+1}$.
- $N_{\text{blocks}} = \sum_\lambda d_\lambda$ (hard, requires rep. theory) scales as $\dim(\mathcal{C})$ = Number of lin. ind. ops. in $\mathcal{C}$ (easy, count solutions to $[\hat{h}_{j,j+1}, \widehat{O}] = 0$)![25]

$N_{blocks} \sim \dim(\mathcal{C})$	**Example**
$\mathcal{O}(1)$	Discrete Global Symmetry
$\mathrm{poly}(L)$	Continuous Global Symmetry
$\exp(L)$	Fragmentation

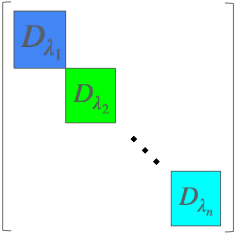
- Symmetries and associated quantum number sectors **uniquely** determined from $\{\hat{h}_{j,j+1}\}$!

[25] **SM, O.I.Motrunich(2022)**

Conventional Symmetries



Simple Examples: Abelian $\mathcal{C}$



- Abelian $\mathcal{C} \implies d_\lambda = 1, \quad N_{\text{blocks}} = \dim(\mathcal{C})$
- No symmetry: Generic $\{\hat{h}_{j,j+1}\}$
  Solve for $[\hat{h}_{j,j+1}, \widehat{O}] = 0$

$$\mathcal{C} = \{\mathbb{1}\}, \; N_{\text{blocks}} = \dim(\mathcal{C}) = 1$$

- Z_2 Symmetry: $\{\hat{h}_{j,j+1}\} = \{X_j X_{j+1}, Z_j\}$.
  Solve for $[X_j X_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

$$\mathcal{C}_{Z_2} = \text{span}\{\mathbb{1}, \prod_j Z_j\} = \langle\!\langle \prod_j Z_j \rangle\!\rangle, \quad N_{\text{blocks}} = \dim(\mathcal{C}_{Z_2}) = 2.$$

- $U(1)$ Symmetry: $\{\hat{h}_{j,j+1}\} = \{X_j X_{j+1} + Y_j Y_{j+1}, Z_j\}$
  Solve for $[X_j X_{j+1} + Y_j Y_{j+1}, \widehat{O}] = 0$ and $[Z_j, \widehat{O}] = 0$

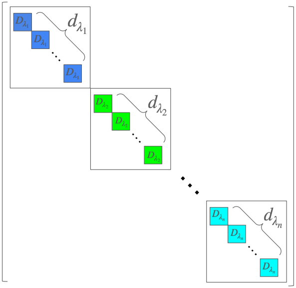
$$\mathcal{C}_{U(1)} = \text{span}\{\mathbb{1}, Z_{\text{tot}}, Z_{\text{tot}}^2, \cdots, Z_{\text{tot}}^L\} = \langle\!\langle Z_{\text{tot}} \rangle\!\rangle = \langle\!\langle \{\prod_j e^{i\alpha Z_j}\} \rangle\!\rangle,$$

$$Z_{\text{tot}} = \sum_j Z_j, \quad N_{\text{blocks}} = \dim(\mathcal{C}_{U(1)}) = L + 1.$$

Simple Examples: Non-Abelian $\mathcal{C}$

- Non-Abelian $\mathcal{C} \implies$ some $d_\lambda > 1$
  $\implies$ degeneracies
- $SU(2)$ Symmetry: $\{\hat{h}_{j,j+1}\} = \{\vec{S}_j \cdot \vec{S}_{j+1}\}$

$$\mathcal{C}_{SU(2)} = \langle\!\langle S^x_{\text{tot}}, S^y_{\text{tot}}, S^z_{\text{tot}} \rangle\!\rangle = \langle\!\langle \{\prod_j e^{i\alpha_\mu S^\mu_j}\} \rangle\!\rangle$$
$$= \text{span}_{p,q,r}\{(S^x_{\text{tot}})^p (S^y_{\text{tot}})^q (S^z_{\text{tot}})^r\}$$

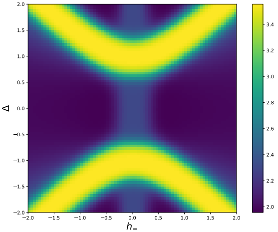


- Block-diagonal form (Schur-Weyl duality):
  $0 \leq \lambda \leq L/2$: S^2 eigenvalues, $d_\lambda = 2\lambda + 1$: irreps of $\mathfrak{su}(2)$
  D_λ: irreps of S_L, $N_{\text{blocks}} \sim \dim(\mathcal{C}_{SU(2)}) \sim poly(L)$
- Another example: Stabilizer codes, e.g., toric code
  - $\mathcal{A}$ is the group algebra of the stabilizer group.
  - $\mathcal{C}$ consists of $\mathcal{A}$ and the non-trivial logical operators.

Numerical Methods & Systematic Searches

- Determining $\mathcal{C}$ given $\mathcal{A}$: Hard in practice, need numerical methods.[26]
    - Simultaneous block diagonalization of generators of $\mathcal{A}$ – can extract operators in $\mathcal{C}$, their irreps, etc.
    - $\mathcal{C}$ frustration-free ground state space of a local superoperator "Hamiltonian" – efficient to solve (at least in one dimension).

- Systematic (numerical) scan through physically relevant families of Hamiltonians[27]
- Discovers unconventional $SU(2)_q$ quantum group symmetries, Strong Zero Modes[28]in *non-integrable* models!



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[26]**SM, O.I.Motrunich (2023)**

[24]**SM, O.I.Motrunich (in preparation)**

[28]P.Fendley (2016); D.V.Else, P.Fendley, J.Kemp, C.Nayak (2017)