Operator growth and "quantum chaos": lessons from SYK

Pratik Nandy

Yukawa Institute for Theoretical Physics (YITP)

1. w/ B. Bhattacharjee (IISc/IBS), X. Cao (ENS), and T. Pathak (IISc), arXiv: 2212.06180 (JHEP)
2. ongoing works…

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Outline

1. “Universal operator growth hypothesis” - Lanczos coefficients and Krylov complexity.

2. Example: large q SYK (moment method).

3. “Generalizing” operator growth hypothesis to open quantum systems - motivating “dissipative quantum chaos”.

4. Numerics: generalizing Lanczos algorithm to Bi-Lanczos algorithm in finite N SYK.

5. Open (Lindbladian) SYK in the large q limit.

6. Motivate to understand some universal properties.

7. Conclusions.

Disclaimer:

In this talk, I’m not comparing between Krylov complexity and OTOC. I just want to motivate the operator growth in generic systems through Krylov complexity.

We want to study the operator growth $\mathcal{O}(t) = e^{i\mathcal{L}^\dagger t}\mathcal{O}$

For unitary evolution: $\mathcal{L}^\dagger = \mathcal{L}$, and $\mathcal{L}\mathcal{O} = [H, \mathcal{O}]$. The evolution is $\mathcal{O}(t) = e^{iHt}\,\mathcal{O}\,e^{-iHt}$

$$\mathcal{L} = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ b_1 & 0 & b_2 & \cdots & 0 \\ 0 & b_2 & 0 & b_3 & \cdots \\ \cdots & \cdots & b_3 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & b_n \\ 0 & 0 & \cdots & b_n & 0 \end{pmatrix}.$$

Input: H and $\mathcal{O}$ —— Lanczos algorithm ——→ Output: $\{b_n\}$ and $\{\mathcal{O}_n\}$

The Lanczos coefficients suppose to capture the chaotic nature of the Hamiltonian.

> *"For chaotic systems, the Lanczos coefficients grow linearly, and this is the maximum growth possible"* $\quad b_n \sim \alpha\, n\,.$

Universal operator growth hypothesis

Parker-Cao-Avdoshkin-Scaffidi-Altman (2018)

In other words, for chaotic systems, K-complexity grows exponentially $\quad C_K(t) \sim e^{2\alpha t}$

The reverse statement is not always true

> *"The linear growth of Lanczos coefficients does not necessary imply chaos".*

Dymarsky-Smolkin (2021)
Bhattacharjee-Cao-**PN**-Pathak (2022)

Lanczos coefficients from moments (generalized version)

Viswanath-Muller (1994)

Parker-Cao-Avdoshkin-Scaffidi-Altman (2018)

Bhattacharjee-Cao-**PN**-Pathak (2022)

Given the autocorrelation function $C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle$

$$C(t) := \sum_{n=0}^{\infty} m_n \frac{(it)^n}{(n)!}$$

Iteratively find the Lanczos coefficients as

$$M_k^{(0)} = (-1)^k m_k, \quad L_k^{(0)} = (-1)^{k+1} m_{k+1}$$

$$M_k^{(n)} = L_k^{(n-1)} - L_{n-1}^{(n-1)} \frac{M_{n-1}^{(n-1)}}{M_k^{(n-1)}} \qquad L_k^{(n)} = \frac{M_{k+1}^{(n)}}{M_n^{(n)}} - \frac{M_{n-1}^{(n-1)}}{M_k^{(n-1)}} \qquad k \geq n\,,$$

$$b_n = \sqrt{M_n^{(n)}}, \quad a_n = -L_n^{(n)}$$

Auto-correlation function

↓

Moments

↓

Lanczos coefficients

↓

Krylov basis wave functions

↓

Krylov complexity

Sachdev-Ye (1993), Kitaev (2015)

Example: Our interest is in large-N and in large-q limit of SYK.

Mean: $\langle j_{i_1\cdots i_q}\rangle = 0$

Hamiltonian $H = i^{q/2} \sum_{1\leq i_1<i_2<\cdots<i_q\leq N} j_{i_1\cdots i_q}\psi_{i_1}\psi_{i_2}\cdots\psi_{i_q}$

Variance: $\langle j^2_{i_1\cdots i_q}\rangle = 2^{q-1}\frac{(q-1)!\mathcal{J}^2}{qN^{q-1}}$

We start with an initial operator $\mathcal{O}(0) = \sqrt{2}\,\psi_1$

We expand the auto-correlation function Maldacena-Stanford (2016)

$$C(\tau) = 1 + \frac{2\ln(\operatorname{sech}\mathcal{J}\tau)}{q} + \cdots$$

$$b_n = \mathcal{J}\sqrt{\frac{2}{q}}\,, \qquad n = 1$$

$$= \mathcal{J}\sqrt{n(n-1)} + O(1/q)\,, \qquad n > 1\,.$$

Parker-Cao-Avdoshkin-Scaffidi-Altman (2018)

Open quantum systems

Lindblad (1976), Gorini-Kossakowski-Sudarshan (1976)

Can we generalize the operator growth hypothesis in generic open system?

The evolution (Markovian dynamics) of system density matrix is governed by the Lindbladian

$$\mathcal{O}(t) = e^{i\mathcal{L}^\dagger t}\mathcal{O}$$

Lindbladian: $$\mathcal{L}^\dagger \mathcal{O} = [H, \mathcal{O}] - i\sum_k \left[\mp L_k^\dagger \mathcal{O} L_k - \frac{1}{2}\{L_k^\dagger L_k, \mathcal{O}\}\right].$$

In generic dissipative system, the Lindbladian in Krylov basis takes an "ideal" tridiagonal form

$$\mathcal{L}_o^\dagger = \begin{pmatrix} i|a_1| & b_1 & 0 & \cdots & 0 \\ c_1 & i|a_2| & b_2 & \cdots & 0 \\ 0 & c_2 & i|a_3| & b_3 & \cdots \\ \cdots & \cdots & c_3 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & b_n \\ 0 & 0 & \cdots & c_n & i|a_n| \end{pmatrix}.$$

However, we need to define Krylov basis in context of open system (non-unitary evolution). The Lanczos algorithm works when the system is closed and the evolution is unitary. However, for non-unitary evolution the Lanczos algorithm fails!

We will see that can be done and our motivation is to understand the asymptotic growth of such coefficients.

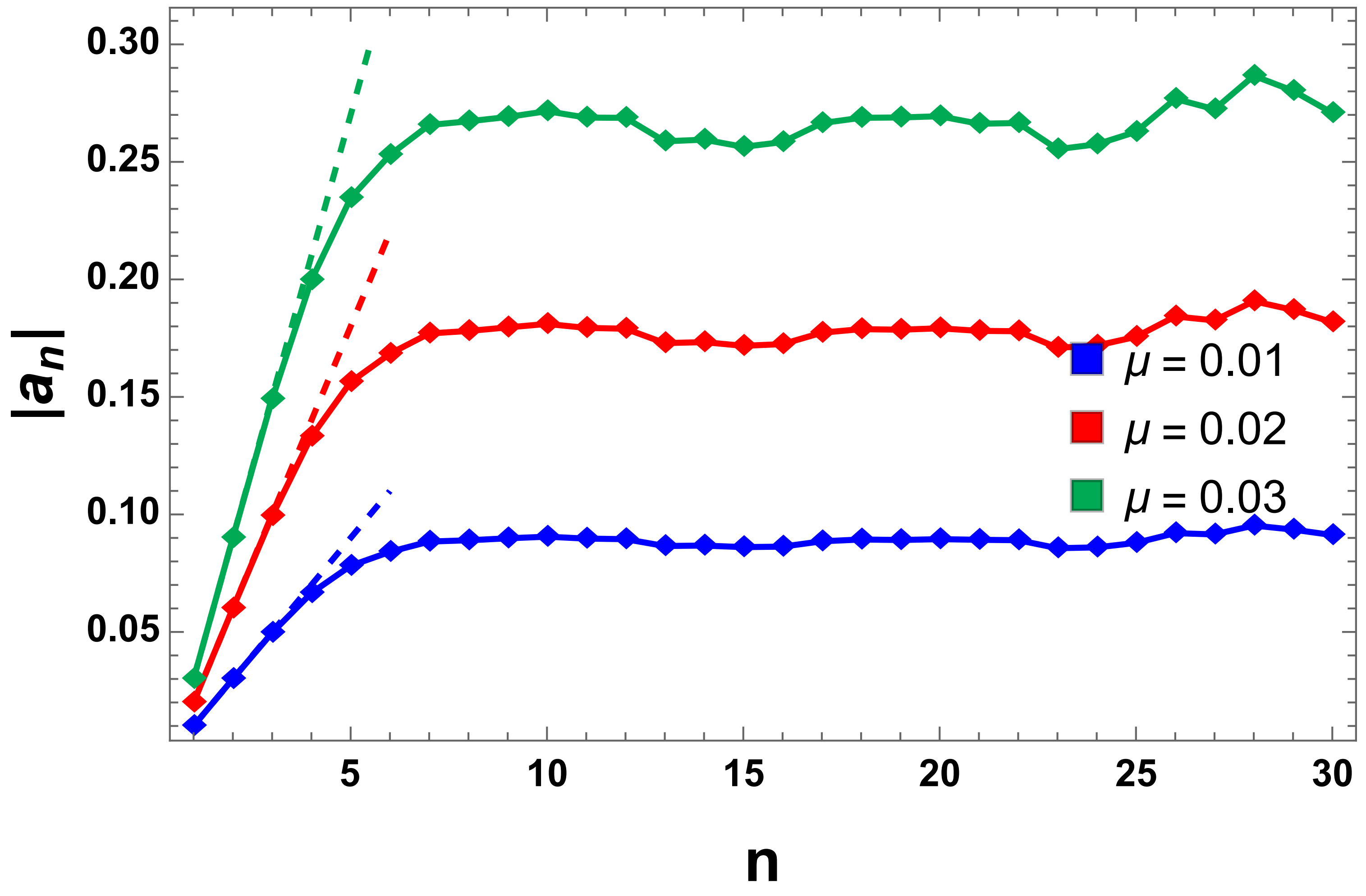
A suitable algorithm is bi-Lanczos algorithm. This is a generalization of Lanczos algorithm suitable to deal with non-unitary evolution.

We create two separate operator space, one is constructed by acting of $\mathscr{L}_o$ while the other is constructed by $\mathscr{L}_o^\dagger$

They are individually not orthonormal but they are bi-orthonormal.

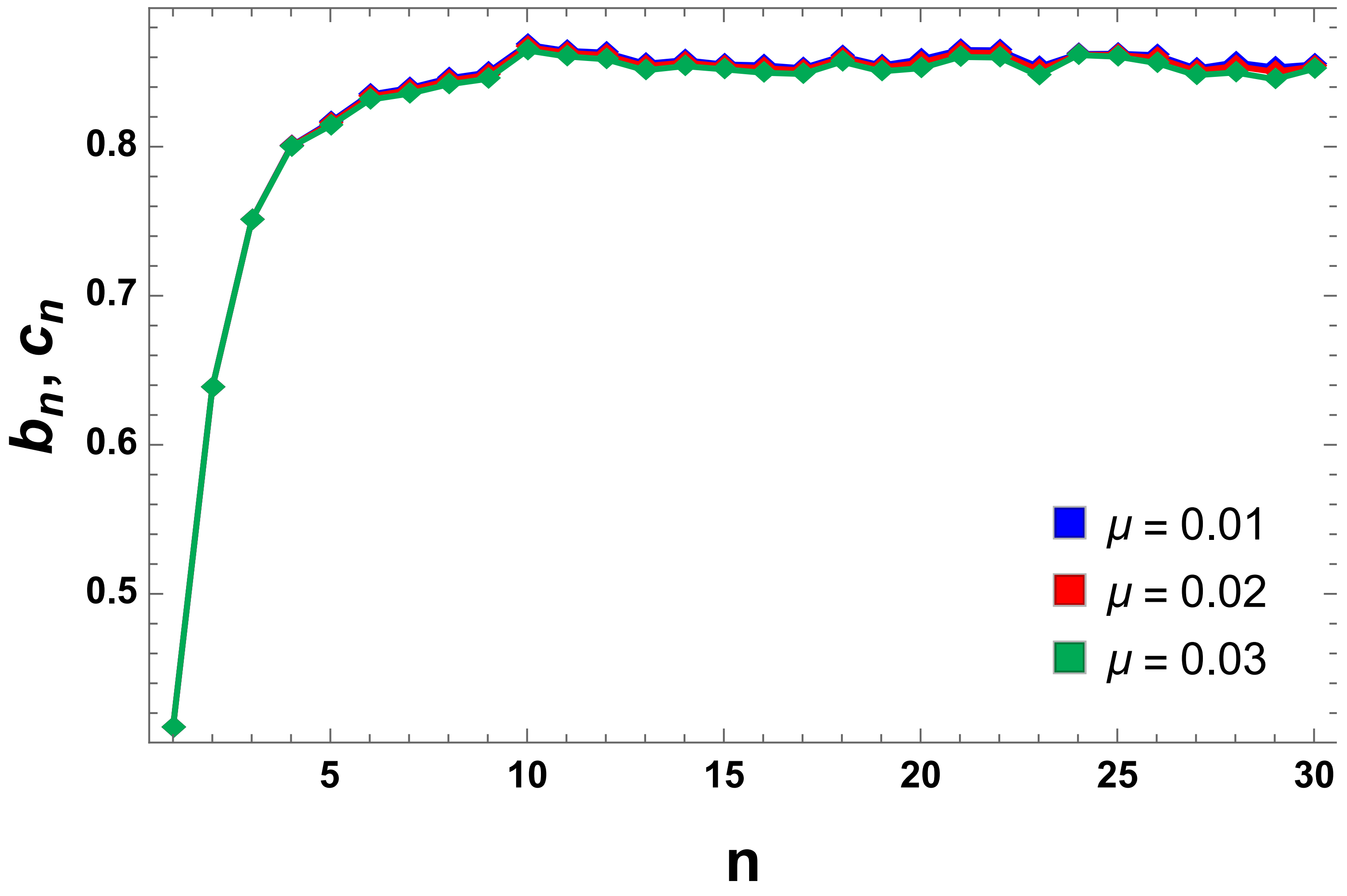
$$\mathscr{L}_o^\dagger = \begin{pmatrix} i|a_1| & b_1 & 0 & \cdots & 0 \\ b_1 & i|a_2| & b_2 & \cdots & 0 \\ 0 & b_2 & i|a_3| & b_3 & \cdots \\ \cdots & \cdots & b_3 & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & b_n \\ 0 & 0 & \cdots & b_n & i|a_n| \end{pmatrix}.$$

We apply this to finite N SYK and find that $b_n = c_n$ (purely real) while the diagonal elements $a_n = i|a_n|$ are purely imaginary



The slope of the diagonal elements is

$$|a_n| = \mu(2n-1)$$



Example: Open SYK

Sachdev-Ye (1993), Kitaev (2015)

Kulkarni-Numasawa-Ryu (2021)

Hamiltonian $$H = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} j_{i_1 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

Mean: $$\langle j_{i_1 \cdots i_q} \rangle = 0$$

Variance: $$\langle j^2_{i_1 \cdots i_q} \rangle = 2^{q-1} \frac{(q-1)! \mathcal{J}^2}{q N^{q-1}}$$

Lindblad operators: $$L_i = \sqrt{\mu}\, \psi_i \,, \qquad i = 1,2,\cdots,N \,.$$

We expand the auto-correlation function $$C(\tau) = 1 + \frac{g(\tau)}{q} + \cdots$$

Initial operator: $\propto \psi_1$

$$g(\tau) = \log\left(\frac{\alpha^2}{\mathcal{J}^2 \cosh^2(\alpha |\tau| + \gamma)} \right) \qquad \tilde{\mu} = \mu q \,, \qquad \alpha = \mathcal{J} \sqrt{\left(\frac{\tilde{\mu}}{2\mathcal{J}}\right)^2 + 1} \,, \qquad \gamma = \sinh^{-1}\left(\frac{\tilde{\mu}}{2\mathcal{J}}\right) .$$

We expand the auto-correlation function and computing moments are straightforward.

We are interested in computing Lanczos coefficients

$$a_n = i\tilde{\mu}n + O(1/q) \qquad n \geq 0\,, \qquad \tilde{\mu} = \mu q\,.$$

$$b_n = \mathcal{J}\sqrt{\frac{2}{q}}\,, \qquad n = 1$$

$$= \mathcal{J}\sqrt{n(n-1)} + O(1/q)\,, \qquad n > 1\,.$$

Bhattacharjee-Cao-**PN**-Pathak (2022)

Comparison to closed system SYK

$$b_n = \mathcal{J}\sqrt{\frac{2}{q}}\,, \qquad n = 1$$

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Parker-Cao-Avdoshkin-Scaffidi-Altman (2018)

Observations for large q SYK

1. $a_n's$ linearly depend on the dissipative factor while the $b_n's$ are independent of it.
2. $a_n's$ are purely imaginary while $b_n's$ are real.
3. For large-n, both a_n and b_n are linear in n.

All are supported by numerics
(bi-Lanczos algorithm)

The analytical and numerical results suggest the asymptotic growth of the Lanczos coefficients

The most general version of "operator growth hypothesis"

$$a_n \sim i\chi\mu n \qquad\qquad b_n \sim \alpha n$$

Bhattacharjee-Cao-**PN**-Pathak (2022)

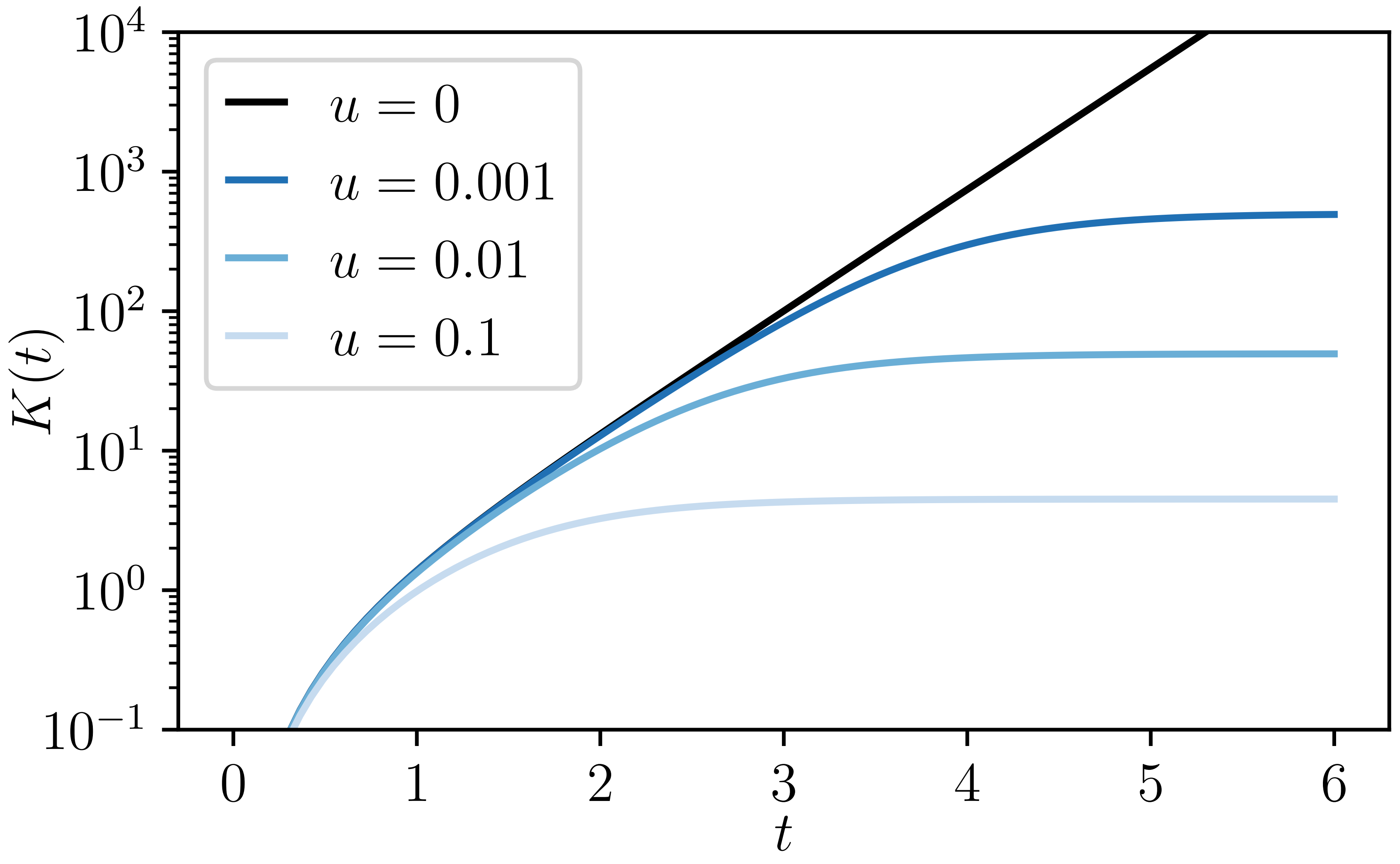
We compute the Krylov complexity by recursively solving the equation $\partial_t \varphi_n(t) = i a_n \varphi_n(t) - b_{n+1}\varphi_{n+1}(t) + b_n \varphi_{n-1}(t)\,.$

We take the coefficients of the form $b_n^2 = (1-u^2)n(n-1+\eta)\,, \quad a_n = iu(2n+\eta)\,.$

Reduces to the asymptotic growth for

$$\alpha = 1-u^2\,, \quad \chi\mu = 2u\,.$$

Krylov complexity $K(t) = \frac{1}{\mathcal{Z}}\sum_n n\,|\varphi_n(t)|^2 = \frac{\eta\left(1-u^2\right)\tanh^2(t)}{1+2u\tanh(t)-\left(1-2u^2\right)\tanh^2(t)}\,.$



Weak dissipation limit $K(t) = \eta\left[\sinh^2(t) - 2u\sinh^3(t)\cosh(t) + O(u^2)\right]\,,$

Asymptotic analysis gives $K(t) \sim 1/u \qquad t_* \sim \ln(1/u)$

Summary and future directions

1. We motivate to understand "dissipative quantum chaos", particularly in SYK. A valid question is to understand how this dissipative time scale is related to the scrambling time.

2. How "universal" is our result? Is the dissipative time scale and the saturation generic and robust for any dissipative chaotic systems?

Ex. Random jump operators

Kulkarni-Numasawa-Ryu (2021)
Sa-Ribeiro-Prosen (2021)

$$L^a = \sum_{1 \le i \le j \le N} K^a_{ij} \, \psi_i \psi_j \,, \qquad a = 1,2,\cdots,M\,.$$

$$H = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} j_{i_1 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

Works are in progress!

$$\langle K^a_{ij} \rangle = 0 \qquad \langle |K^a_{ij}|^2 \rangle = \frac{K^2}{N^2} \quad \forall i, j, a$$

Analytically solvable in $N, M \to \infty$ limit keeping $R = M/N$ finite (a special "double scaling limit").

3. Holographic interpretation in terms of wormholes.

4. Application to non-Hermitian systems?

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Thank you for your attention!