Time-like Entanglement Entropy in AdS/BCFT

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Time-like entanglement entropy in AdS$_3$/CFT$_2$

- The standard entanglement entropy for a generic space-like interval A in a CFT$_2$ may be expressed in terms of twist field correlators using the replica technique as

$$S_A = \lim_{n\to 1} \frac{1}{1-n} \log \langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle .$$

- The time-like entanglement entropy S_A^T for a purely time-like interval A is obtained by analytically continuing the space-like interval to a time-like interval [Doi et.al '22].
- The entanglement entropy of a generic space-like interval A having time-like width T_0 and space-like width X_0 may be obtained using the replica technique as [Calabrese and Cardy '04]

$$S_A = \frac{c}{3} \log \frac{\sqrt{X_0^2 - T_0^2}}{\epsilon}.$$

- Then, the time-like entanglement entropy S_A^T for a purely time-like interval A is obtained by analytically continuing the space-like interval to a time-like interval followed by taking $X_0 = 0$ in above equation as follows [Doi et.al '22]

$$S_A^T = \frac{c}{3} \log \left(\frac{T_0}{\epsilon}\right) + \frac{i\pi c}{6}.$$

The TEE takes complex value in a CFT$_2$ in contrast to real valued standard entanglement entropy.

Holographic time-like entanglement entropy

- Consider the Poincaré patch of a AdS$_3$ spacetime (with AdS radius $R_{\text{AdS}} = 1$) whose metric is given by

$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2}.$$

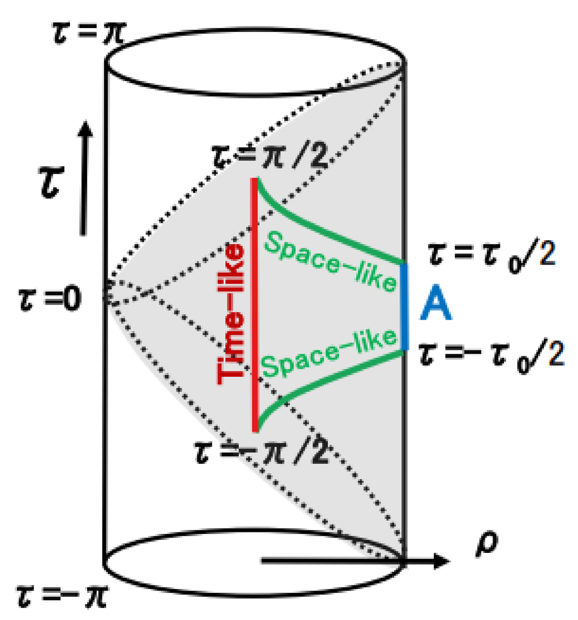

Figure 1. Schematic of RT surface for the holographic TEE. [Credit: Doi et.al '22]

- The holographic time-like entanglement entropy for the time-like interval A is given by three geodesics where two space-like geodesics connects the endpoints of A and null infinities, and a time-like geodesic which connects the endpoints of two space-like geodesics.
- The real part of the time-like entanglement entropy obtained from the length of space-like geodesic is given by [Doi et.al '22]

$$S_A^T = \frac{c}{3} \log \frac{T_0}{\epsilon}.$$

- The imaginary part of the time-like entanglement entropy is obtained by embedding the Poincaré patch in the global patch and the length of this time-like geodesic is $\pi c/6$ which matches with the imaginary part of dual CFT$_2$ result.

TEE in BCFT$_2$

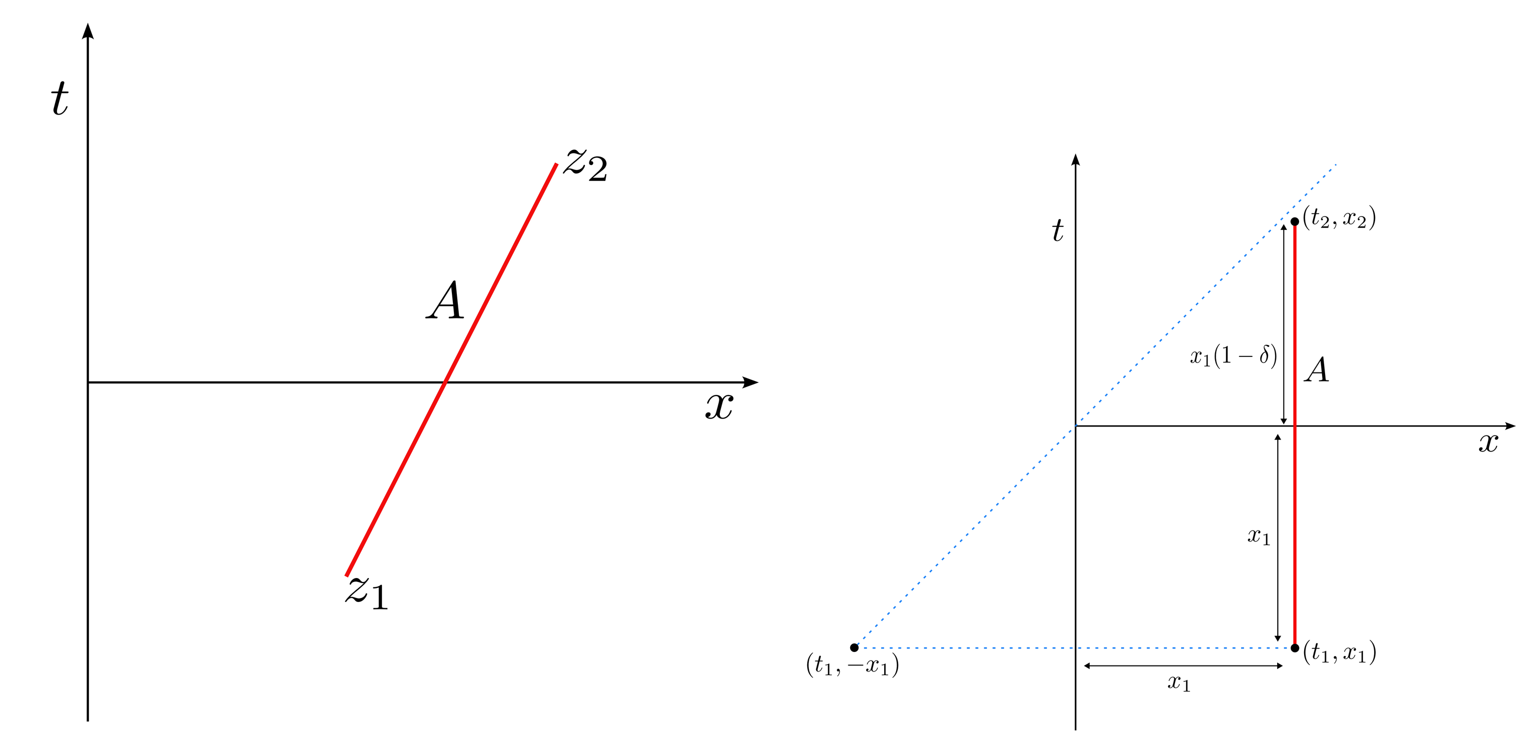

Figure 2. (a) Schematic of a single interval in a BCFT. (b) Regge limit configuration.

- The two-point function of scalar operators in a BCFT behaves kinematically like a CFT four-point function

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = \frac{1}{|4x_\perp y_\perp|^{\Delta}} \mathcal{G}(\xi).$$

- The function of cross ratio is known explicitly in some limits of ξ as follows:

I. Bulk limit

- The interval is far away from the boundary in this limit and $\xi \to 0$. The two point twist correlator in the bulk limit may be written in the following form

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle = \frac{\epsilon^{2\Delta_n}}{|z_1 - z_2|^{2\Delta_n}}.$$

- For a purely time-like interval with $x_1 = x_2$, the time-like entanglement entropy S_A^T in the bulk limit after an analytical continuation is given by

$$S_A^T = \frac{c}{3} \log \frac{T_0}{\epsilon} + \frac{i\pi c}{6} := S^B.$$

II. Regge limit

- The Regge limit ($\xi \to -1$) occurs when the second end point of the interval lie on the light cone of the mirror image of the first end point of the interval. The two point function in the Regge limit is given by

$$\langle \sigma_n(z_1)\bar{\sigma}_n(z_2)\rangle = \frac{1}{|2x_1|^{2\Delta_n}} \frac{1}{(1+\xi)^{\Delta_n}}.$$

- The time-like entanglement entropy for the time-like interval A is given by

$$S_A^T = \begin{cases} \frac{c}{3} \log \left(\frac{2x_1}{\epsilon} \sqrt{2 - \frac{T_0}{x_1}} \right), & \text{for } T_0 \to 2x_1^-, \quad \text{i.e. } \xi \to -1^+, \\ \frac{c}{3} \log \left(\frac{2x_1}{\epsilon} \sqrt{\frac{T_0}{x_1} - 2} \right) + \frac{i\pi c}{6}, & \text{for } T_0 \to 2x_1^+, \quad \text{i.e. } \xi \to -1^-, \end{cases} := S^R.$$

III. Boundary limit

- In the boundary limit ($\xi \to \infty$), the interval is closer to the boundary. Similarly, the time-like entanglement entropy for a pure time-like interval is given by

$$S_A^T = \frac{c}{3} \log \frac{2x_1}{\epsilon} + 2 \log g_b := S^b.$$

The behaviour of the time-like entropy is shown in the following figure.

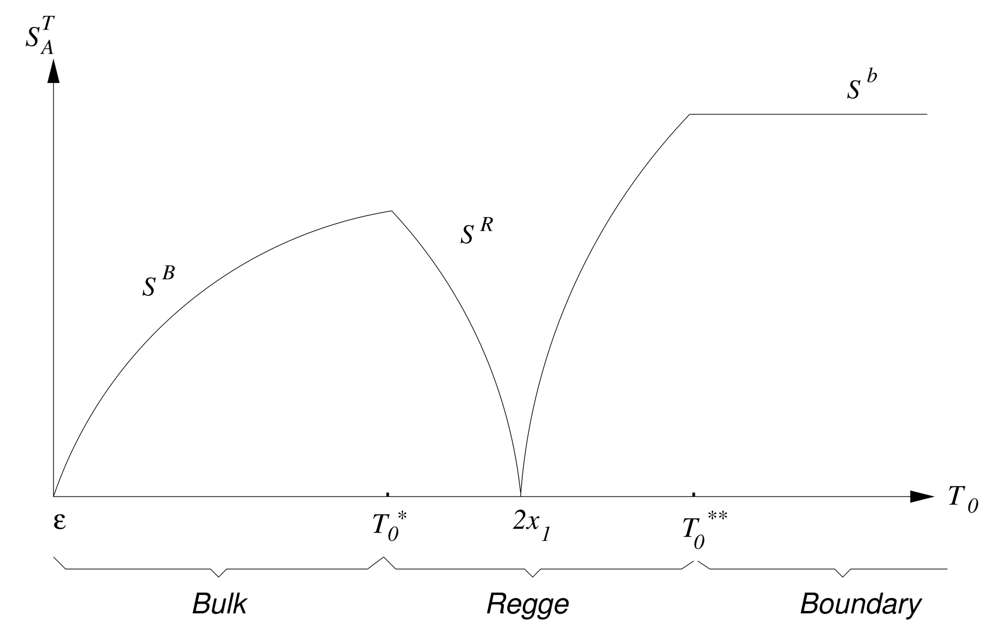

Figure 3. Phases of the time-like entanglement entropy S_A^T in a BCFT.

Holographic TEE in AdS/BCFT

- We have three possible choices of RT surface for this configuration depending on the size and distance from the boundary described as follows:

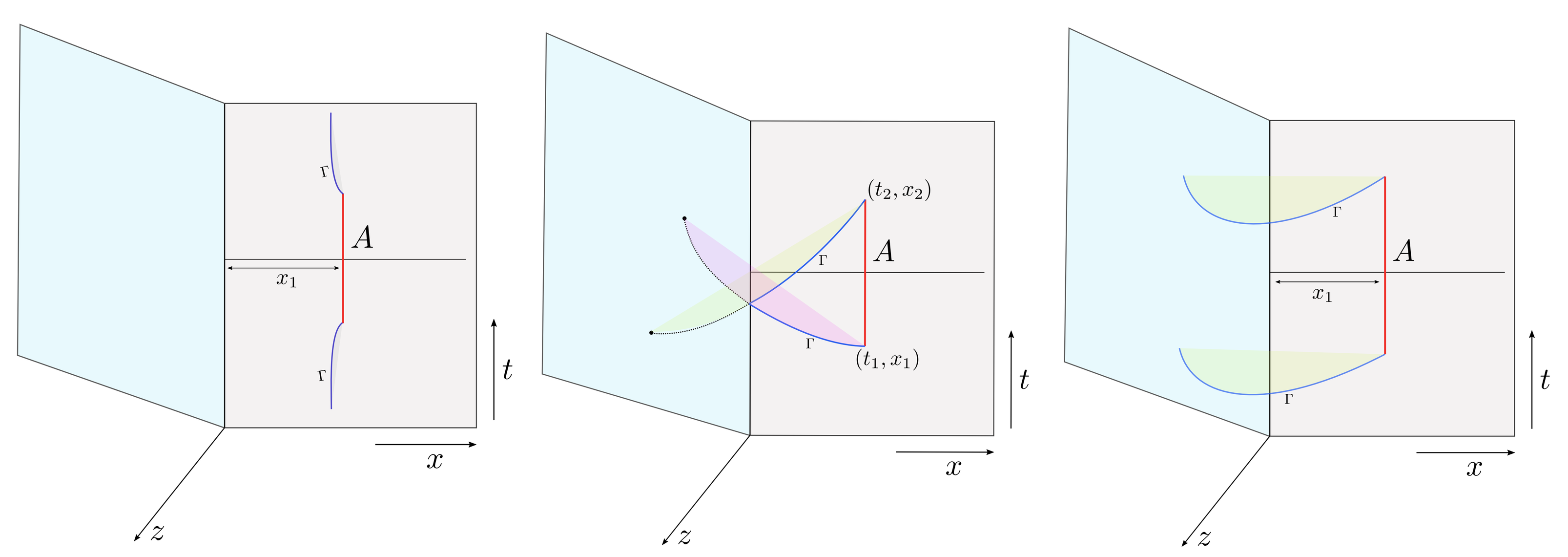

Figure 4. (a) RT surface for Bulk phase. (b) RT surface for Regge phase. (c) RT surface for boundary phase.

- The RT surface in the bulk phase consists of two space-like geodesics and one time-like geodesic. So, the holographic TEE in this phase is given by

$$S_A^T = \frac{L}{4G} = \frac{c}{3} \log \frac{T_0}{\epsilon} + \frac{i\pi c}{6}.$$

- In the Regge phase, the RT surface is given by two geodesics where each geodesic joins the one end point of the interval and ends on the plane perpendicular to the boundary.

$$S_A^T = \frac{c}{3} \log \left(\frac{2x_1}{\epsilon} \sqrt{2 - \frac{T_0}{x_1}} \right).$$

- The RT surface in the boundary phase ends on the EOW brane.

$$S_A^T = \frac{c}{3} \log \frac{2x_1}{\epsilon} + \frac{c}{3}\rho_0.$$

- The holographic TEE obtained using the RT formula matches exactly with the corresponding dual field theory results for each of the phases.

Summary

- We have obtained the time-like entanglement entropy for a pure time-like interval in the context of AdS$_3$/BCFT$_2$.
- We observed that the TEE of a time-like interval at zero temperature has three phases in contrast to the two phases of the standard entanglement entropy in a BCFT.
- The new Regge phase is unique to the time-like interval when one end point of the interval approaches the light cone of the mirror reflection of other end point.
- We also computed the TEE holographically and observed that it agrees precisely with the dual field theory results.
- Similarly, the TEE of an interval at finite temperature may also be obtained.

Reference