Classical sampling algorithms for estimating quantum computing resources for bosonic systems

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[Polchinski, arxiv:1010.6134]

Matrix Quantum Mechanics

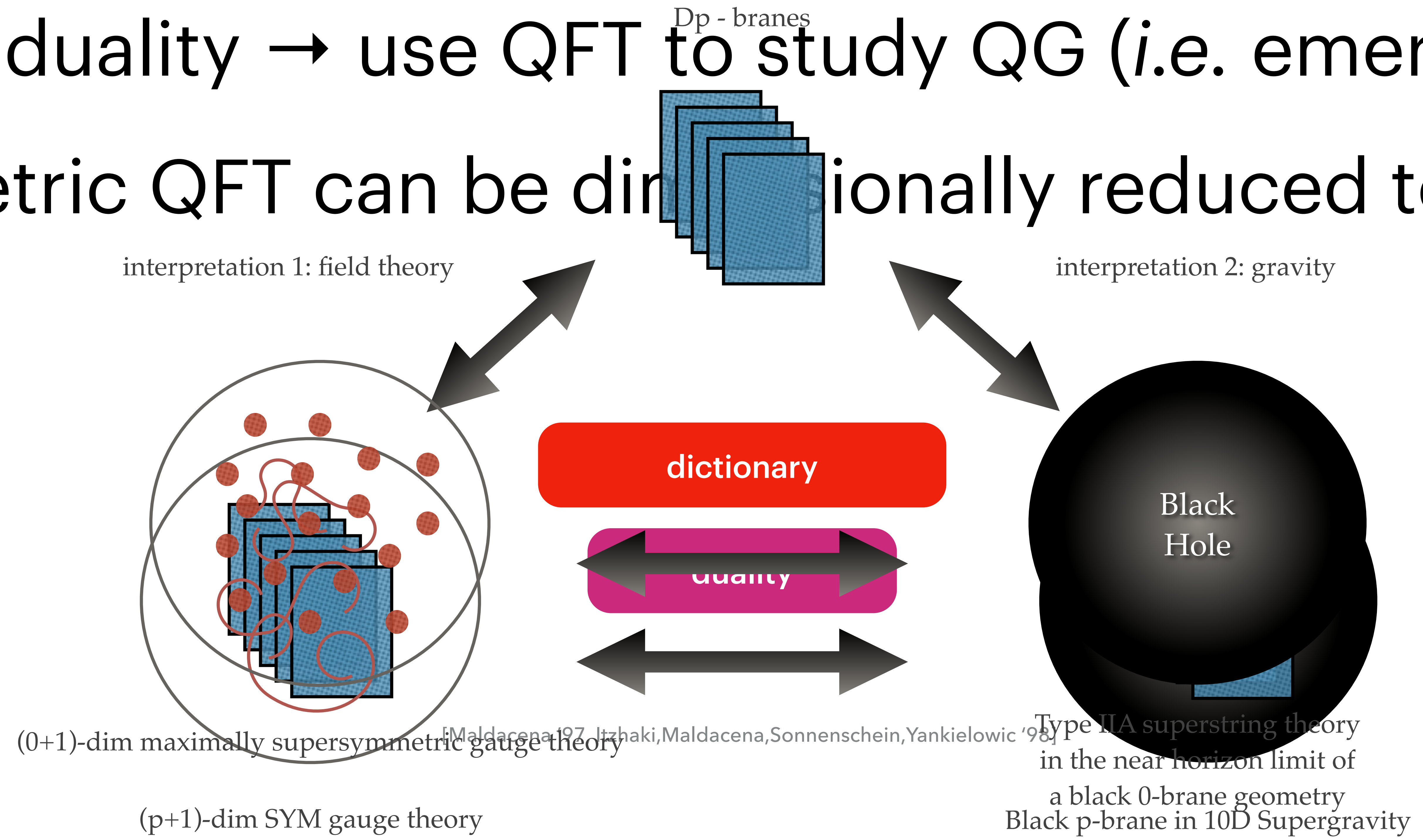
Motivations

★Holographic duality → a quantum field theory **"is"** a gravitational theory

- D0-branes and open strings ⇔ Black hole in Type IIA superstring

★Gauge/gravity duality → use QFT to study QG (*i.e.* emergent geometry)

- Supersymmetric QFT can be dimensionally reduced to matrix QM (BFSS)



[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

[Rinaldi et al., PRX Quantum 3 (2022) 010324]

Numerical Methods for MQM

Prototype: small-scale system

Bosonic Model

FREE | BOS. INTERACTION

$$\hat{H}_B = \mathrm{Tr}\left(\frac{1}{2}\hat{P}_I^2 + \frac{m^2}{2}\hat{X}_I^2 - \frac{g^2}{4}\left[\hat{X}_I, \hat{X}_J\right]^2\right)$$

Physical states are invariant under SU(N) Gauge Symmetry

$$\hat{X}_I = \sum_{\alpha=1}^{N^2-1} \hat{X}_I^{\alpha} \tau_{\alpha} \qquad I = 1,\ldots,D$$

$\hat{X}_I^{\alpha}$ → bosonic degrees of freedom

τ_{α} → generators of SU(N) group

Example: N=2, D=2

$$\hat{H}_B = \sum_{\alpha,I}\left(\frac{1}{2}\hat{P}_{I\alpha}^2 + \frac{m^2}{2}\hat{X}_{I\alpha}^2\right) + \frac{g^2}{4}\sum_{\gamma,I,J}\left(\sum_{\alpha,\beta} f_{\alpha\beta\gamma}\hat{X}_I^{\alpha}\hat{X}_J^{\beta}\right)^2 \qquad I = 1,2 \qquad \alpha = 1,2,3$$

Simplest: N_{bos}

Independent an-harmonic oscillators

$$\hat{H} = \frac{1}{2}\sum_{i=1}^{N_{\mathrm{bos}}}\hat{p}_i^2 + V\left(\hat{x}_1, \cdots, \hat{x}_{N_{\mathrm{bos}}}\right) \qquad V(\hat{x}_i) = \frac{\lambda}{4}\hat{x}_i^4 + \frac{m^2}{2}\hat{x}_i^2$$

[Pateloudis, Bergner, Hanada, Rinaldi, et al., arxiv:2210.04881]

Numerical Methods

★Partition Function:

★ $Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}}$

★Expectation value:

★ $\langle f(\vec{x}) \rangle_\beta = \frac{1}{Z(\beta)} \mathrm{Tr}\, f(\vec{x}) e^{-\beta \hat{H}}$

- Many classical methods related to Feynman Path Integral can treat continuous variables
- **What do we do when the "coordinates" are digitized?**

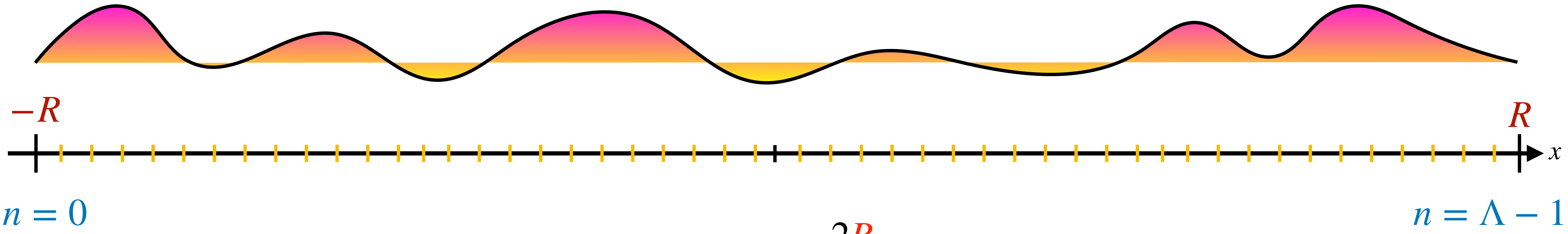
Credits:RIKEN

Quantum Computers
→ Represent a state using quantum bits (qubits)

Hilbert space regularization

Truncation in the coordinate basis

$$\hat{x}|x\rangle = x|x\rangle$$



$$x(n) = -R + n a_{\rm dig}, \quad a_{\rm dig} = \frac{2R}{\Lambda - 1}, \quad n = 0, 1, \cdots, \Lambda - 1$$

- R is the "infrared" cutoff $\rightarrow \infty$
- $a_{\rm dig}$ is the **digitization** spacing $\rightarrow 0$
- Λ is the number of **digitization** points $\rightarrow \infty$

Note: can extend this to many bosons $N_{bos} > 1$!

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Hilbert space regularization

Truncation in the coordinate basis

$$\hat{x}|x\rangle = x|x\rangle$$

$$x_i(n_i) = -R + n_i a_{\rm dig}, \quad a_{\rm dig} = \frac{2R}{\Lambda - 1}, \quad n_i = 0,1,\cdots,\Lambda-1, \quad i = 0,1,\cdots,N_{bos}$$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Partition function regularization

$$Z(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}}$$

$$\tau = \beta = \frac{1}{T} = K\Delta$$

$$Z(\beta) = \sum_{\vec{n}^{(1)},\dots,\vec{n}^{(K)}} \left\langle \vec{n}^{(1)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(2)} \right\rangle \cdot \left\langle \vec{n}^{(2)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(3)} \right\rangle \cdots \left\langle \vec{n}^{(K)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(1)} \right\rangle$$

$$\left\langle \vec{n}^{(j)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(j+1)} \right\rangle \simeq \left\langle \vec{n}^{(j)} \middle| e^{-\Delta\cdot\sum_i \frac{\hat{p}_i^2}{2}} \middle| \vec{n}^{(j+1)} \right\rangle \cdot e^{-\Delta\cdot V\left(\vec{n}^{(j+1)}\right)}$$

$$\left\langle \vec{n}^{(j)} \middle| \left(1 - \Delta\sum_i \frac{\hat{p}_i^2}{2}\right) \middle| \vec{n}^{(j+1)} \right\rangle \cdot e^{-\Delta\cdot V\left(\vec{n}^{(j+1)}\right)}$$

$$= \left\{ \left(1 - \frac{N_{\mathrm{bos}}\Delta}{a_{\mathrm{dig}}^2}\right) \delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}} + \frac{\Delta}{2a_{\mathrm{dig}}^2} \sum_{i=1}^{N_{\mathrm{bos}}} \left(\delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}+\hat{i}} + \delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}-\hat{i}} \right) \right\} \cdot e^{-\Delta\cdot V\left(\vec{n}^{(j+1)}\right)}$$

Needs to be positive

Needs to be small

Δ

$-R$ $\quad R \quad x$

$n = 0 \qquad n = \Lambda - 1$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Sampling configurations

$$\langle f(\vec{x})\rangle_\beta = \frac{1}{Z(\beta)}\mathrm{Tr}\, f(\vec{x})e^{-\beta\hat{H}}$$

$$\tau = \beta = \frac{1}{T} = K\Delta$$

$$Z(\beta) = \sum_{\vec{n}^{(1)},\cdots,\vec{n}^{(K)}} \left\langle \vec{n}^{(1)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(2)} \right\rangle \cdot \left\langle \vec{n}^{(2)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(3)} \right\rangle \cdots \left\langle \vec{n}^{(K)} \middle| e^{-\Delta\cdot\hat{H}} \middle| \vec{n}^{(1)} \right\rangle$$

$$\sum_{\vec{n}^{(1)},\cdots,\vec{n}^{(K)}} \prod_{j=1}^{K} \left\{ \left(1 - \frac{N_{\mathrm{bos}}\Delta}{a_{\mathrm{dig}}^2}\right)\delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}} + \frac{\Delta}{2a_{\mathrm{dig}}^2}\sum_{i=1}^{N_{\mathrm{bos}}}\left(\delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}+\hat{i}} + \delta_{\vec{n}^{(j)},\vec{n}^{(j+1)}-\hat{i}}\right)\right\} \cdot e^{-\Delta\cdot V\left(\vec{n}^{(j+1)}\right)}$$

Sum over “configurations”

Configurations have positive weights.

We can use Markov chain Monte Carlo!

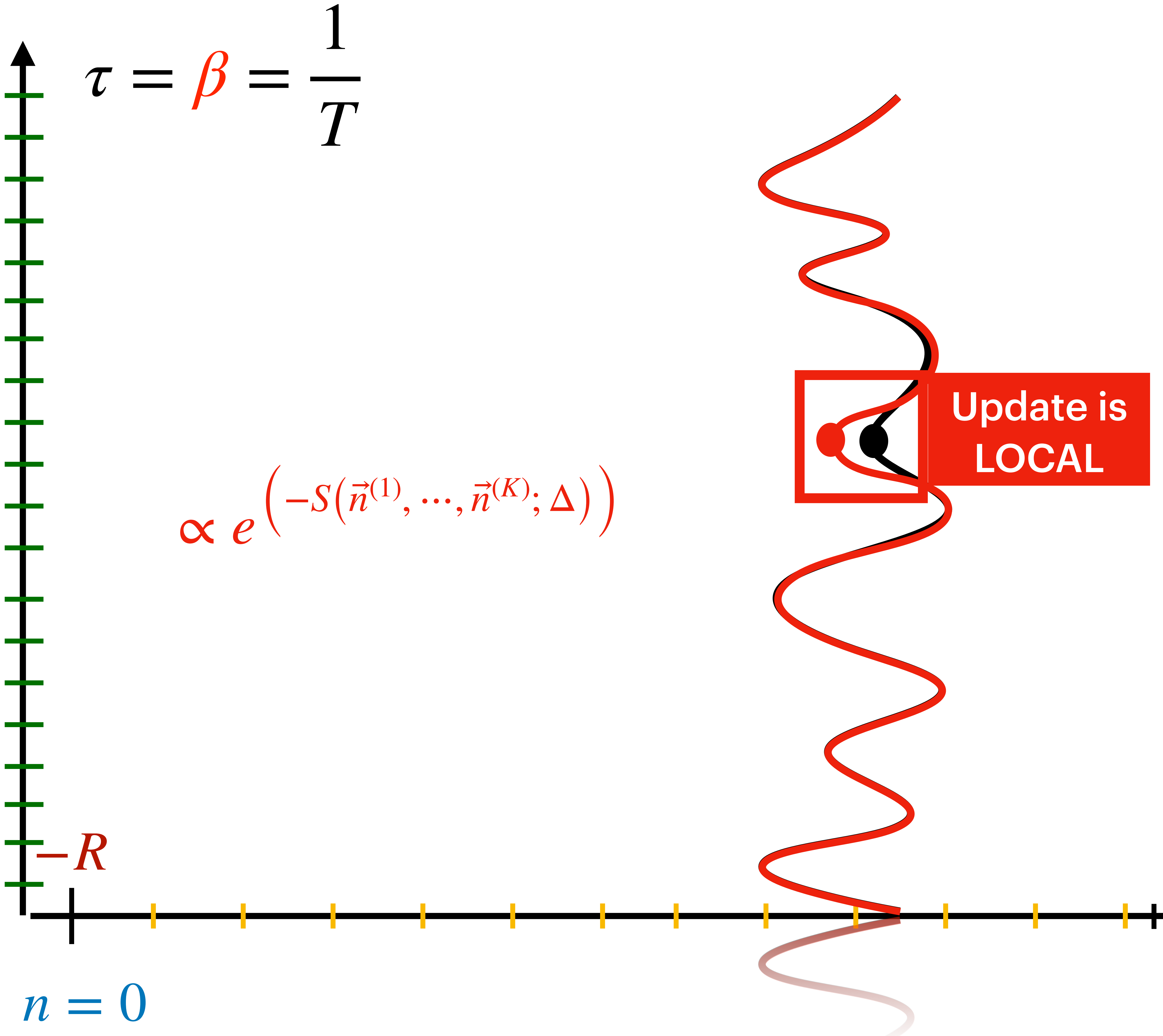
Δ

$-R$ R x

$n = 0$ $n = \Lambda - 1$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Metropolis algorithm

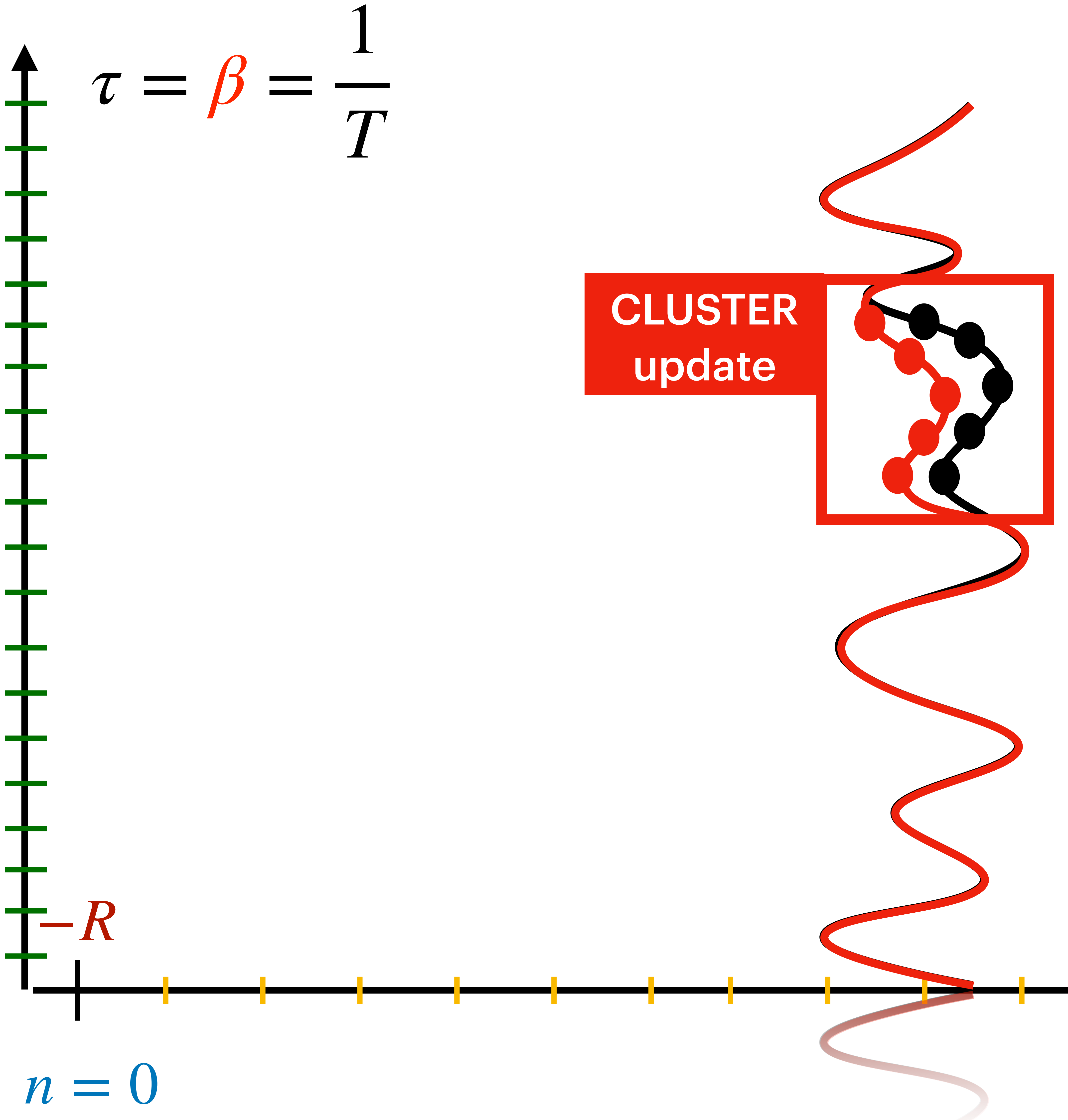


Simple

- Choose one "imaginary time" $j = 1, \ldots, K$
- Choose one "boson" $i = 1, \ldots, N_{\text{bos}}$
- Update site: $n' \equiv n_i^{(j)} \pm 1$ with prob. 0.5
- Accept with prob. $\min\left(1, e^{-\delta S}\right)$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Metropolis algorithm



Improved

- Choose a “cluster” size B
- Choose one “imaginary time block” $l = j, j+1, \cdots, j+B$
- Choose one “boson” $i = 1, \ldots, N_{\text{bos}}$
- Update “block”: $n'^{(l)} \equiv n_i^{(l)} \pm 1$ with prob. 0.5
- Accept with prob. $\min\left(1, e^{-\delta S}\right)$

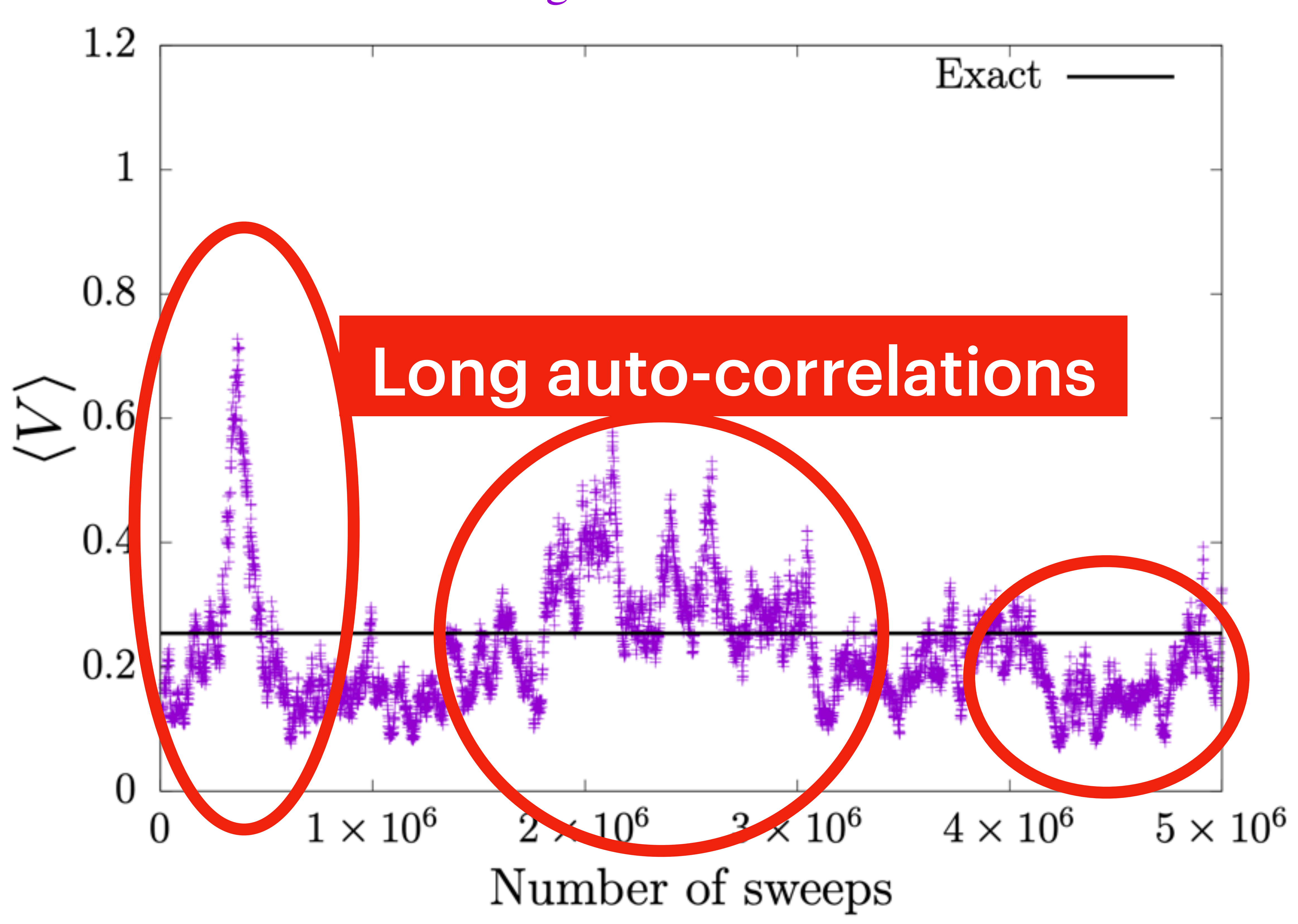
[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Results

An-harmonic oscillator: $N_{\rm bos} = 1$

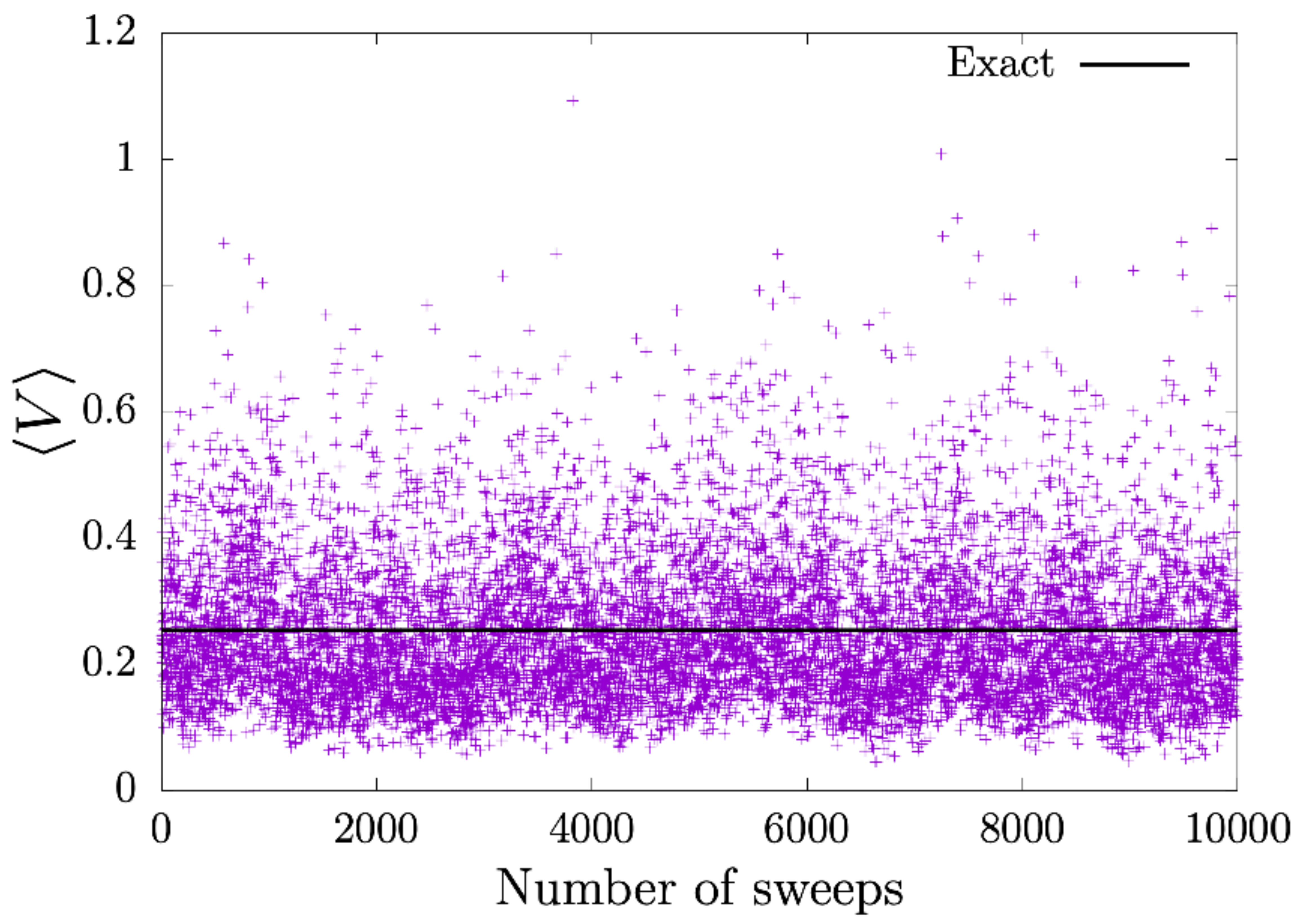
Simple

$T = 0.1, a_{\rm dig} = 0.5, \Delta = 0.001$



Improved

$T = 0.1, a_{\rm dig} = 0.5, \Delta = 0.001, B_{\rm max} = K/2 = 5000$



[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Results

An-harmonic oscillator:

$N_{\text{bos}} = 1$

$$V(\hat{x}) = \frac{\lambda}{4}\hat{x}^4 + \frac{m^2}{2}\hat{x}^2, \quad \lambda = 1$$

a_{dig}	$m^2 = 1.0$, MC	$m^2 = 1.0$, Exact	$m^2 = -1.0$, MC	$m^2 = -1.0$, Exact
0.3	0.2626(26)	0.2618	−0.06326(68)	-0.06354
0.5	0.2533(53)	0.2539	−0.0664(27)	-0.06633
0.7	0.2482(70)	0.2414	−0.0717(18)	-0.07024

$T = 0.1, R = 1000a_{\text{dig}}, \Lambda = 2001, \Delta = 0.001$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Results

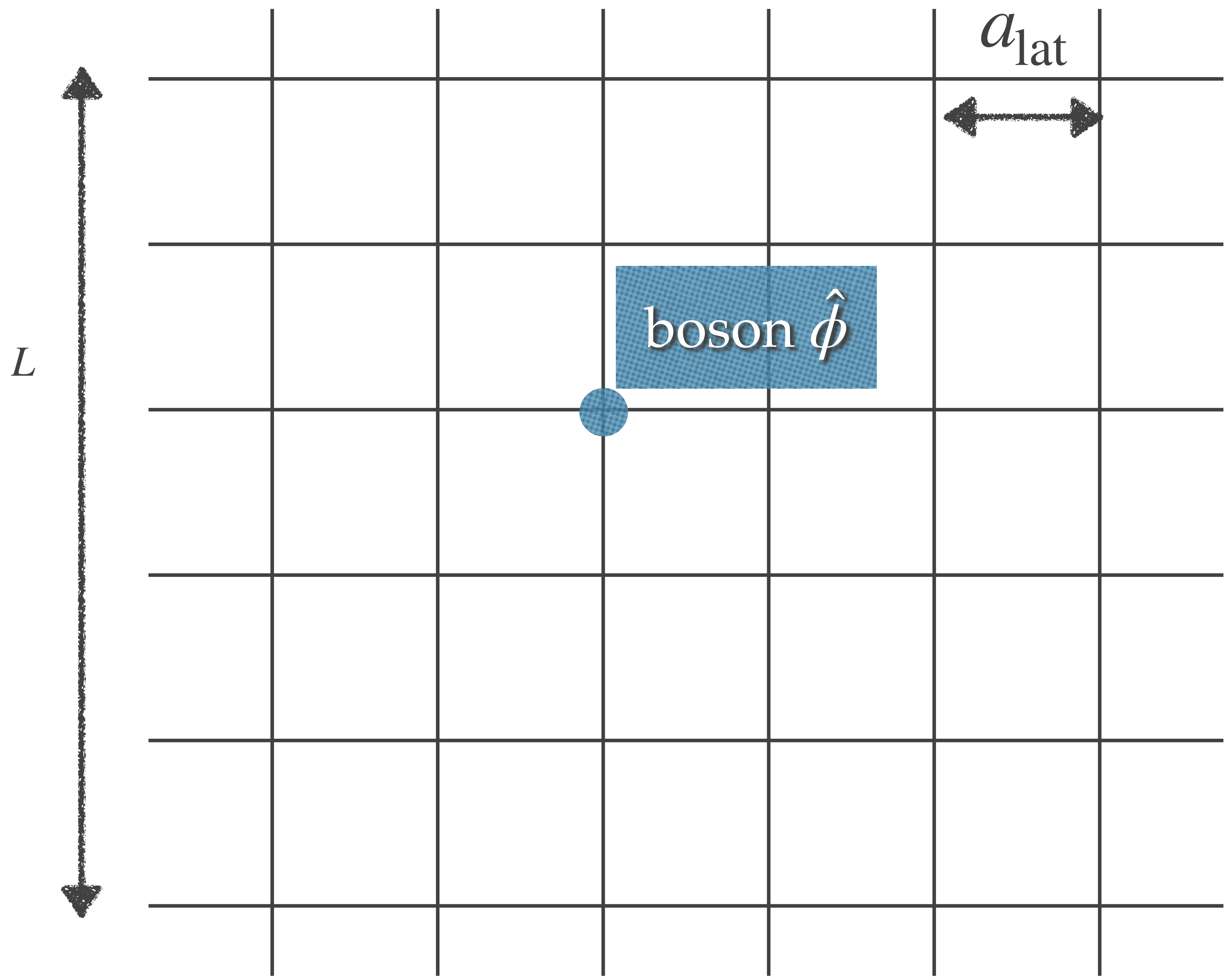
Free Scalar Lattice Field Theory

$$\hat{H}_{\rm lat} = a_{\rm lat}\hat{H} = \sum_{\vec{n}_{\rm lat}} \left(\frac{1}{2}\hat{\pi}^2_{\vec{n}_{\rm lat}} + \frac{1}{2}\sum_{\mu=1}^{d}\left(\hat{\phi}_{\vec{n}_{\rm lat}+\hat{\mu}} - \hat{\phi}_{\vec{n}_{\rm lat}}\right)^2 + V\left(\hat{\phi}_{\vec{n}_{\rm lat}}\right)\right), V(\overrightarrow{\phi}) = \frac{(ma_{\rm lat})^2}{2}\sum_{\vec{n}}\hat{\phi}^2_{\vec{n}}$$

Dimension of

the Hilbert space

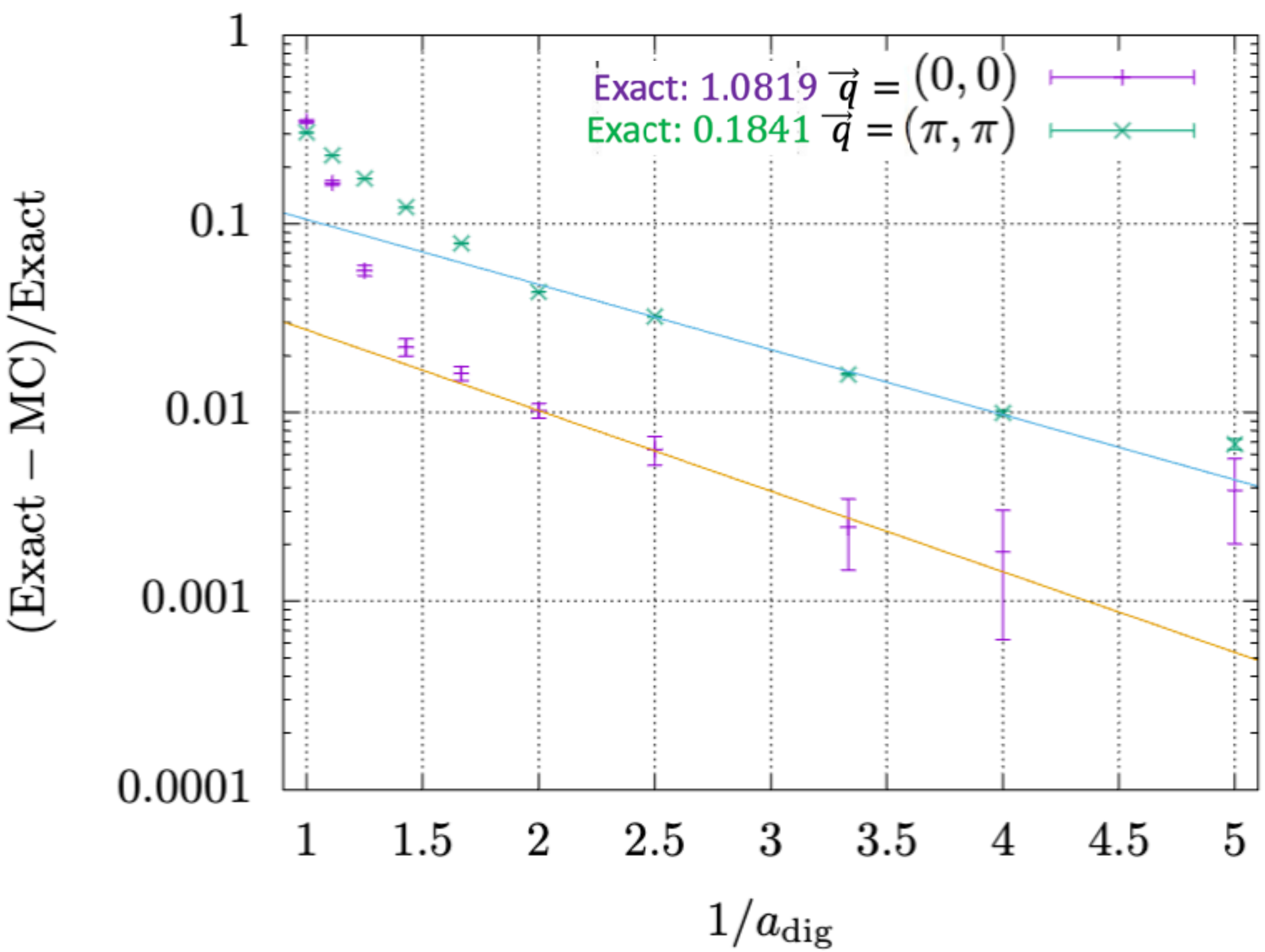
$$\Lambda^{N_{\rm bos}} = \Lambda^{L^d}$$



Results

Free Scalar Lattice Field Theory

$N_{\mathrm{bos}} = 4 \times 4 = 16$



a_{dig}	Δ	$K = \beta/\Delta$	$\vec{q} = (0,0)$	$\vec{q} = (\pi,\pi)$
0.20	0.0008	1250	1.0778(20)	0.18288(8)
0.25	0.00125	800	1.0800(13)	0.18230(5)
0.30	0.0015	667	1.0793(11)	0.18120(4)
0.40	0.002	500	1.0751(12)	0.17819(4)
0.50	0.002	500	1.0709(10)	0.17611(3)
0.60	0.005	200	1.0645(15)	0.16961(4)
0.70	0.005	200	1.0579(26)	0.16156(6)
0.80	0.005	200	1.0208(39)	0.15214(10)
0.90	0.005	200	0.9039(55)	0.14167(17)
1.00	0.005	200	0.7038(70)	0.12792(25)

$T = 1.0, R = 1000a_{\mathrm{dig}}, \Lambda = 2001, \frac{\Delta}{2a_{\mathrm{dig}}} < 0.01$

[Hanada, Liu, Rinaldi, Tezuka, arxiv:2212.08546]

Conclusions

✓ Quantum simulations can be used for addressing **Quantum Gravity** problems, using the holographic duality

✦ Classical sampling methods (MCMC) can be used to study digitization errors for quantum systems with many bosons!

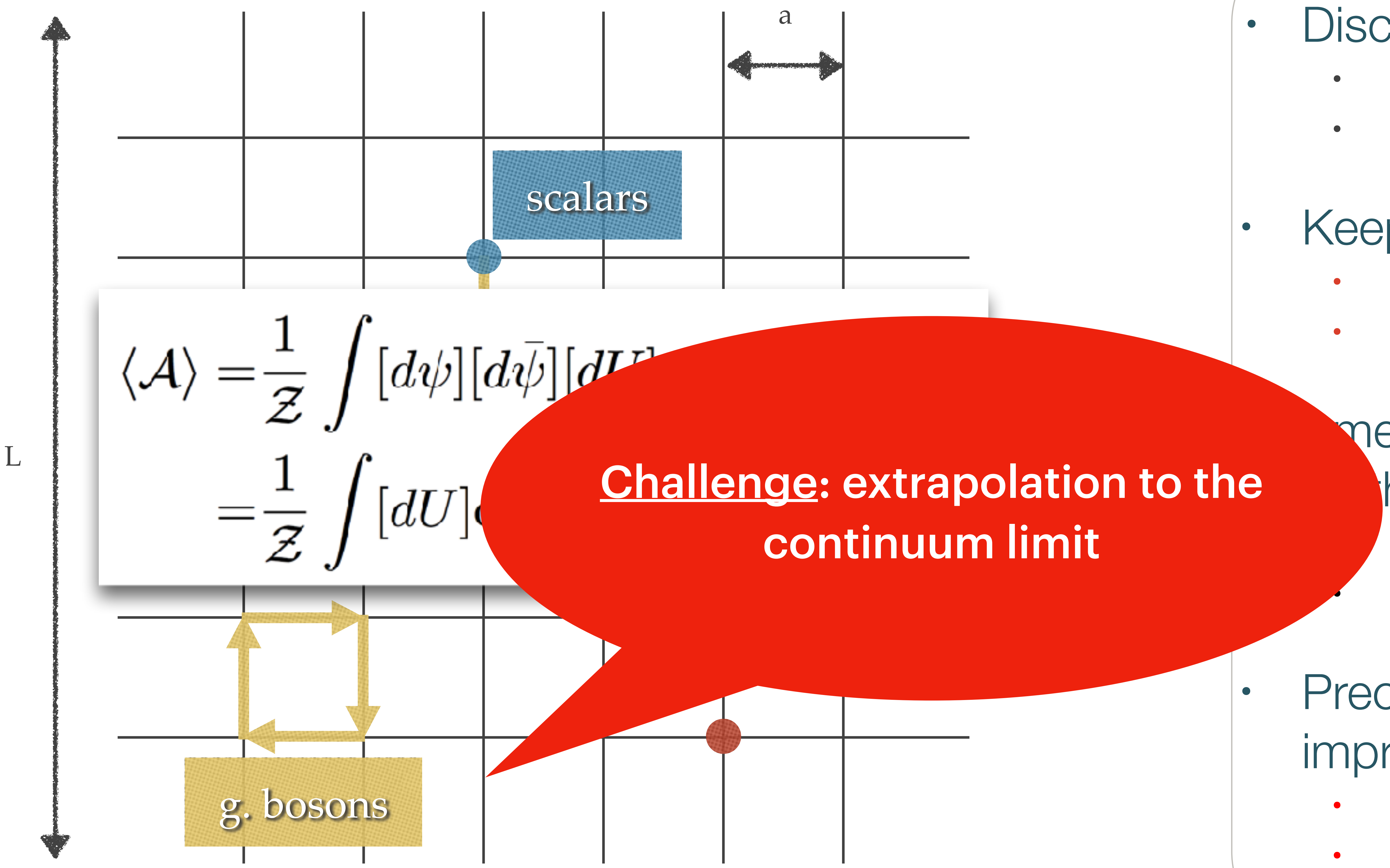
✦ Expectation values of observables for digitized lattice theories in high dimensions can be computed: relevant for resource estimations and validation

➡ Our method can validate the exponential decrease of digitization errors in many theories

➡ Similarly, we can validate the “infrared” cutoff effects

Path Integral Monte Carlo

Lattice Gauge Theory Primer



- Discretize space and time
  - lattice spacing "a"
  - lattice size "L"
- Keep all d.o.f. of the theory
  - not a model!
  - no simplifications
- Amenable to numerical methods
  - Monte Carlo sampling
  - use supercomputers
- Precisely quantifiable and improvable errors
  - Systematic
  - Statistical

LATTICE QUANTUM FIELD THEORY – MATHEMATICS

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} + m)\psi$$

MICROSCOPIC THEORY OF FIELDS

ψ: quark field

U: gauge field

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}U\, e^{-S[\bar{\psi},\psi,U]}\, \mathcal{O}$$

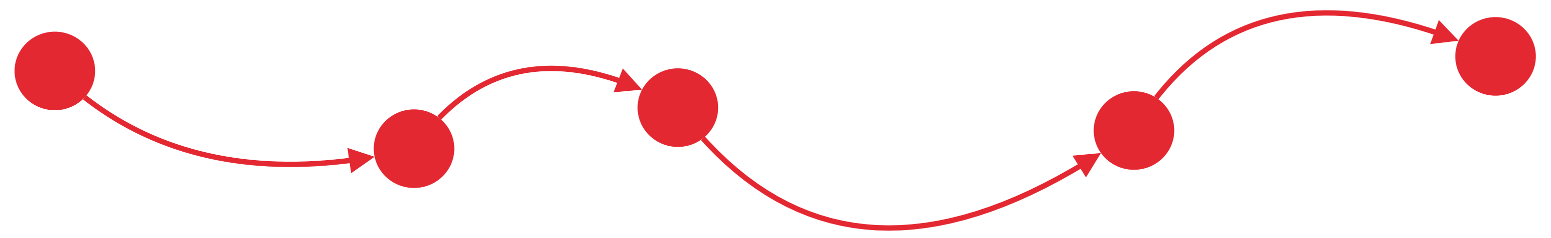
QUANTUM FEYNMAN PATH INTEGRAL

Physical observable

DISCRETIZE

Makes integral finite dimens.

$$\{U_1, U_2, U_3, \ldots, U_N\}$$



move in configuration space with prob.

MARKOV CHAIN MONTE CARLO

Sampling

$$\approx \frac{1}{N}\sum_{i=1}^{N} \mathcal{O}[U_i] + O\left(\frac{1}{\sqrt{N}}\right)$$

statistical error

IMPORTANCE SAMPLING

Estimator