Going beyond two
- multipartite entanglement and multi inner product -

Shinsei Ryu
(Princeton)

YKIS conference "Foundations and Developments of Quantum Information Theory" 2023/09/22

Introduction

- One of the most fundamental concepts in quantum mechanics is the inner product: $\langle\psi|\phi\rangle$. Represents a transition amplitude.
- It'd also be useful to think it in terms of entanglement:

$$\big(\langle\psi| \otimes \langle\phi|^*\big)|\Omega\rangle = \langle\psi|\phi\rangle$$

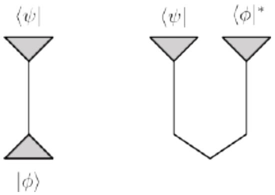
$$|\Omega\rangle = \text{maximally entangled state}$$

- E.g.

$$|Bell\rangle = (1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle),$$

$$|TFD\rangle = \sqrt{Z}^{-1} \sum_n e^{-\beta E_n/2} |E_n\rangle_L |E_n\rangle_R$$

- Maybe useful to consider in terms of diagrams



- Or in terms of geometry; "ER=EPR" [Maldacena-Susskind (13)]

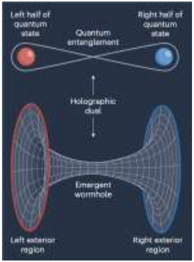
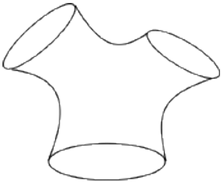


Image: [Brown-Susskind].

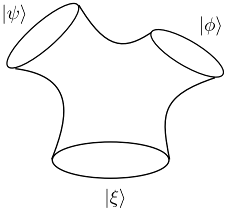
Multiboundary and multipartite entanglement



$$|V\rangle = \sum C_{ijk}|i\rangle|j\rangle|k\rangle$$

E.g.,[Balasubramanian-Hayden-Maloney-Marolf-Ross, "Multiboundary Wormholes and Holographic Entanglement,"(14)].

$\langle \psi, \phi, \xi \rangle \sim$



We go beyond biparite entanglement and bi inner product.

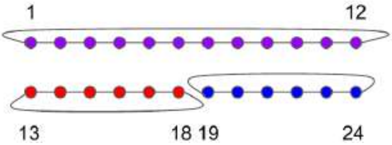
- (1+1)d critical spin chain (CFT) with Yuhan Liu and Yijian Zou (22)
- (1+1)d gapped system (higher Berry phase) with Shuhei Ohyama (23)
- Tripartite entanglement in (2+1)d topologically-ordered states with Yuhan Liu, Ramanjit Sohal, Jonah Kudler-Flam, Yuya Kusuki (21-23)

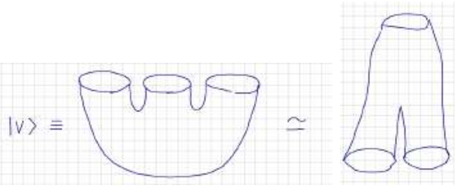
Triple inner product in critical spin chain

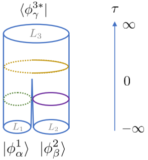
- Lattice quantum many-body systems, e.g., $H = -\sum_i X_i X_{i+1} - \sum_i Z_i$, $H = \sum(X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1})$.

- Triple wave function overlap $A_{\alpha\beta\gamma}$

$$A_{\alpha\beta\gamma} = \langle\phi_\gamma|\big(|\phi_\alpha\rangle|\phi_\beta\rangle\big) \quad \Big[= \langle V|\big(|\phi^1_\alpha\rangle|\phi^2_\beta\rangle|\phi^3_\gamma\rangle^*\big)\Big]$$







Numerical approaches to critical "spin" chains

- Exact diagonalization, DMRG, Monte Carlo, etc.
- A quantum spin chain may be at conformal critical point $\Rightarrow$ How do we identify CFT?
  - Central charge $\Leftarrow$ entanglement entropy scaling
  - Dimensions $\Leftarrow$ numerical exact diagonalization
- These may not be enough to nail down full information of CFT
- $A_{\alpha\beta\gamma}$ can be used to extract OPE coefficients (and more) of (1+1)d lattice quantum systems at conformal critical point [Zou-Vidal (21), Liu-Zou-SR (22)]

$$\phi_\alpha(z,\bar{z})\phi_\beta(w,\bar{w}) = \sum_\gamma |z-w|^{-\Delta_\alpha-\Delta_\beta+\Delta_\gamma} C_{\alpha\beta\gamma}\phi_\gamma(z,\bar{z}) + \cdots$$

Wavefunction overlap and OPE

- Wave function overlap can be mapped on to a 3pt function on a plane: [Zou-Vidal (21)]

$$\frac{A_{\alpha\beta\gamma}}{A_{\mathbb{1}\mathbb{1}\mathbb{1}}} = \langle \phi^1_\alpha(-\infty)\phi^2_\beta(-\infty)\phi^3_\gamma(+\infty)\rangle_{\text{Pants}}$$
$$= \text{Jacobian} \cdot \langle \phi^1_\alpha(w_1)\phi^2_\beta(w_2)\phi^3_\gamma(w_3)\rangle_{\mathbb{C}} \quad \propto C_{\alpha\beta\gamma}$$

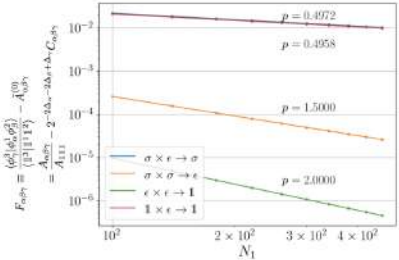
- Finite size corrections [Liu-Zou-SR (22)] :

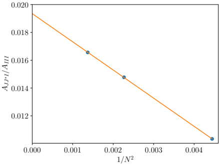
$$A_{\alpha\beta\gamma} = \sum_{\delta,\hat{\chi}=0,1} a_{(\delta,\hat{\chi})}\, 2^{-2\Delta_\alpha - 2\Delta_\beta + 2\Delta_\gamma} C_{\alpha\beta\gamma}\, L^{-\Delta_{(\delta,\hat{\chi})}}$$

$\Delta_{(\hat{\delta},\chi)}$: operator content of the $\mathbb{Z}_2$ orbifold theory
E.g. $A_{\mathbb{1}\mathbb{1}\mathbb{1}} \propto L^{-c/8}$

Applications

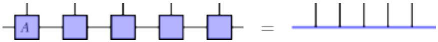
- Critical Ising chain, XXZ model
- Interacting anyon chain for the Haagerup fusion category ($c \sim 2$) [Huang-Lin-Ohmori-Tachikawa-Tezuka (21); Vanhove-Lootens-Damme-Wolf-Osborne-Haegeman-Verstraete (21)]
  Tested a conjectured CFT
- "Buliding block" for tensor network [Ueda-Yamazaki (23)]





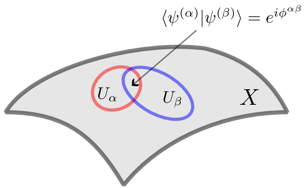
Gapped states in (1+1)d and (higher) Berry phase

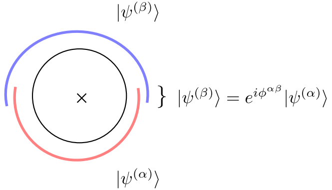
- Short-range entangled or invertible states in (1+1)d
- Is there Berry phase associated with the triple inner product?
- We will use matrix product states (MPS)



[Previous and related works on higher Berry phase:
Kitaev (2019); Cordova-Freed-Lam-Seiberg (19); Kapustin-Spodyneiko (20); Kapustin-Sopenko (22); Hsin-Kapustin-Thorngren (20); Artymowicz-Kapustin-Sopenko (20); Choi-Ohmori (22); Shiozaki (21); Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21); Beaudry-Hermele-Moreno-Pflaum-Qi-Spiegel (23); Ohyama-Shiozaki-Sato (22); Ohyama-Terashima-Shiozaki (23)]

Regular Berry phase for single-particle wavefunctions





- Parameterized state $|\psi(x)\rangle$, $x = (x^1, x^2, \cdots) \in X$.
- Berry connection $\langle\psi(x)|\psi(x+dx)\rangle = \mathcal{A}_\mu(x)dx^\mu$, "associated with" the gauge transformation $|\psi(x)\rangle \to e^{i\phi(x)}|\psi(x)\rangle$
- Mathematical framework: complex line bundle;
  Classified by Chern class, $H^2(X,\mathbb{Z})$
- Wu-Yang description of magnetic monopole
  At the double intersection $U_\alpha \cap U_\beta$, $\langle\phi^{(\alpha)}|\phi^{(\beta)}\rangle = e^{i\phi^{\alpha\beta}}$, which defines $[e^{i\phi^{\alpha\beta}}] \in H^1(X, U(1)) \simeq H^2(X,\mathbb{Z})$.

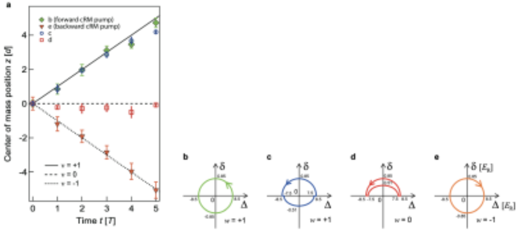
Thouless pump

[Thouless (1983)]

Topological Thouless Pumping of Ultracold Fermions

Shuta Nakajima[1],* Takafumi Tomita[1], Shintaro Taie[1], Tomohiro Ichinose[1], Hideki Ozawa[1], Lei Wang[2], Matthias Troyer[2], and Yoshiro Takahashi[1]
[1] *Department of Physics, Graduate School of Science, Kyoto University, Japan 606-8502*
[2] *Theoretische Physik, ETH Zurich, 8093 Zurich, Switzerland*

[Nakajima et al (16)]

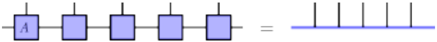


$$H = \sum_i \left[-(J+\delta) f_i^\dagger d_i - (J-\delta) f_i^\dagger d_{i+1} + h.c. + \Delta(f_i^\dagger f_i - d_i^\dagger d_i) \right]$$

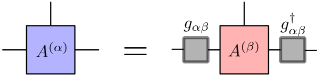
$$\nu = \frac{1}{2\pi} \int_0^T dt \int_{-\pi}^{\pi} dk\, \Omega(k,t) = \text{integer}$$

MPS and MPS gauge transformation

- Parameterized injective MPS $\{A^i(x)\}$ over some parameter space X.



- MPS gauge transformation, $A \to gAg^\dagger$: [Perez-Garcia-Verstraete-Wolf-Cirac]



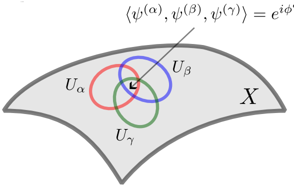
  where $g_{\alpha\beta} \in PU(\text{bond dim})$.

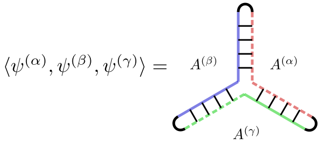
- Plays a crucial role in the classification of SPT phases in (1+1)d [Pollmann-Turner-Berg-Oshikawa (10), Chen-Gu-Wen (11), Schuch-Perez-Garcia-Cirac(11)]

- Is there "Berry phase" associated with the MPS gauge transformation?

Main results

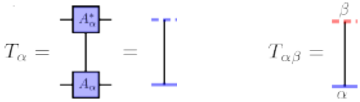
[Ohyama-SR (23)]





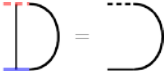
- Defined and computed triple inner product of three MPSs.
- At triple intersection, $U_\alpha \cap U_\beta \cap U_\gamma$, $\langle \psi^{(\alpha)}, \psi^{(\beta)}, \psi^{(\gamma)} \rangle = e^{i\phi^{\alpha\beta\gamma}}$ where $g_{\alpha\beta} g_{\beta\gamma} = e^{i\phi^{\alpha\beta\gamma}} g_{\alpha\gamma}$.
- $e^{i\phi^{\alpha\beta\gamma}}$ satisfies the cocycle condition, $\delta e^{i\phi^{\alpha\beta\gamma}} = 1$, and defines $[e^{i\phi^{\alpha\beta\gamma}}] \in H^2(X, U(1)) \simeq H^3(X, \mathbb{Z})$
- Mathematical framework: *gerbe* – a generalization of complex line bundle and classified by Diximier-Douady class, $H^3(X, \mathbb{Z})$ See also [Qi-Stephen-Wen-Spiegel-Markus-Pflaum-Beaudry-Hermele (23)]

- Transfer matrix and mixed transfer matrix



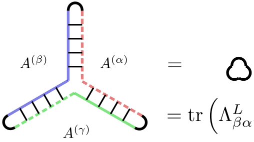
- (Right and left) eigen vector $\Lambda^{R,L}_{\alpha\beta}$:

$$T_{\alpha\beta} \cdot \Lambda^R_{\alpha\beta} = \Lambda^R_{\alpha\beta} \iff$$



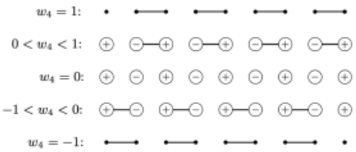
Related to the gauge transformation as $\Lambda^R_{\alpha\beta} = g_{\alpha\beta}\Lambda^R_\beta$, $\Lambda^L_{\alpha\beta} = g_{\alpha\beta}\Lambda^L_\beta$.

- Triple inner product:



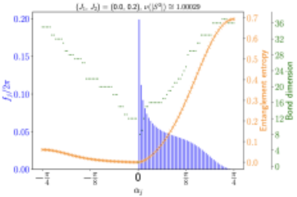
$$= \mathrm{tr}\left(\Lambda^L_{\beta\alpha}\Lambda^R_{\beta\gamma}\Lambda^R_{\gamma\alpha}\right) = \mathrm{tr}\left(\Lambda^L_\alpha \hat{g}_{\alpha\beta} 1_n \hat{g}_{\beta\gamma} 1_n \hat{g}_{\gamma\alpha}\right)$$

- Topological classification of parameterized family of states (“adiabatic process”)
  Many-body generalization of Thouless pump (“higher Thouless pump”)



- A concrete model parameterized over $X = S^3$: [Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21)]

- What about the Berry connection and curvature? Practical method to calculate topological invariant? [Shiozaki-Heinsdorf-Ohyama (23)]



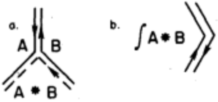
- "Aharonov-Bohm phase" of MPS: 2-form gauge field
- Possible because of "internal structure" of MPS, can go beyond single-particles
- 2-form gauge field couples naturally to 1d extended objects; strings
- No fundamental string in condensed matter physics, but we could have emergent ones.
- More (formal) direct link with string theory (string field theory):

NON-COMMUTATIVE GEOMETRY AND STRING FIELD THEORY

Edward WITTEN*

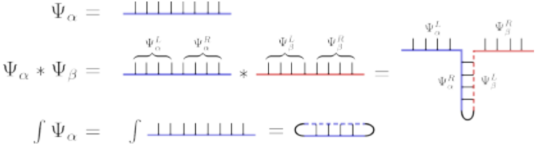
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540, USA

Received 2 December 1985

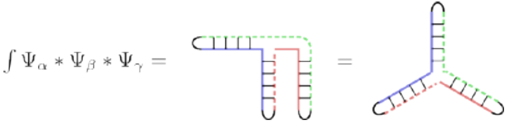


String ⋆ product and triple inner product

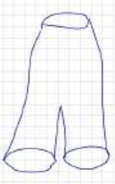
- Following [Witten (85)], we can introduce "⋆" product and "integration" for MPSs

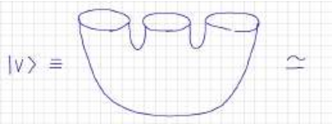


- Triple inner product is given by "string field theory vertex"



Multipartite entanglement?





- $|V\rangle \in \mathcal{H}_{\mathrm{chiralCFT}}^{\otimes 3}$
- What "type" of tripartite states?

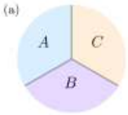
$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle\right], \quad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\left[|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right]$$

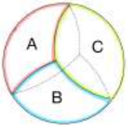
"Three qubits can be entangled in two inequivalent ways" [Dür-Vidal-Cirac (00)]

- Reflected entropy $R_{A:B}$ and Markov gap $h_{A:B} = R_{A:B} - MI_{A:B}$

[Dutta-Faulkner (19)][Akers-Rath (19)][Hayden-Parrikar-Sorce (21)]

Reflected entropy in 2d topological liquids





- By bulk-boundary correspondence, the problem turns out to be related to tripartite entanglement of (2+1)d topological liquids
  C.f. Topological entanglement entropy

- Chiral topological liquids with chiral central charge c:

$$h_{A:B} = R_{A:B} - I_{A:B} = \frac{c}{3}\ln 2, \quad c\text{: central charge.}$$

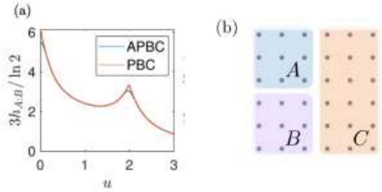
[Liu-Sohal-Kudler-Flam-SR (21)],[Liu-Kusuki-Sohal-Kudler-Flam-SR (23)]

C.f. [Zou-Siva-Soejima-Mong-Zaletel (21)]

- Conjecture [Zou-Siva-Soejima-Mong-Zaletel (21)] :

$$h_{A:B} \leq (c/3)\ln 2$$

where c is the central charge of the total ungappable degrees of freedom



Lattice Chern insulator calculation [Liu et al (21)]

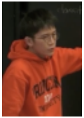
- May have an implication on numerics; non-zero $h_{A:B}$ may be an obstruction to have a finite dim PEPS representation
C.f. diverging correlation length of chiral PEPS

Summary/Outlook

- Introduced generalized inner product (multi-inner product)
- Applications in critical/gapped 1d many-body systems
- Multipartite entanglement of 2d topological liquids

Yuhan Liu (U Chicago/Princeton/Max Planck), Yijian Zou (Stanford/Perimeter), Shuhei Ohyama (YITP, Kyoto),





Ramanjit Sohal (Princeton), Yuya Kusuki (Caltech), Jonah Kudler-Flam (IAS, Princeton),