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Dimensional Reduction of the S^3/WZW Duality

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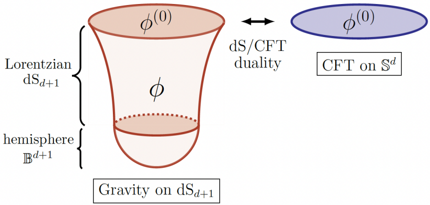
1. S^3 Einstein gravity/$SU(2)_k$ WZW duality

• [Hikida, Nishioka, Takayanagi & Taki '21] proposed an interesting duality between (classical) Einstein gravity on S^3 and (non-chiral) $SU(2)_k$ WZW model in the critical limit of the level $k \to -2$.

• Although their original motivation for this duality was to study the Lorentzian dS_3 gravity [c.f. Takayanagi's Talk], this Euclidean version of the duality provides an interesting example of a duality without a boundary.

$$\Psi_{\text{dS}}\left[\phi^{(0)}\right] = Z_{\text{CFT}}\left[\phi^{(0)}\right]$$



Einstein gravity on S^3

- For the 3D gravity on S^3, we consider

$$I_G = -\frac{1}{16\pi G_N}\int d^3x\sqrt{g_3}\,(R_3 - 2)$$

- The Euclidean black hole solution is given by

$$ds^2 = (r_0^2 - r^2)d\tau^2 + \frac{dr^2}{r_0^2 - r^2} + r^2 d\varphi^2$$

with the black hole inverse temperature

$$\tau \sim \tau + \beta\,, \qquad \beta = \frac{2\pi}{r_0}$$

- The (classical) on-shell action is given by

$$I_G^{\text{on-shell}} = -\frac{\pi\, r_0}{2G_N} \propto \text{Temperature}$$

The CFT side

- For the CFT side, we consider chiral and anti-chiral $SU(2)_k$ WZW models:

$$I_{\rm CFT}[g_+, g_-] = I^{(+)}_{\rm WZW}[g_+] - I^{(-)}_{\rm WZW}[g_-] \,,$$

with

$$I^{(\pm)}_{\rm WZW}[g_\pm] = \mp\frac{k}{2\pi}\int_{\partial M_3} d\tau d\varphi {\rm Tr}\big(g_\pm^{-1}\partial_\tau g_\pm g_\pm^{-1}\partial_\pm g_\pm\big) - \frac{k}{12\pi}\int_{M_3} {\rm Tr}\big(G_\pm^{-1} dG_\pm\big)^3$$

- The partition function is given by [Witten '89]

$$Z_{\rm CFT}(\beta) = \left(\frac{2}{k+2}\right)\left|\sin\left(\frac{\pi r_0}{k+2}\right)\right|^2$$

Critical limit $k \to -2$ in the $SU(2)_k$ WZW model

- If we consider the critical limit of the level [Hikida, Nishioka, Takayanagi & Taki '21]

$$k \to -2 + \frac{6i}{c^{(g)}} + \mathcal{O}\big((c^{(g)})^2\big)$$

with the central charge is identified as

$$c^{(g)} = \frac{3}{2G_N}$$

the partition function becomes

$$Z_{\mathrm{CFT}}(\beta) \approx \left|\frac{c^{(g)}}{3}\right|^{\frac{1}{2}} \exp\left[\frac{\pi c^{(g)} r_0}{3}\right]$$

2. Dimensional reduction (low temperature limit)

• We note that

$$I_G^{\text{on-shell}} = -\frac{\pi\, r_0}{2G_N} \propto \text{Temperature}$$

so the dynamics of the above duality can be completely captured in the low temperature limit.

• Also the near-horizon geometry $r \to r_0$ becomes $S^2 \times S^1$:

$$ds^2 \approx (r_0^2 - r^2)d\tau^2 + \frac{dr^2}{r_0^2 - r^2} + r_0^2\, d\varphi^2$$

• This reminds us about the NAdS$_2$/NCFT$_1$ story for near-extremal higher dimensional black holes.

• We try to formulate a similar reduction for this duality.

JT gravity on S^2 with NN^* boundary condition

- For the gravity side, we consider the following dimensional reduction ansatz

$$ds_3^2 = g_{\mu\nu}^{(2)}\, dx^\mu dx^\nu + \Phi^2 d\varphi^2$$

where $g_{\mu\nu}^{(2)}$ and Φ are functions of x^μ and independent of φ. With this ansatz, the 3D Ricci scalar is reduced as

$$R_3 = R_2 - 2\Phi^{-1}\Box_2\Phi$$

- Since there is no boundary term in the 3D action, we find

$$I_G = -\frac{1}{8G_N}\int d^2x\sqrt{g_2}\,\Phi\big(R_2 - 2\big) + \frac{1}{4G_N}\int_0^\beta d\tau\left[\sqrt{g_2}\, g_2^{rr}\,\partial_r\Phi\right]_{r=0}^{r=r_0}$$

This boundary term corresponds to the NN^* boundary condition, according to [Goel, Iliesiu, Kruthoff & Yang '20]

On-shell action in JT gravity

- The corresponding static background solution is given by

$$\begin{aligned} ds_2^2 &= (r_0^2 - r^2)d\tau^2 + \frac{dr^2}{r_0^2 - r^2} \\ \Phi &= r \end{aligned}$$

where we chose the coefficient of the dilaton solution to match with the 3D solution [c.f. Trivedi's Talk & Zenoni's Talk].

- Since $R_2 = 2$ on-shell, we have $I^{\text{on-shell}}_{\text{bulk}} = 0$. Then, the on-shell action is given by the boundary term as

$$I_G^{\text{on-shell}} = I^{\text{on-shell}}_{\text{bdy}} = -\frac{\pi\, r_0}{2G_N}$$

BF theory and higher spin extension

• We can formulate in terms of BF theory as

$$I_G = \frac{1}{2G_N}\int_{M_2} \mathrm{Tr}[bf] - \frac{1}{4G_N}\int_{M_2} d\,\mathrm{Tr}[ba]$$

where $f = da + a \wedge a$ and a, b are $SU(2)$ valued 1-form and 0-form, respectively.

• With this BF theory formalism, we can easily generalize to higher-spin theory by just replacing $SU(2)$ by $SU(N)$.

• This gives a dimensional reduced version of the Gaberdiel-Gopakumar duality.

Rewriting of WZW model to Alekseev-Shatashvili theory

• With AdS_3 boundary condition, it is known that the Hamiltonian reduction reduces the $SL(2,\mathbb{R})$ WZW model to a Liouville field theory [Coussaert, Henneaux & van Driel '95].

• However, [Cotler & Jensen '18] pointed out that this is not enough, and the correct reduction leads to the Alekseev-Shatashvili theory.

• Here, we assume a simple analytical continuation is possible from AdS_3, even though the boundary condition for our case is a little bit unclear.

$$I_{\mathrm{CFT}}[X,\bar{X}] = I_{\mathrm{AK}}^{(+)}[X] + I_{\mathrm{AK}}^{(-)}[\bar{X}]$$

with

$$I_{\mathrm{AK}}^{(\pm)}[X] = \frac{2k}{\pi}\int d\tau d\varphi \left[\frac{X''\partial_\pm X'}{X'^2} - \mu\, X'\partial_\pm X\right]$$

where $' = \partial_\tau$, $X, \bar{X}, \mu \in \mathbb{C}$ and $X \sim X + 2\pi$, $\bar{X} \sim \bar{X} + 2\pi$.

Dimensional reduction of Alekseev-Shatashvili theory

• As in the gravity side, we regard the field X (and $\bar{X}$) independent of the spacial coordinate φ. Further, if we introduce the Liouville variable by $X' = e^{\frac{i\phi}{2}}$, then we arrive at the Liouville QM:

$$I_{\rm LQM}[\phi] \,=\, I^{(+)}_{\rm AK}[X] \,=\, -k\int_0^\beta d\tau\left[\frac{1}{2}\,(\partial_\tau\phi)^2 + \,2\mu\,e^{i\phi}\right]$$

• If we promote the Liouville cosmological constant to a dynamical variable $\pi_f = -4ik^2\mu$, we can introduce (or recover) the $SU(2)$ gauge symmetry:

$$\hat{\ell}_{-1} \,=\, i\pi_f\,, \qquad \hat{\ell}_0 \,=\, -f\pi_f \,+\, i\pi_\phi\,, \qquad \hat{\ell}_{+1} \,=\, if^2\pi_f + 2f\pi_\phi + e^{i\phi}$$

and the Hamiltonian is the $SU(2)$ quadratic Casimir

$$H \,=\, \frac{1}{2k}\left(\hat{\ell}_0^2 + \frac{1}{2}\{\hat{\ell}_{-1},\hat{\ell}_{+1}\}\right) \,=\, \frac{1}{2k}\Big(-\pi_\phi^2 \,+\, i\pi_f\,e^{i\phi}\Big)$$

Derivation from Schwarzian theory

- We can also derive the Liouville QM from the Schwarzian theory:

$$L = C\,S(f;\tau) = C\left[\frac{f'''(\tau)}{f''(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2\right]$$

- Now we promote $SL(2;\mathbb{R})$ to $SL(2;\mathbb{C})$:

$$f(\tau) \to \frac{\alpha f(\tau)+\beta}{\gamma f(\tau)+\delta} \in \mathbb{C}\,, \qquad (\alpha,\beta,\gamma,\delta \in \mathbb{C})$$

and pick a particular direction, we get $SU(2)$:

$$\delta_- f(\tau) = i\varepsilon_-\,, \qquad \delta_0\, f(\tau) = -\varepsilon_0\, f(\tau)\,, \qquad \delta_+ f(\tau) = i\varepsilon_+\, f(\tau)^2\,,$$

- For the Lagrangian, we choose $f' = ie^{i\phi}$, then

$$L = \pi_\phi \phi' + \pi_f f' - \left(\frac{\pi_\phi^2}{2C} + i\pi_f e^{i\phi}\right)$$

Partition function in the Liouville QM

• Since the Hamiltonian is the $SU(2)$ quadratic Casimir, the partition function can be written in terms of the $SU(2)$ characters as

$$Z_j(\beta) = \mathrm{Tr}_{R_j}\left[e^{-\beta H}\right] = \chi_j\left(\frac{i\beta}{4\pi k}\right)$$

• Since we are interested in the low temperature limit $\beta \to \infty$, it is more useful to consider the S-modular transform of the character:

$$\chi_j(\tau) = \sum_\ell S_j{}^\ell\, \chi_\ell(-1/\tau)\,, \qquad S_j{}^\ell = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{\pi}{k+2}(j+1)(\ell+1)\right)$$

Critical limit $k \to -2$ in the Liouville QM

• Since

$$\chi_\ell(-1/\tau) = \chi_j\left(\frac{4\pi k i}{\beta}\right) = \mathrm{Tr}_{R_j}\left[e^{-\frac{8\pi^2 k H}{\beta}}\right]$$

the ground state dominates among the summation in the S-transform in the low temperature limit $\beta \to \infty$. Therefore, in the low temperature limit, the partition function is approximated by

$$Z_j(\beta) \approx S_j{}^0 = \sqrt{\frac{2}{k+2}}\,\sin\left(\frac{\pi r_0}{k+2}\right)$$

• This matches with the WZW result, so the critical limit $k \to -2 + 6i/c^{(g)}$ agrees with the JT (or BF) result.

• These results establish our proposed duality between S^2 JT gravity (or SU(2) BF theory) and $SU(2)$ gauge Liouviile QM.

Thank you!