On the unreasonable effectiveness of modular flow

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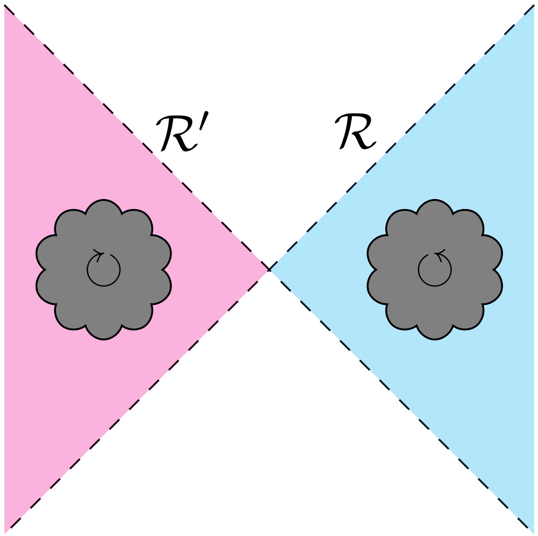
What is modular flow?

- Modular flow is **a hidden symmetry of a subsystem**.
- Given a state $|\Omega\rangle$ in a QFT and a subregion $\mathcal{R}$, the modular Hamiltonian K_Ω generates a flow $e^{iK_\Omega t}$ with:

$$e^{-iK_\Omega t}|\Omega\rangle = |\Omega\rangle ,$$
$$e^{iK_\Omega t}\mathcal{R}e^{-iK_\Omega t} = \mathcal{R},$$
$$e^{iK_\Omega t}\mathcal{R}'e^{-iK_\Omega t} = \mathcal{R}'.$$

And for which **the state** $|\Omega\rangle$ **looks thermal in** $\mathcal{R}$.

What is modular flow? (figure)



A survey of applications

- Proof of the generalized second law [Wall 2011]
- Proof of the ANEC [Faulkner, Leigh, Parrikar, Wang 2016]
- Proof of the QNEC [Ceyhan, Faulkner 2018]
- Constraints on QFT correlators [Lashkari 2018]
- UV-finite formulas for black hole entropy [Witten 2021; Chandrasekaran, Longo, Penington, Witten 2022; Jensen, JS, Speranza 2023]

Why is modular flow so useful?

- Modular flow has a reputation for being very mathematically finicky: proving anything requires a lot of highly technical lemmas.
- The point of this talk is to explain that modular flow's usefulness is not miraculous, but in fact follows naturally from the principles of statistical mechanics.
- **If a subsystem in a QFT is to look thermal with respect to some arrow of time, then the only possible arrow of time is modular flow.**

The standard treatment of modular flow

- Hilbert space $\mathcal{H}$, state $|\Omega\rangle$, von Neumann algebra $\mathcal{A}$.
- "Tomita operator" an antilinear, unbounded operator S defined via
$$S(a\,|\Omega\rangle) = a^\dagger\,|\Omega\rangle\,.$$
- Modular operator $\Delta_\Omega = S^\dagger S$, modular Hamiltonian $K_\Omega = -\log(\Delta_\Omega)$.
- Define modular flow as $e^{iK_\Omega t}$; prove it is a symmetry of $|\Omega\rangle$ and $\mathcal{A}$, and satisfies the "KMS condition."

A perspective from thermal physics

- On a lattice with Hamiltonian H, a **thermal state** is

$$\rho = \frac{e^{-\beta H}}{Z}.$$

- The two-point function is

$$\langle ab \rangle_\beta = \mathrm{tr}(\rho a b).$$

- The time-dependence of the two point function is

$$\langle e^{iHt} a e^{-iHt} b \rangle_\beta = \frac{1}{Z} \mathrm{tr}\left(e^{(it-\beta)H} a e^{-iHt} b \right)$$

The KMS condition

- Kubo, Martin, and Schwinger observed that thermal correlators admit analytic continuations in time:

$$\langle e^{zH} a e^{-zH} b \rangle_\beta = \frac{1}{Z} \operatorname{tr}\left(e^{(z-\beta)H} a e^{-zH} b\right),$$

and

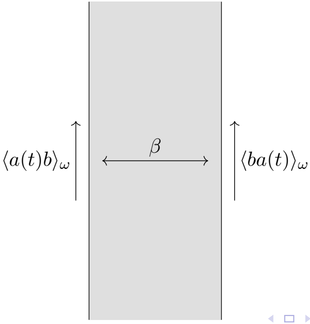
$$\begin{aligned} \langle e^{(it+\beta)H} a e^{-(it+\beta)H} b \rangle_\beta &= \frac{1}{Z} \operatorname{tr}\left(e^{itH} a e^{-(\beta+it)H} b\right) \\ &= \langle b e^{itH} a e^{-itH} \rangle_\beta. \end{aligned}$$

A general definition of thermality

- Let $\mathcal{A}$ be an algebra of operators for a quantum system, $\langle\cdot\rangle_\omega$ a state, and H a Hamiltonian.
- We say $\langle\cdot\rangle_\omega$ **looks thermal with respect to** H if the two-point function

$$\langle a(t)b\rangle_\omega \equiv \langle e^{iHt} a e^{-iHt} b\rangle_\omega$$

admits an analytic continuation like this:



Is this a good definition of thermality?

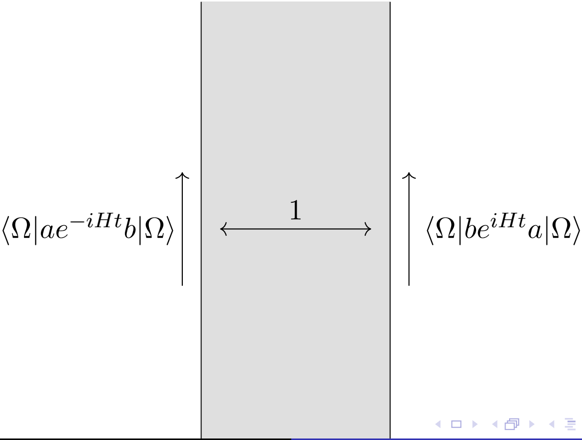
- We have to worry: given a state on a system, could there be multiple Hamiltonians that make it look thermal?
- Should we expect there to exist even one Hamiltonian that makes it look thermal?
- We will now discuss the following theorem:

Theorem

Fix a Hilbert space $\mathcal{H}$, a state $|\Omega\rangle$, and a subsystem described by the algebra $\mathcal{A}$. Suppose there exists a Hamiltonian H on $\mathcal{H}$ for which $|\Omega\rangle$ satisfies the KMS condition on $\mathcal{A}$ at temperature $\beta = 1$. Then H **must be** $-\log \Delta_{\Omega}$ **where** Δ_{Ω} **is the modular operator**.

Sketching the theorem

Concretely, think of $|\Omega\rangle$ as a QFT state, and $\mathcal{A}$ the von Neumann algebra of operators in a subregion. H is some (self-adjoint, unbounded) operator on $\mathcal{H}$. For every $a, b \in \mathcal{A}$, there is an analytic function:



Sketching the theorem

- Morally speaking, the analytic continuation of the two-point function should be

$$F_{ab}(z) = \langle\Omega| a e^{-zH} b |\Omega\rangle ,$$

  except that because H is unbounded, this expression is not well defined.

- Being physicists and taking it seriously for the moment, we see that by putting $z = 1$, the operator H must in some sense satisfy

$$\langle\Omega| a e^{-H} b |\Omega\rangle = \langle\Omega| ba |\Omega\rangle .$$

Sketching the theorem

- Another way of writing:

$$\langle e^{-H/2} a^\dagger \Omega | e^{-H/2} b\Omega \rangle = \langle b^\dagger \Omega | a\Omega \rangle.$$

- This would be satisfied if $e^{-H/2}$ were an antilinear operator satisfying

$$e^{-H/2} a \left|\Omega\right\rangle = a^\dagger \left|\Omega\right\rangle .$$

OR, crucially, if it were the positive piece in the polar decomposition of an antilinear operator S with this property: $S = Je^{-H/2}$

Sketching the theorem

- This motivates the introduction of the Tomita operator S satisfying
$$Sa|\Omega\rangle = a^{\dagger}|\Omega\rangle\,. \tag{1}$$
- This operator — which was invented by mathematicians as a useful trick — shows up **naturally** if you want to reproduce thermal physics in a general setting.
- Define $e^{-K} = S^{\dagger}S$, and ask: is $K = H$?

Sketching the theorem

- At this point, it's just an analyticity argument.
- The modular Hamiltonian K can be studied using the formula for S, and basic arguments about the uniqueness of analytic continuations show $H = K$.
- There are some details that you have to be careful about, having to do with domains of unbounded operators, but otherwise the argument is straightforward.

An important caveat

- This argument doesn't actually show that the modular Hamiltonian **does** satisfy the KMS condition; it says that if there **is** a Hamiltonian satisfying the KMS condition, then it must be the modular Hamiltonian.
- But because the modular Hamiltonian is always well defined, we can ask: does it always satisfy KMS?
- The answer is yes, but this proof is much harder than the one I've just sketched:

Summary

- If you want to apply thermodynamic reasoning to a far-from-equilibrium state, you need an arrow of time with respect to which that state looks thermal.
- The only arrow of time that could work is modular flow.
- **Modular flow arises inevitably whenever you want to think thermodynamically about a far-from-equilibrium state; this is why it is so helpful for understanding entropy and energy in general states in QFT and quantum gravity.**