Thermodynamics of Near-Extremal Rotating black holes

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MOTIVATION

- ✴ The conventional picture of (non-supersymmetric) extremal black holes, viz., one where there is non-trivial degeneracy at zero temperature, is somewhat curious.

- ✴ Not only does this naively violate the third law of thermodynamics, it also poses challenges for how near-extremal black holes Hawking radiate.

$$E = E_0 + \frac{T^2}{T_{\text{gap}}}\,, \qquad S = S_0 + \frac{T}{T_{\text{gap}}}\, S_1$$

- ✴ The fact that energy departures from extremality are quadratic, has important implications: at temperatures below the gap, the black hole is unable to decharge by emitting even a single Hawking quantum. Preskill, Schwarz, Shapere, Trivedi, Wilczek '91

- ✴ Various attempts have been made to address, eg., black hole pair production, attractor mechanism, etc. Hawking, Horowitz, Ross '94 Dabholkar, Sen, Trivedi '06

- ✴ The modern understanding of this situation is somewhat prosaic: the non-trivial degeneracy is illusory, and in fact near-extremal black holes have a vanishingly small degeneracy. They behave like a conventional quantum mechanical system with few low-lying excitations. Ghosh, Maxfield, Turiaci, '19 Iliesiu, Turiaci '20

MOTIVATION

* The essential point is that the semiclassical analysis needs to carried out with care. While the black hole is a dominant saddle, fluctuations around it are important.

$$\mathcal{Z} \sim \det_0 e^{I_0} + \det_1 e^{I_1} + \cdots$$

* The modification in the low temperature thermodynamics arises from the presence of zero modes of the extremal geometry localized in the near-horizon region.

$$\mathcal{Z} \sim T^{\frac{3}{2}} \exp\left(\underbrace{S_0 + S_1 \frac{T}{T_{\rm gap}}} + c \log S_0 \right)$$

1-loop det of gapless modes

classical result

1-loop corrections from gapped and gapless modes

Sen '11-'12

Iliesiu, Murthy, Turiaci '22

* The zero mode contribution can be nicely isolated by examining how they get gapped in the near-extremal solution.

PLAN OF THE TALK

- ✴ The fastest way to derive this picture is to appeal to the enhanced SL(2,R) symmetry of the near-horizon region

  Kunduri, Lucietti, Reall '07

- ✴ Dimensional reduction to this AdS_2 spacetime gives and effective JT gravity description, where quantum effects can be understood.

  Moitra, Trivedi, Vishal '18

- ✴ We will try to understand this in two examples, both involving rotation, motivated by two different questions:

- ✤ The dimensional reduction is not quite natural in the case of rotating black holes, where the AdS_2 factor is typically warped and fibered over some compact base space. Would therefore be preferable to test the ideas directly without reduction. Upshot: mostly works, but poses a puzzle regarding rotational zero modes.

  Castro, Larsen '09 Moitra, Trivedi, Sake, Vishal '19

- ✤ The effects of the zero modes contributing to the 1-loop determinant should be a robust prediction of any sensible theory of quantum gravity. How does it work in string theory (viz., account for finite string length effects).
  Upshot: one calculable example to demonstrate the universal low temperature thermodynamics.

Near-extremal Kerr and its entropy

Ilija Rakic, MR, Joaquin Turiaci

EUCLIDEAN QUANTUM GRAVITY: SADDLE ANALYSIS

* Focus on gravity with vanishing cosmological constant and understand the general computation of one-loop fluctuations around a classical saddle.

$$S_{\text{grav}} = -\frac{1}{16\pi G_N}\int d^dx\,\sqrt{g}\,R + S_{\text{bdy}} + S_{\text{GF}}$$

* Saddle analysis proceeds as usual: we find solutions to Einstein's equations with given boundary conditions and evaluate the on-shell action.

* Fluctuations around the background: pick suitable gauge fixing term (harmonic gauge) to simplify the quadratic fluctuation operator. For spin-2 fields we have the Lichnerowicz operator

$$S^{(2)}_{\text{grav}} = \frac{1}{16\pi G_N}\int d^dx\,\sqrt{g}\,\frac{1}{4}\,\tilde{h}^{\mu\nu}\Delta_L h_{\mu\nu}\,, \qquad \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}\;h^\rho_\rho$$

$$\Delta_L h_{\mu\nu} = -\nabla_\rho\nabla^\rho h_{\mu\nu} - 2\,R_{\mu\rho\nu\sigma}\,h^{\rho\sigma}$$

* Construct eigenbasis of Lichnerowicz operator (may have zero modes) and compute the determinant.

EVALUATING THE ONE-LOOP DETERMINANT

* Evaluation of the determinant follows from the eigenbasis using standard heat-kernel techniques.

$$K^{ij}(x,x';s) = \sum_n e^{-\lambda_n s} f_n^{(i)}(x)\, \bar{f}_n^{(j)}(x')$$

quantum propagator

$$\log Z_{\text{1-loop}} = \frac{1}{2}\int_{\Lambda_{\text{UV}}}^{\infty} \frac{ds}{s} K(s)\,, \qquad K(s) = \int d^d x\, \sqrt{g}\, K^{ii}(x,x,s)$$

UV cut-off

* K(s) has a Laurent expansion with the polar terms capturing the UV divergences, which have to be countertermed away.

* Physical information in the constant piece, which is present in even dimensions. It can be computed directly using the conformal anomaly without recourse to details about the actual eigenbasis.

* Assuming the black hole saddle configurations have a single macroscopic length scale (eg., horizon size) gives the one-loop correction to black hole entropy.

* *Ensemble matters*: Legendre transform between ensembles with macroscopic parameters also gives logarithmic corrections.

Sen '11-'12

KERR ENTROPY

- For the non-extreme Kerr black hole, one evaluates the conformal anomaly, which gives the result in the grand canonical ensemble. Transforming to the canonical ensemble (fixed angular momentum) one finds for pure gravity: Sen '12

$$S = \frac{A}{4\,G_N} + \frac{334}{180}\log\frac{A}{4\,G_N}$$

- Extremal Kerr: zoom into the near-horizon where the canonical ensemble is natural. Accounting for the fact that there are 4 zero modes in the geometry:

$$S = S_0 + \frac{64}{180}\log S_0$$

Sen '11

- Near-extremal Kerr: account for the quantum dynamics of the zero modes (TBD) leading to:

$$S = S_0 + \frac{154}{180}\log S_0 + 4\pi^2\frac{T}{T_q} + \frac{3}{2}\log\frac{T}{T_q}$$

$$S_0 = 2\pi\, J$$

$$T_q = \frac{\pi}{\sqrt{G_N}\, J^{\frac{3}{2}}}$$

- Result valid for low temperatures, but not exponentially low (eg., when other saddles start to dominate).

Rakic, MR, Turiaci /wip

NHEK: SCHWARZIAN ZERO MODES

- Near-horizon geometry of extreme Kerr: Bardeen, Horowitz '99

$$ds_0^2 = J^2\,(1+\cos^2\theta)\left[(y^2-1)\,dt^2 + \frac{dy^2}{y^2-1} + d\theta^2\right] + \frac{4\,J^2\,\sin^2\theta}{1+\cos^2\theta}\,(d\varphi + i\,(y-1)\,dt)^2$$

- Natural boundary conditions: freeze angular momentum.
- Expectation for zero modes: 3 come from near AdS_2 or the Schwarzian dynamics (breaking of SL(2)) and rotation should give one.
- One can look for the modes by solving for metric fluctuations generated by large diffeomorphisms that are annihilated by the Lichnerowicz operator:

$$\xi_{\rm Sch} = \partial_t\Phi\,\frac{\partial}{\partial y} - \partial_y\Phi\,\frac{\partial}{\partial t} + i\,((y-1)\,\partial_y - 1)\,\Phi\,\frac{\partial}{\partial\varphi} \qquad \Phi = (n+y)\left(\frac{y-1}{y+1}\right)^{\frac{n}{2}} e^{i\,n\,t}$$

$$\Delta_L \nabla_{(a}\xi^{\rm Sch}_{b)} = 0$$

- Normalize the zero modes with the ultralocal measure on the space of fluctuations.

REGULATED ACTION

* To compute the determinant, one regulates the zero modes by consider a small departure from extremality.

* Focus on the near-horizon of a near-extremal black hole, including the zero modes:

$$g_{\mu\nu} = g^{0}_{\mu\nu} + T\, g^{1}_{\mu\nu} + \epsilon\, h^{\rm Sch}_{\mu\nu}$$

* The on-shell action gives a thermally regulated action for the zero modes:

$$I_{\rm EH} = 2\pi\, J + 4\pi^2\, \frac{T}{T_q} - 2\pi^2\, \frac{T}{T_q} \sum_{|n|>1} \frac{n}{8\pi}\, |\epsilon_n|^2$$

* Including the conformal anomaly contribution, and accounting for the logarithmic terms we end up with the quoted result:

$$S = S_0 + \frac{154}{180}\, \log S_0 + 4\pi^2\, \frac{T}{T_q} + \frac{3}{2}\, \log \frac{T}{T_q}$$

ROTATIONAL ZERO MODE: A PUZZLE

* The near-horizon region was expected to have a rotational zero mode as well.
* Intuition: dimensional reduction gives an Abelian gauge field which has a large gauge transformation by a function

$$\Lambda = e^{i\,n\,t}\left(\frac{y-1}{y+1}\right)^{\frac{n}{2}}$$

* Working directly in four dimensions we find that this does not generate a zero mode for the spin-2 Lichnerowicz operator.

$$\xi_{\text{rot}} = \Lambda\,\frac{\partial}{\partial\varphi} \qquad\qquad \Delta_L \nabla_{(a}\xi^{\text{rot}}_{b)} \neq 0$$

* Issue: gauge condition not being respected. Can prove that there is no non-singular gauge transformation that brings us to the harmonic gauge.
* There do not *appear* to be zero modes which are generated by large diffeos.
* In the canonical ensemble, the absence of this mode affects the log(Area) term, but not the temperature correction.

ROTATIONAL ZERO MODE: A PUZZLE

* We can try to benchmark the calculation by looking at other examples: near-extremal rotating BTZ is a simple example to study.

$$ds_0^2 = (y^2-1)\,dt^2 + \frac{dy^2}{y^2-1} + 4\,r_0^2\left(d\varphi + i\,\frac{y-1}{r_0}\,dt\right)^2$$

* Once again the Schwarzian zero modes are easy to find in this geometry, but there appears to be no rotational zero mode.

* In fact, as we will argue shortly, the absence is consistent with the microscopic picture, for either the canonical or the grand canonical ensemble.

* This would have seemed definitive, but for the fact that the matching of the semiclassical calculation with microscopic data in the BMPV black hole depends on the existence of the rotational zero mode:

$$ds_0^2 = (y^2-1)\,dt^2 + \frac{dy^2}{y^2-1} + d\psi^2 + \sin^2\psi\,d\chi^2 + \cosh^2\alpha\,(d\varphi + \cos\psi\,d\chi - i\,\tanh\alpha\,(y-1)\,dt)^2$$

* Need 7 zero modes: 3 Schwarzian, 3 from the two-sphere, and one rotational.

Seeking the Schwarzian in the string

Christian Ferko, Sameer Murthy, MR

LOW TEMPERATURE, HIGH SPIN UNIVERSALITY

- Asymptotic high spin density of states in a 2d CFT has a nice universal limit that can be recognized as the Schwarzian contribution.

Ghosh, Maxfield, Turiaci, '19

- Consider a 2d CFT with Virasoro symmetry (no conserved currents), with a modular invariant partition function that has a character decomposition:

$$Z_{\rm CFT}(\tau,\bar\tau) = \chi_{\rm vac}(\tau)\,\overline{\chi}_{\rm vac}(\bar\tau) + \sum_{h,\bar h} \chi_h(\tau)\,\overline{\chi}_{\bar h}(\bar\tau)$$

$$\chi_{\rm vac}(\tau) = q^{-\frac{c-1}{24}}\,\frac{1-q}{\eta(\tau)}\,, \qquad \chi_h(\tau) = q^{h-\frac{c-1}{24}}\,\frac{1}{\eta(\tau)}\,, \qquad q = e^{2\pi i\tau}$$

- The limit of interest is low temperatures and fixed angular momentum:

$$T \sim \mathcal{O}\big(c^{-1}\big)\,, \qquad J \sim \mathcal{O}\big(c^{3}\big)\,, \qquad c \gg 1$$

$$Z(T,J) \sim \left(\frac{\pi\, c}{12}\,T\right)^{\frac{3}{2}}\, J^{-\frac{3}{4}}\, \exp\left[2\pi\sqrt{\frac{c}{6}}\,J - \beta\left(J - \frac{1}{12}\right) + \frac{\pi\, c}{12}\,T\right]$$

- The temperature dependence is the same in the grand canonical ensemble.

LOW TEMPERATURE, HIGH SPIN UNIVERSALITY

- Result follows using modular invariance and the nature of the vacuum character.
- Original result inspired by BTZ black hole thermodynamics, but can be argued to hold in an extended domain. Pal, Qiao '23
- The semiclassical computation proceeds as in the near-extremal Kerr case.
- The fact that temperature dependence is unchanged between canonical and grand canonical ensembles is consistent with there being a Schwarzian zero mode, but no rotational zero mode in the near-horizon geometry.
- Alternately, one can start with the gravitational action in AdS_3 to compute the thermal graviton partition function, which gives the Virasoro vacuum character, and use modular properties to extract the BTZ result Giombi, Maloney, Yin '08
- We will follow this route for the string, working directly with the thermal AdS_3 geometry for simplicity.

STRINGS IN AdS_3

* Focus on bosonic string theory with target space AdS_3 x X.

* The AdS_3 part with pure NS-NS flux is a SL(2,R) WZW model, and we will simply represent X by a unitary CFT of the appropriate central charge. Giveon, Kutasov, Seiberg '98

de Boer, Ooguri, et al '98

Maldacena, Ooguri '00

$$k = \frac{\ell_{\mathrm{AdS}}}{\ell_s} \qquad\qquad \frac{3\,k}{k-2} + c_X = 26$$

* We will for the most part stick to the semiclassical limit (large *k*); the worldsheet theory is expected to be dual to a spacetime CFT, though one with somewhat strange properties.

* There are some special features at small *k*, where there has been recent progress:

  ✦ The spacetime CFT dual to the bosonic theory at *k=3* has an infinite tower of higher spin states in its spectrum. Gaberdiel, Gopakumar, Hull '18

  ✦ The superstring theory at *k =1* is special; it has been argued to be dual to the symmetric product orbifold. Eberhardt, Gaberdiel, Gopakumar '18

STRING SPECTRUM IN AdS_3

- The physical states of the string comprise of a discretuum, a continuum, and their spectral flows (for the SL(2,R) WZW model).

Maldacena, Ooguri '00

- In the sector without spectral flow one has the tachyon (from the continuous representation) and set of discrete states with spacetime CFT dimension

$$h_{\rm CFT} = \frac{1}{2} + \sqrt{\underbrace{\frac{1}{4} + (k-2)\,(N + h_X - 1)}_{\in \frac{(k-2)^2}{4}[0,1]}} + \mathsf{n}\,, \qquad \mathsf{n} \geq -N$$

unitarity constraint

- In the spectrally flowed sector, the discretuum and continuum take the form:

discretuum

$$h_{\rm CFT} = \frac{1}{2} + w + \sqrt{\underbrace{\frac{1}{4} + (k-2)\,(N + h_X - w\,\mathsf{n} - \frac{w\,(w+1)}{2} - 1)}_{\in \frac{(k-2)^2}{4}[w^2, (w+1)^2]}} + \mathsf{n}$$

continuum

$$h_{\rm CFT} = \frac{w\,k}{4} + \frac{1}{w}\left(\frac{s^2 + \frac{1}{4}}{k-2} + N + h_X - 1\right), \qquad s \in \mathbb{R}$$

THERMAL ADS$_3$ STRING PARTITION FUNCTION

- From the string spectrum the spacetime CFT satisfies the assumptions under which the large spin, low temperature asymptotics were derived.
- Useful to examine how this works directly from the string partition function, and examine whether the behaviour persists for finite string tension, and further see how the behaviour changes as we approach the aforementioned special points.

$$\mathcal{Z}_{ws}(\beta,\mu) = -\beta\, F(\beta,\mu) = \frac{\beta\sqrt{k-2}}{8\pi}\int_0^\infty \frac{d\mathfrak{t}_2}{\mathfrak{t}_2^{\frac{3}{2}}}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\mathfrak{t}_1\, e^{4\pi\, a_k\, \mathfrak{t}_2}\, \mathcal{Z}_X(\mathfrak{t})\, \mathcal{Z}_{ads}(\mathfrak{t})$$

$$\mathcal{Z}_{ads} = \sum_{m=1}^{\infty} \frac{e^{-\frac{(k-2)\, m^2\, \beta^2}{4\pi\, \mathfrak{t}_2}}}{\left|\sinh\frac{m\boldsymbol{\beta}}{2}\right|^2}\left|\prod_{n=1}^{\infty}\frac{1-\mathfrak{z}^n}{(1-e^{m\boldsymbol{\beta}}\,\mathfrak{z}^n)\,(1-e^{-m\boldsymbol{\beta}}\,\mathfrak{z}^n)}\right|^2$$

$$\mathfrak{z} = e^{2\pi i \mathfrak{t}} \qquad \boldsymbol{\beta} = 2\pi i\,\overline{\tau} \qquad a_k = 1 - \frac{1}{4\,(k-2)}$$

worldsheet modulus — spacetime CFT data — vacuum energy

- Note: Contribution from a single geometry, and isn't likely to give a spacetime modular invariant partition function.
- But it should suffice to allow us extract the asymptotics, assuming it can be modular completed (as for the thermal graviton partition sum)

Maldacena, Ooguri, Son '00

A SPACETIME EXPRESSION FROM THE STRING

- Recognizing the free energy expression, as resulting from multiparticling a single string free energy, it is useful to first decompose the result for the single string free energy, and then reassemble (ellipsis denote other geometries).

$$Z_{\rm CFT}(\tau,\overline{\tau}) = \exp(-\beta\, F + \cdots) = \exp\left(-\beta \sum_{m=1}^{\infty} f(m\,\beta,\mu) + \cdots\right)$$

$$f(\beta,\mu) = \frac{1}{\beta} \sum_{\mathcal{H}_{\text{single-string}}} e^{-\beta\, E - i\,\beta\,\mu\,\ell}$$

- Carrying out the worldsheet modular integral one can re-express the string partition function in terms of the CFT characters, resulting in

$$\beta\, f(\beta,\mu) = \left(1+\frac{q^2}{1-q}\right)\left(1+\frac{\overline{q}^2}{1-\overline{q}}\right) + \frac{1+q^4}{1-q}\,\frac{1+\overline{q}^4}{1-\overline{q}}\,(q\overline{q})^{-\frac{3}{2}+\sqrt{(k-2)+\frac{1}{4}}} + \cdots$$

- The first term is the building block of the Virasoro vacuum character: the single string spectrum only has states where a single spacetime Virasoro raising operator hits the vacuum. This part already captures the piece we seek.

- We have elided over the higher excitations and the spectrally flowed states (and dropped the tachyon divergence).

SUMMARY & OPEN QUESTIONS

- ✴ Thermodynamics of near-extremal black holes is akin to low temperature quantum systems with low degeneracy.
- ✴ Results have been derived in the context of semiclassical quantum gravity, but also can be motivated from in semiclassical string theory.

Open Questions:

- ✦ Discrepancy of the zero modes in near-horizon for warped, fibered AdS_2 geometries.
- ✦ Generalizations to other rotating geometries: can one leverage systematic study of (extremal) near-horizon spacetimes to construct near-horizon near-extremal geometries.

  cf., Kunduri, Lucietti

- ✦ Understand the dual of the string for general *k, cf.,* proposal as symmetric orbifold of Liouville times internal CFT. In particular, how does the CFT partition function get expressed as a sum of holomorphic and anti-holomorphic characters?

  Gaberdiel, Eberhardt ’19; Eberhardt ’21

- ✴ Analysis of string propagation on asymptotically AdS_2 geometries…

Thank You!