Classification of 2D topological phases with strict area law

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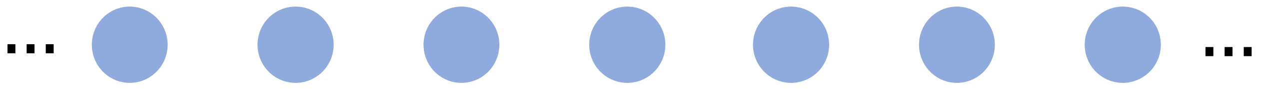
with Isaac Kim [UC Davis]
Speaker: Daniel Ranard [MIT]

What can a ground state look like?

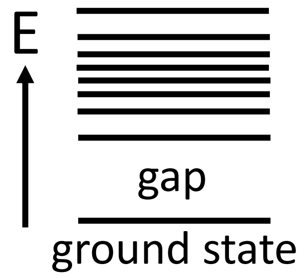
for a local lattice Hamiltonian

E.g. 1D spin chain.

$$H = \sum_i \sigma_i^z$$



$$|0\rangle \; |0\rangle \; |0\rangle \; |0\rangle \; |0\rangle \; |0\rangle \; |0\rangle$$



H is *gapped*: energy gap between ground state and next state.

All gapped 1D ground states “look the same”: like a product state.

Not true in higher dimensions!

Quantum phases of matter

Two gapped systems $(|\psi_1\rangle, H_1)$ and $(|\psi_2\rangle, H_2)$ are in **the same phase** iff

Equivalent conditions:

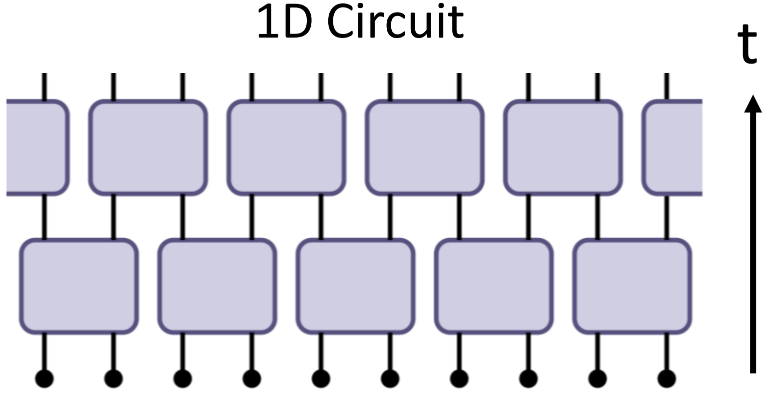
1. **Circuits:** $|\psi_1\rangle = U\,|\psi_2\rangle$ for a constant-depth, local unitary circuit U.
"Same entanglement structure."

2. **Deformations:** ∃ path of gapped Hamiltonians from H_1 to H_2.
"No phase transition."

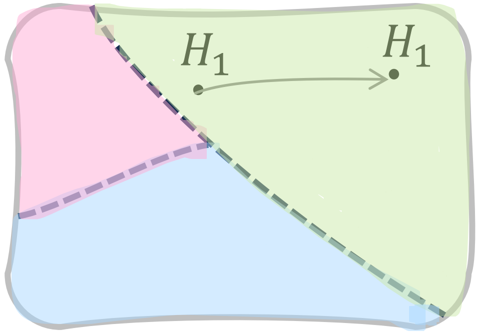
Phases are ***equivalence classes***.

We don't impose symmetry: "topological phases."

1D Circuit



Space of Hamiltonians



What can gapped ground states look like?

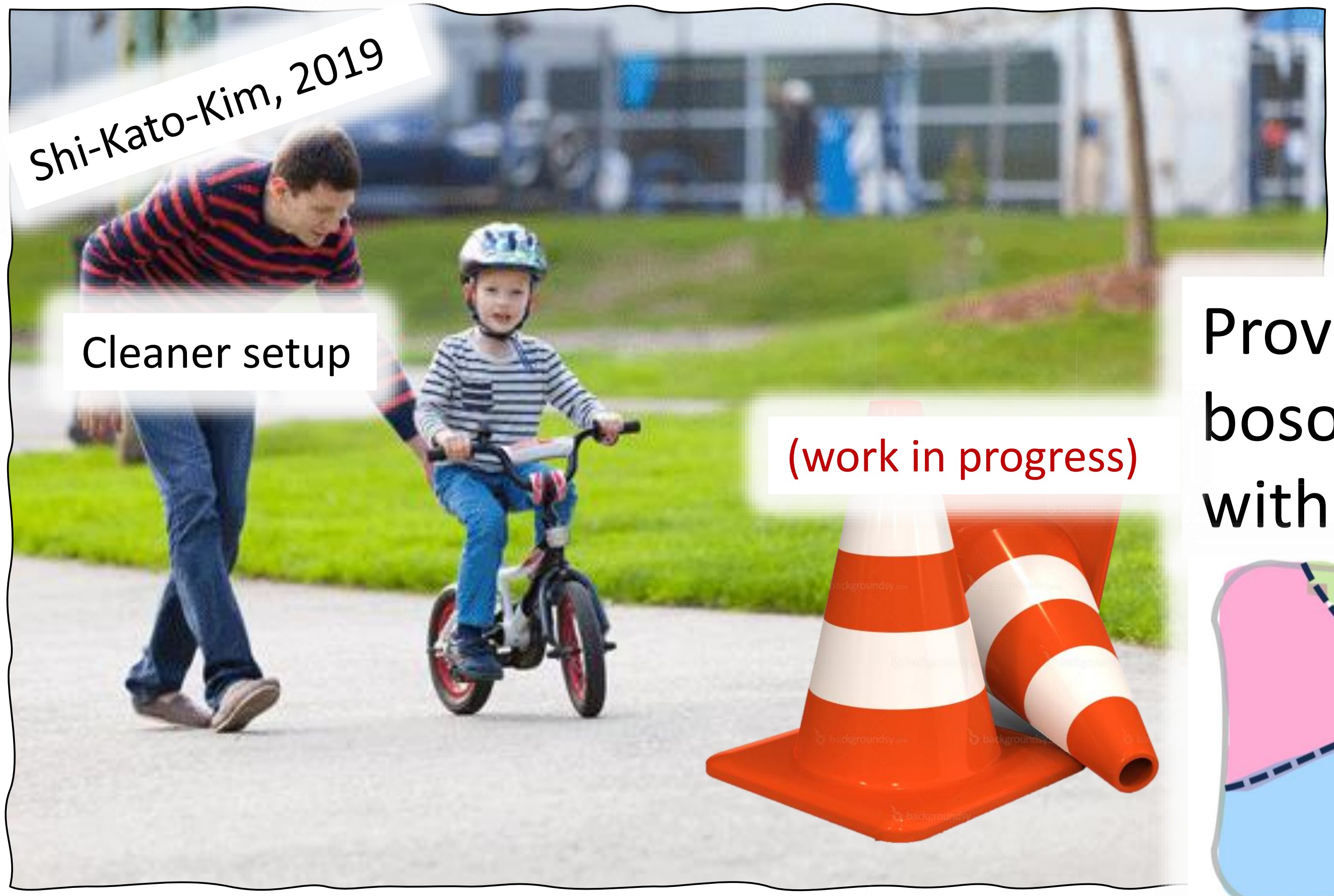
i.e.

Can we classify gapped quantum phases?

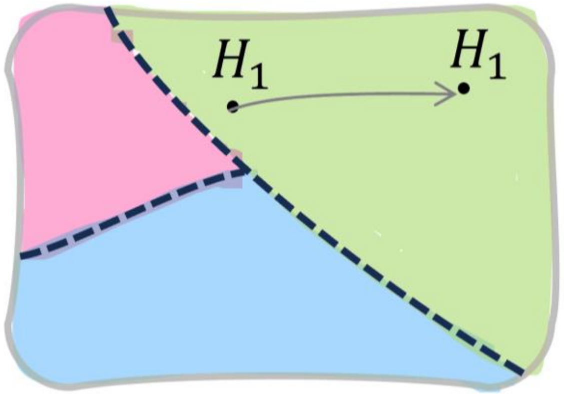
Much is well-understood.

Few solid arguments.

This work



Prove classification of 2D bosonic gapped phases with gapped boundary?



Summary

Classify 2D bosonic gapped ground states with gapped boundary?

Example of non-trivial phase: toric code. Or: string-net states.

Anyon content described by *tensor category.*

Conjectured classification:

States in the same phase ⇔ states have same *anyon content.*

1. (⇒) Define circuit-invariant anyon data. **(Most previous work.)**
2. (⇐) Show two states with same anyon data are connected by short circuit.

Baby version: trivial phase ⇔ no anyons.

Our training wheels: states with **strict area law**. "Entanglement bootstrap."

Key technical task: convert these states to string-nets with short circuits.

⇒ Full classification of 2D states with strict area law.

Previous work

Algebraic theory of anyons:
Levin-Wen, Kitaev, Kitaev-Kong.

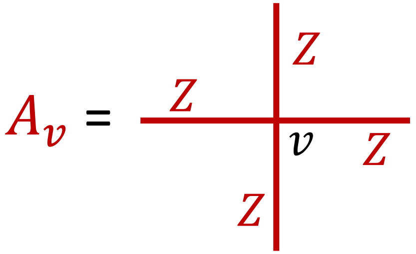
Circuit-invariant anyon data:
Haah, Kato-Naaijkens, Cha-Naaijkens-Nachtergaele, Ogata.

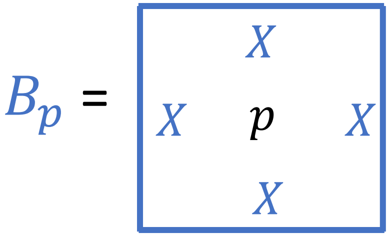
Entanglement bootstrap tools:
Shi-Kato-Kim, Shi-Kim.

Showing same anyon content implies same phase: little previous work?

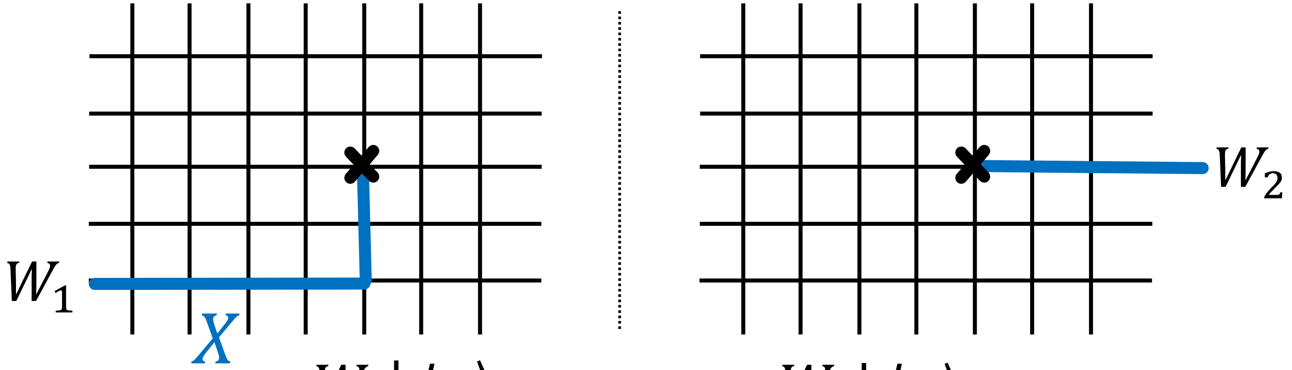
An example: the toric code

$$H = \sum_{\text{vertex } v} A_v + \sum_{\text{plaquette } p} B_p$$





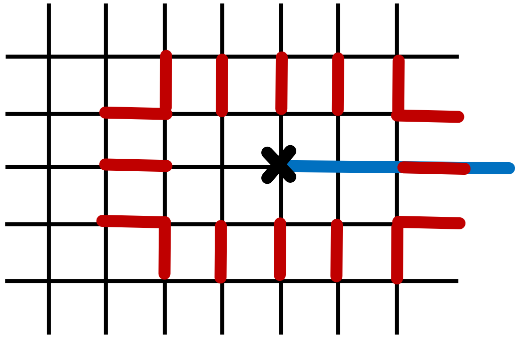
Qubit on each edge



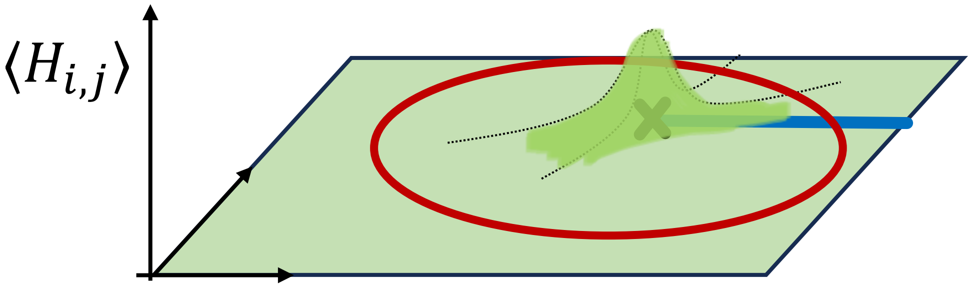
$W_1|\psi_0\rangle = W_2|\psi_0\rangle$

Deformable string operator

Unique gapped ground state $|\psi_0\rangle$ on plane.



Localized excitation



Excitation not *locally* detectable far away.
But *globally* detectable on distant annulus.

Warm-up: a coarse classification

Claim: trivial phase $\Leftrightarrow$ no anyons.

1. ($\Rightarrow$) O(1)-complexity (“trivial”) ground states cannot support anyons.
2. ($\Leftarrow$) States which cannot support anyons are O(1)-complexity.

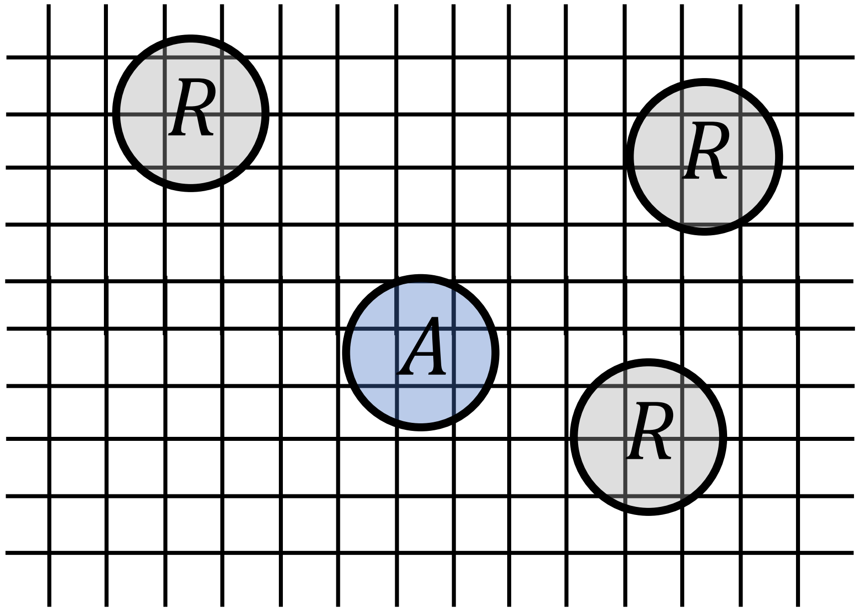
First, need better notion of anyons.

Want to ask whether a ground state “supports anyonic excitations,” without referring to a parent Hamiltonian.

What are anyons?

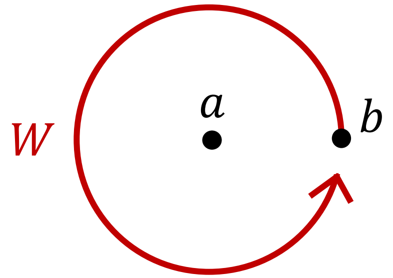
ρ_0= ground state
ρ_1 = state with single excitation
localized to A

"localized to A" means ρ_1 looks like ρ_0 locally outside A:
$\rho_0^R = \rho_1^R$ $\forall$ O(1)-sized disks $R \subset \bar{A}$.



When $\rho_1 = U_A \rho_0 U_A^{-1}$ for some U_A, we say:
the excitation can be "created locally."

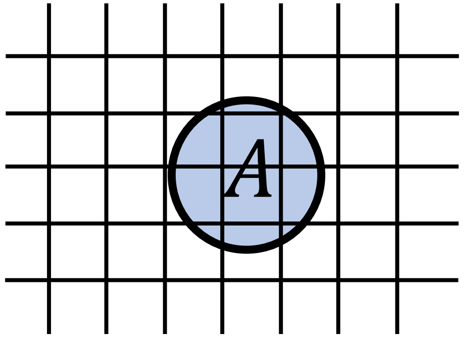
Braid excitation b around a with unitary W.
If a can be created locally using U_A then $[W, U_A] = 0$,
so applying W cannot produce a phase depending on presence of a.
⇒ anyons cannot be created locally!



Take this as a definition of an anyon, or "topological excitation":
a localized excitation that cannot be locally created.

Note we don't need to refer to a Hamiltonian. Just say: state ρ_0 supports a topological excitation ρ_1 if [...]

Warm-up: a coarse classification



Claim: trivial phase $\Leftrightarrow$ no anyons.

1. **($\Rightarrow$) O(1)-complexity ground states cannot support anyons.**
2. ($\Leftarrow$) States without anyons are O(1)-complexity.

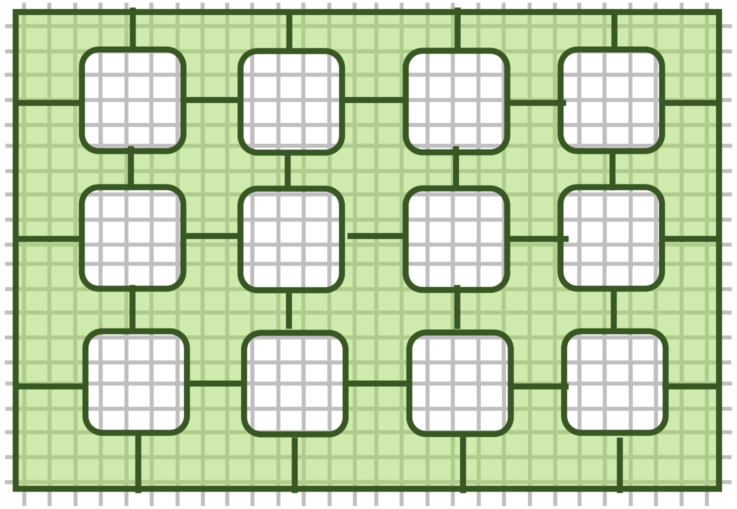
Proof of (1):

First argue *product state* $\rho_0 = |00\ldots\rangle\langle 00\ldots|$ cannot support anyons.
Let ρ_1 be state with single excitation w.r.t ρ_0, localized to A.
So ρ_1 looks like $|0\rangle\langle 0|$ *locally* outside A. But then ρ_1 matches ρ_0 *globally* outside A.
Then A and $\bar{A}$ disentangled, so $|\psi_1\rangle = U_A|\psi_0\rangle$ for some U_A. Excitation can be created locally.
Likewise, take arbitrary trivial state $|\psi_0\rangle = U|0\rangle^{\otimes n}$. Consider state $|\psi_1\rangle$ with excitation localized to A . Exercise: show $|\psi_1\rangle = U_{A_+}|\psi_0\rangle$ for some U_{A_+}.
So $|\psi_0\rangle$ does not support anyons.

Warm-up: a coarse classification

Claim: trivial phase ⇔ no anyons.

1. (⇒) O(1)-complexity ground states cannot support anyons.
2. **(⇐) States without anyons are O(1)-complexity.**



Proof of (2):

Ground state $|\psi_0\rangle$, does not support anyons.
Take coarse-grained green regions.
Coarse-grained Hamiltonian on green is 2-local.
2-local Hamiltonians have a trivial groundstate, $|\psi_1\rangle$.
(similar to 1D…) [Bravyi-Vyalyi.]
$|\psi_1\rangle$ is like $|\psi_0\rangle$ with some localized excitations.
$|\psi_0\rangle$ doesn't support anyons, so* excitations of $|\psi_1\rangle$ can be removed with local unitaries.

Then $|\psi_0\rangle$ is trivial state.

*something swept under rug

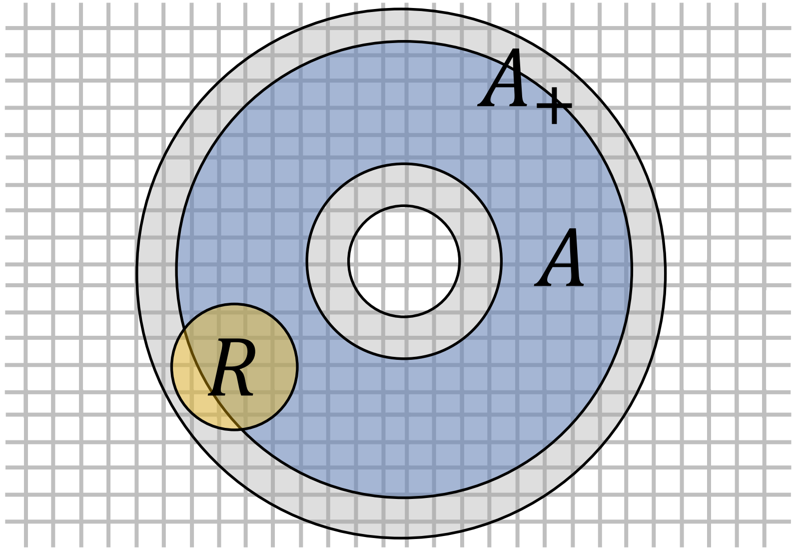
A more sophisticated approach to anyons

Information convex set

ρ = ground state

Consider **all** states σ_{A_+} on A_+ that "locally look the same as ρ," i.e look the same as ρ on all O(1)-sized disks $R \subset A_+$, $\mathrm{Tr}_{\bar{R}}\sigma_{A_+} = \mathrm{Tr}_{\bar{R}}\rho$.
This set of states is convex.

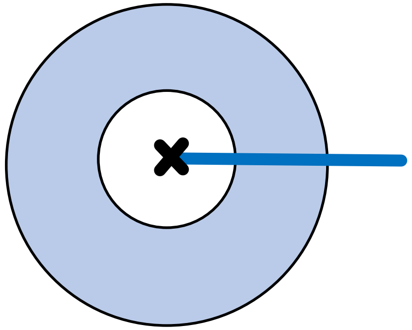
Annulus $A \subset A_+$.



Erase $A_+ \setminus A$ to obtain some convex set of states on A.
Called the **information convex set** of A, denoted $\Sigma(A)$.

For ρ = toric code ground state,
$\Sigma(A) = \mathrm{conv}\{\rho_A, \rho_A^e, \rho_A^m, \rho_A^{em}\}$.

Recovered list of anyons from the ground state!



The entanglement bootstrap program

Shi, Kato, & Kim, 2019.

Recovers anyon data from the ground state.

Given state, they obtained: list of anyons, fusion rules, string operators.

Using methods from Levin-Kawagoe 2019, we also can recover the full braided tensor category.

Lots of power, at a price:
Requires some assumptions about the state.

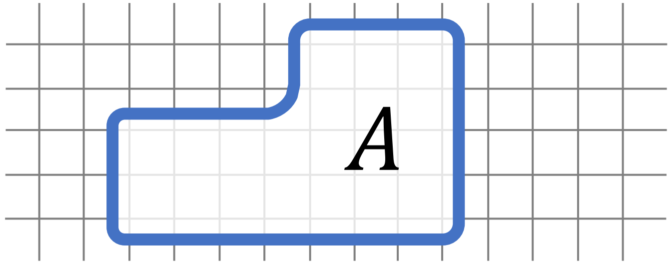
What assumptions?

Strict area law

Sufficient to assume entanglement entropy obeys **strict area law**,

$$S_A = \alpha|\partial A| + \gamma,$$

for some constants α, γ, for all regions A.



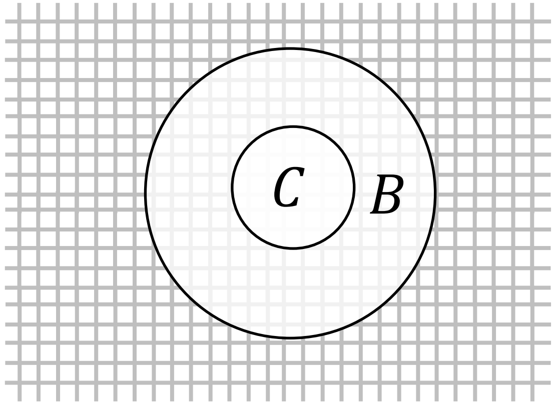
Unnatural for non-translation-invariant states.
Empirically: Approximately true for translation-invariant, gapped states.

Imagine coarse-graining lattice first, so it holds with high precision.

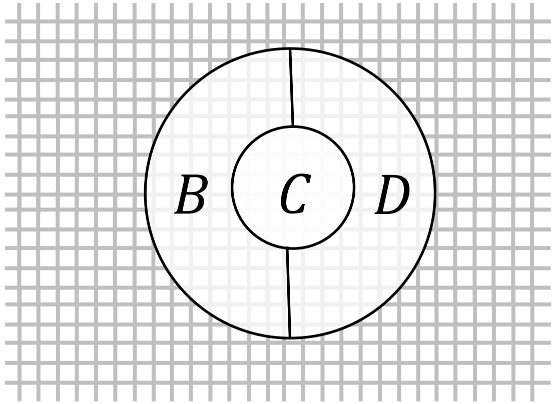
Entanglement bootstrap (EB) axioms

Also sufficient to assume the two entropy equalities below, for all O(1)-sized regions that have these topologies.

$$S(C) + S(BC) - S(B) = 0$$



$$S(BC) + S(CD) - S(B) - S(D) = 0$$



Imagine coarse-graining lattice first, so regions are O(1) but large.

Equalities follow from strict area law, but do not require it.

They're "empirically" true to high accuracy.

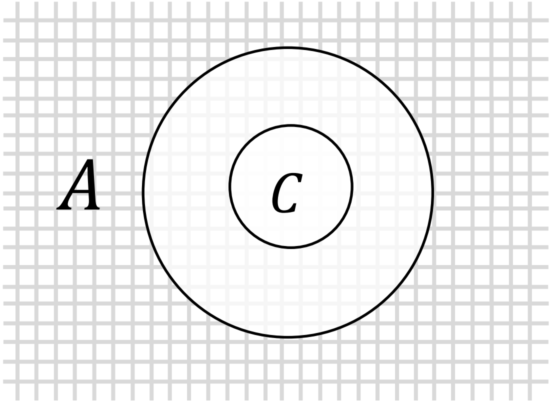
Entanglement bootstrap axioms

Equivalent to assuming:

(1) No long-range mutual information, &

(2) No long-range *conditional* mutual information.

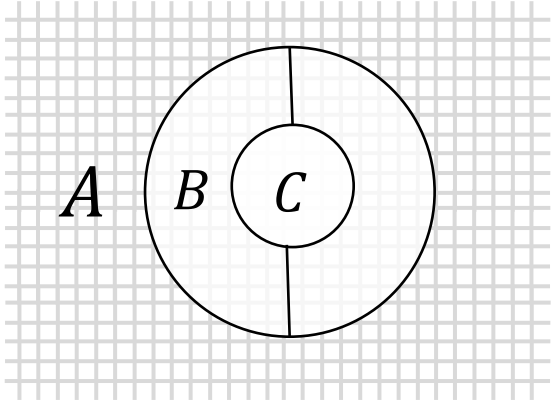
$I(A\!:\!C) = 0$



These states:

- Have zero correlation length
- Have gapped parent Hamiltonians
- Approximately encompass all gapped ground states after sufficient coarse-graining?

$I(A\!:\!C|B) = 0$



"Gapped boundary"

Given a gapped Hamiltonian on the plane, if you place it on a disk with boundary, the system may become gapless.

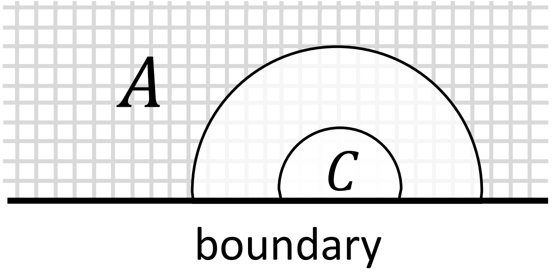
Many possible choices of Hamiltonian terms used at the boundary. Sometimes, *no* choice of boundary terms maintains the gap.

We only consider systems that have "gapped boundary."

This restriction is baked into our assumptions:

We assume the state still satisfies the EB conditions along the boundary.

(Shi & Kim, 2021)



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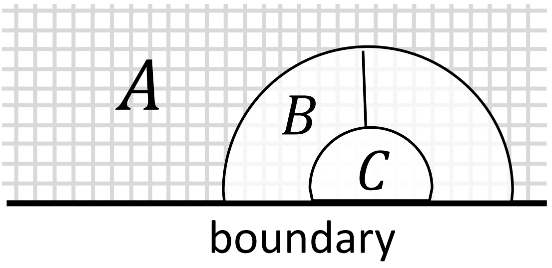
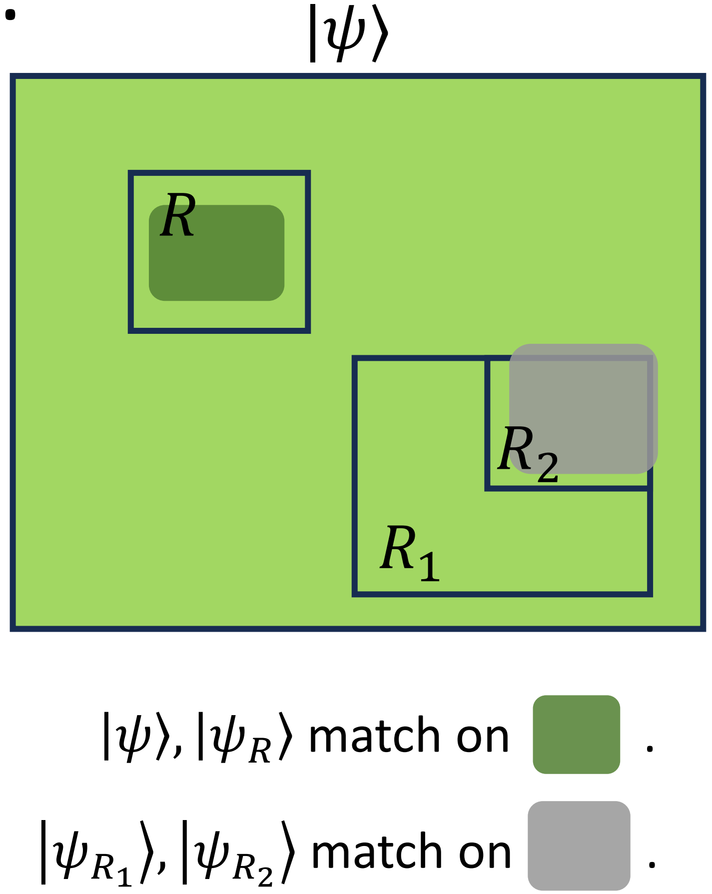


$I(A:C|B) = 0$

Our setup: a "nice" gapped state.



Given pure state $|\psi\rangle$ on large disk-like region with boundary.

Assume:

1. EB conditions hold (zero MI, zero CMI).
2. For every disk-like* subregion R, there exists $|\psi_R\rangle$, with boundary, satisfying EB conditions, that matches $|\psi\rangle$ away from ∂R.
3. These $|\psi_R\rangle$ match each other wherever they overlap (including along their boundaries).

(Does not require translation-invariance!)

We suggest these assumptions are reasonable for a gapped Hamiltonian with gapped boundary terms that can be placed anywhere on the lattice. Do you agree?

Our results

Call states from previous slide “nice” 2D gapped states with gapped boundary.

Core result:
A “nice” state can be transformed to a string-net by a constant-depth unitary circuit.

Corollary:
Two nice states can be mapped to each other in the bulk iff they have the same bulk anyon data (same UMTC).
(Uses Lootens et al. result about Morita-equivalent string-nets.)

Corollary:
Assume any 2D gapped groundstate with gappable boundary can be mapped to a nice state by quasi-local flow. Then the equivalence classes of 2D gapped phases with gappable boundary are given by the doubled UMTCs.
(Uses Ogata’s result about invariance of anyon data under quasi-local flow.)

Background for proof technique

Information convex set

ρ = ground state

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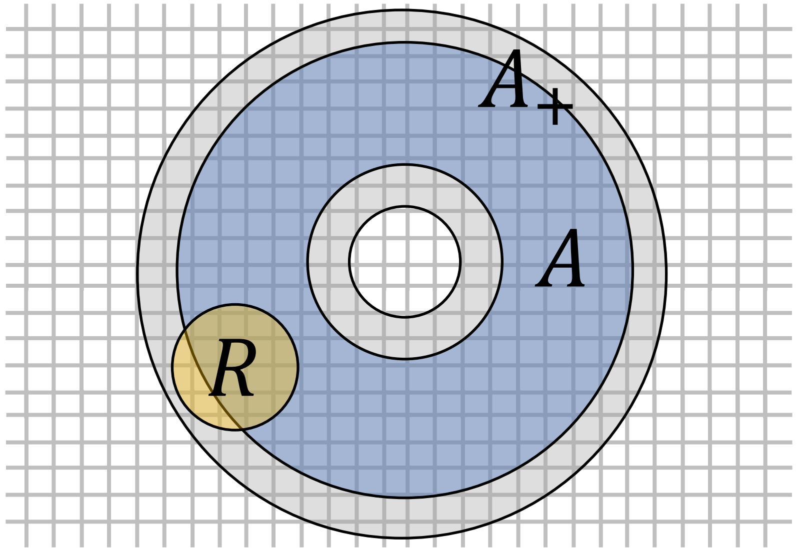

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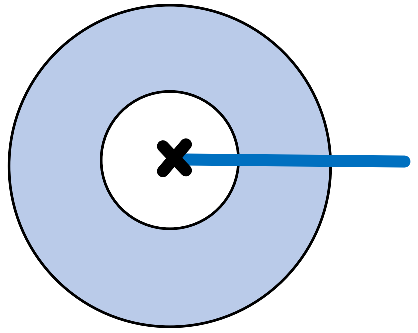
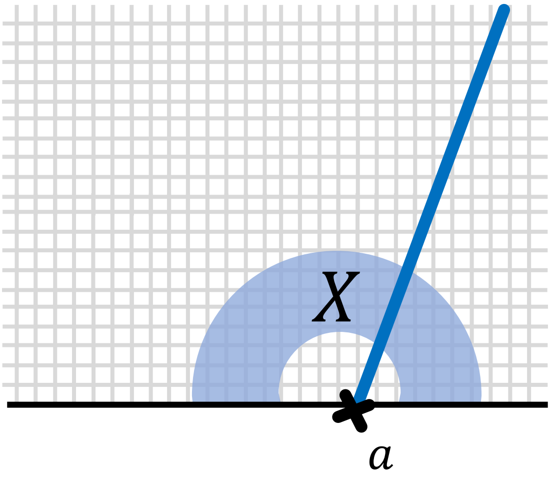


Boundary anyons



Can define information convex set for any region.

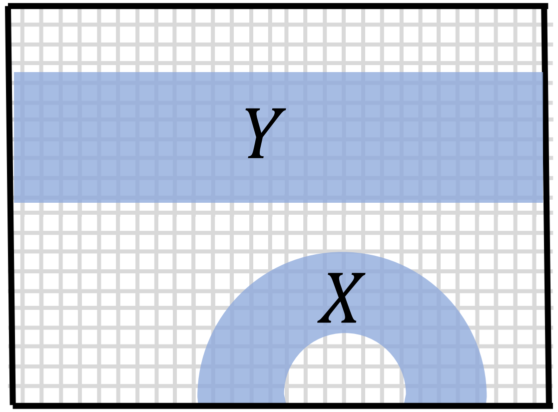
For boundary half-annulus X,

$$\Sigma(X) = \mathrm{conv}\{\rho_A^a\}_a$$

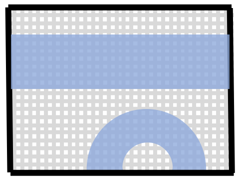
with one sector for each anyon type a that lives at the boundary.

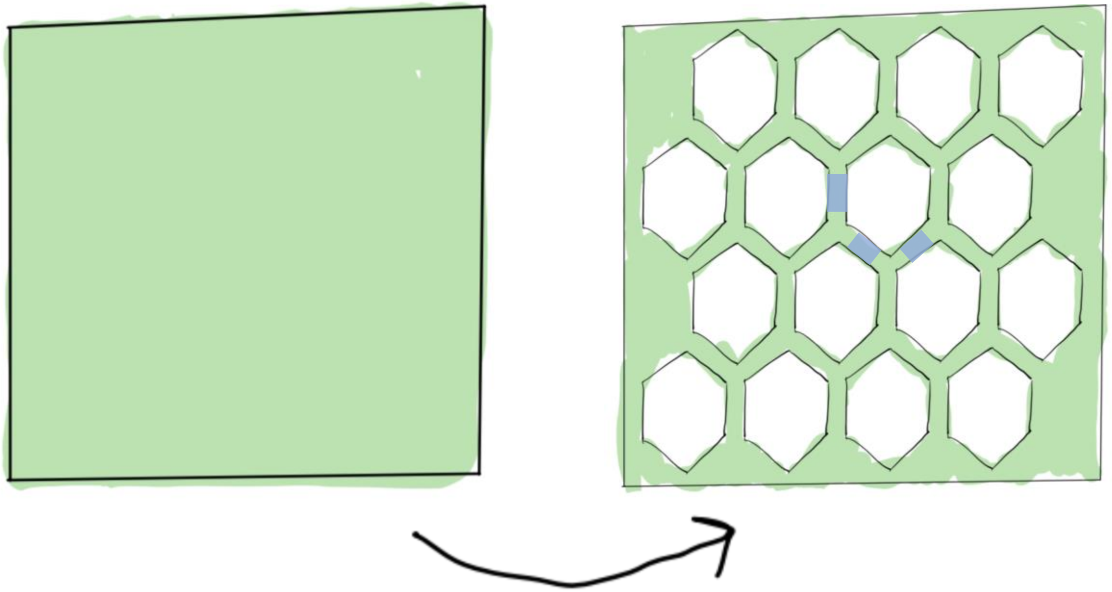
Structure of IC only depends on *topology* of region.

E.g. $\Sigma(X) \cong \Sigma(Y)$.



Proof idea





Apply depth-1 circuit to "punch holes" in the bulk. The holes have gapped boundary.

What does state look like on a blue region X ?

Locally looks like ground state, so it's in $\Sigma(X)$.
⇒ it's a mixture of orthogonal sectors labeled by boundary anyon types.

Consider blue region as single qudit, with basis labeled by boundary anyon types.

Looks like string-net on hexagonal lattice!
Edge DOF labeled by boundary fusion category.

We coarse-grain, locally disentangle. Then show there exists a parent Hamiltonian identical to string-net parent Hamiltonian.

Outlook

The classification of 2D bosonic gapped phases with gapped boundary appears to follow the conjecture!

Was it pre-ordained? Maybe not, otherwise the proof would have been easier. (This wasn't even the "ϵ, δ" version.)
Or give me a simpler argument!

One non-trivial "check": we obtained generalized string-nets, not the original (slightly more restrictive) string nets.

Next:
Let's do nonzero correlation length, higher dimensions, symmetries, etc.

Thank you!