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Complexity phase transitions in monitored random circuits

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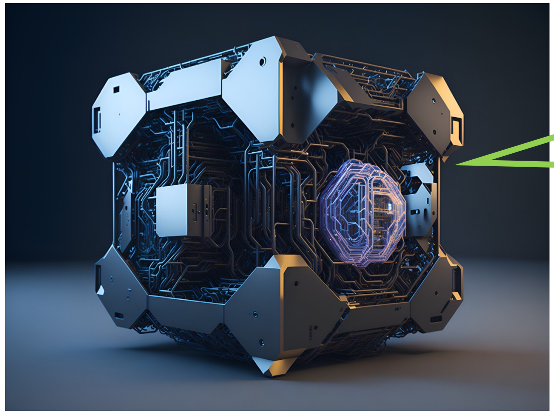
arXiv:2305.15475

Outline

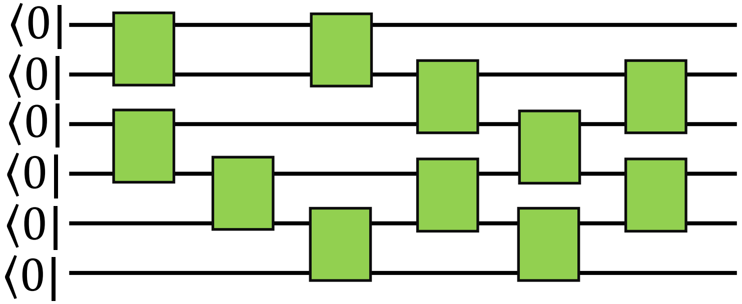
- Introduction
  - Quantum complexity
  - Monitored quantum circuits
- Complexity phase transition in monitored random circuits
  - Setting
  - Results
- Proof sketch

Complexity of quantum state

Assume that we have a fault-tolerant QC...



$$|\psi\rangle =$$



solving problem, Hamiltonian simulation...

Prepare a quantum state and measure observables

Basic question

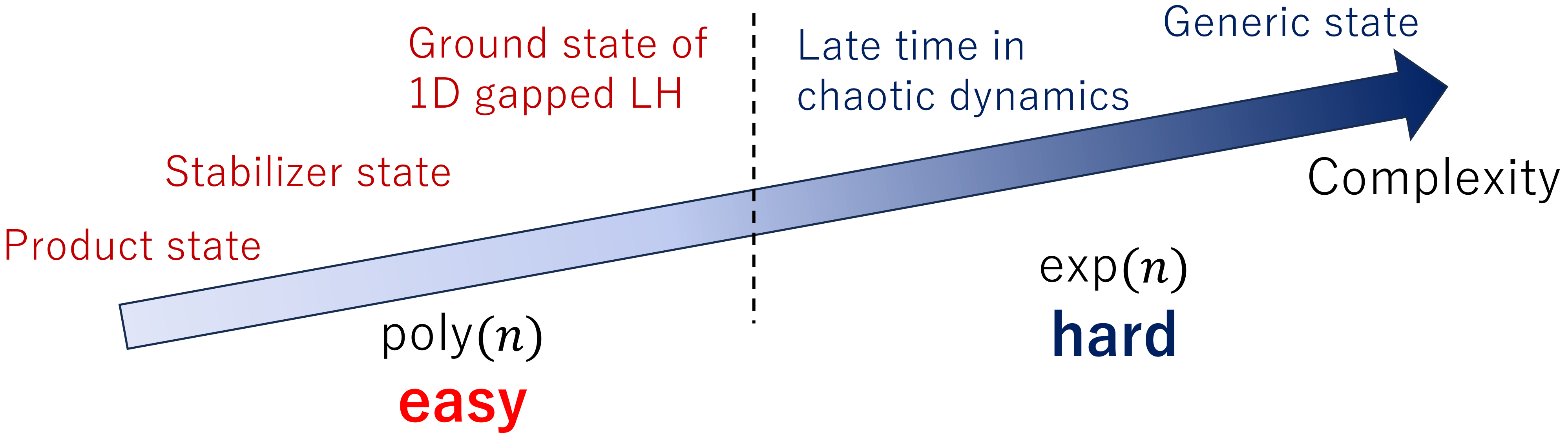
What kind of quantum states are easy (hard) to prepare by quantum computer?

Complexity of quantum state

Quantum state complexity $C(|\psi\rangle)$ of n-qubit state $|\psi\rangle$

$C(|\psi\rangle)$ is minimum number of two-qubit gates to generate $|\psi\rangle$ by a unitary circuit from $|0^n\rangle$ (approximately or exactly).

- ✓ Clear operational meaning
- ✓ Classical counterpart (circuit complexity) is one of the most pervasive topics in CS.
- ✓ Def. of topological ordered states

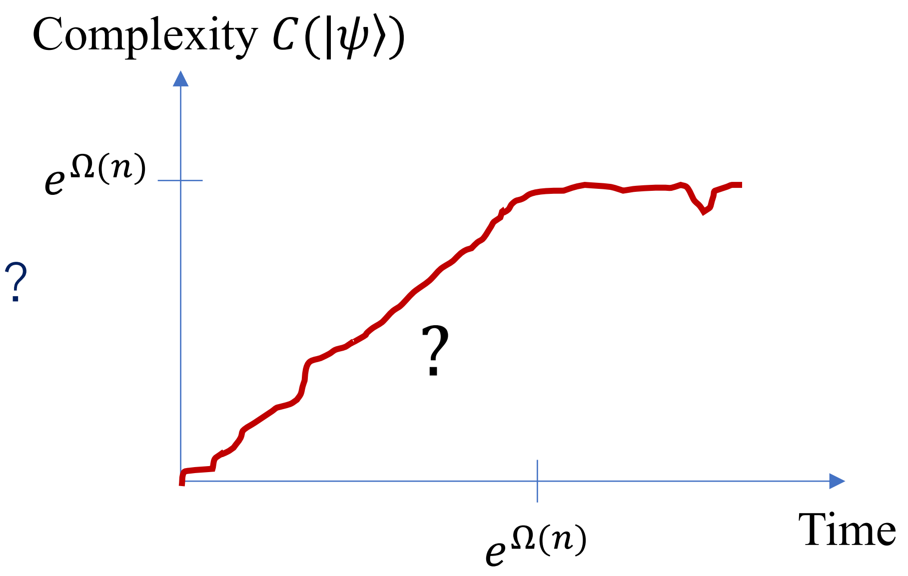


Complexity growth: Brown & Susskind conjecture

How do typical local dynamics produce the state complexity?

$$U = \int_0^t e^{-iH\tau} d\tau$$

$|\psi(t=0)\rangle \rightarrow |\psi(t)\rangle$

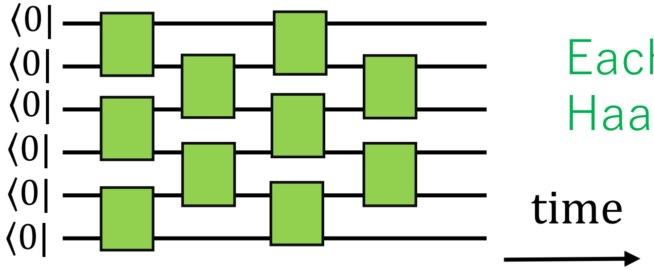


Complexity should grow linearly until an exponential time?

[Brown, Susskind]

In contrast to local observables and entanglement entropy, which saturate quickly.

Toy model: Local random circuits



Each unitary gate is Haar random: $U \in U(4)$

Sublinear growth of approximate state complexity

[Brandão, Harrow, Horodecki 2012] [Roberts, Yoshida 2016] [Brandão, Chemissany, Hunter-Jones, Küng, Preskill 2019] [Oszmaniec, Horodecki, Hunter-Jones 2022]

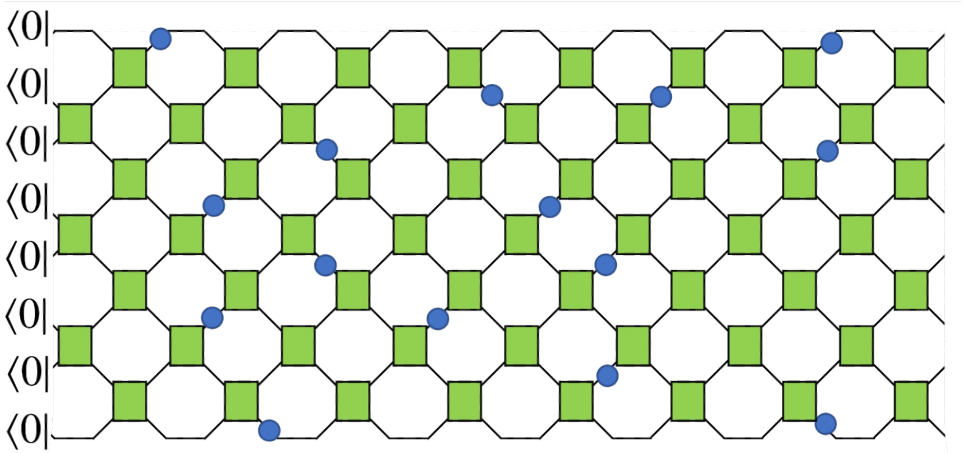
Linear growth of exact state complexity

[Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern, 2022]

Monitored quantum circuits

[Li-Chen-Fisher; Skinner-Ruhman-Nahum (2018)]

Unitary circuit + Measurements: Toy model of entanglement phase transition

$|\psi\rangle =$ 

● : 1-qubit measurement in the computational basis $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$ with **rate *p***

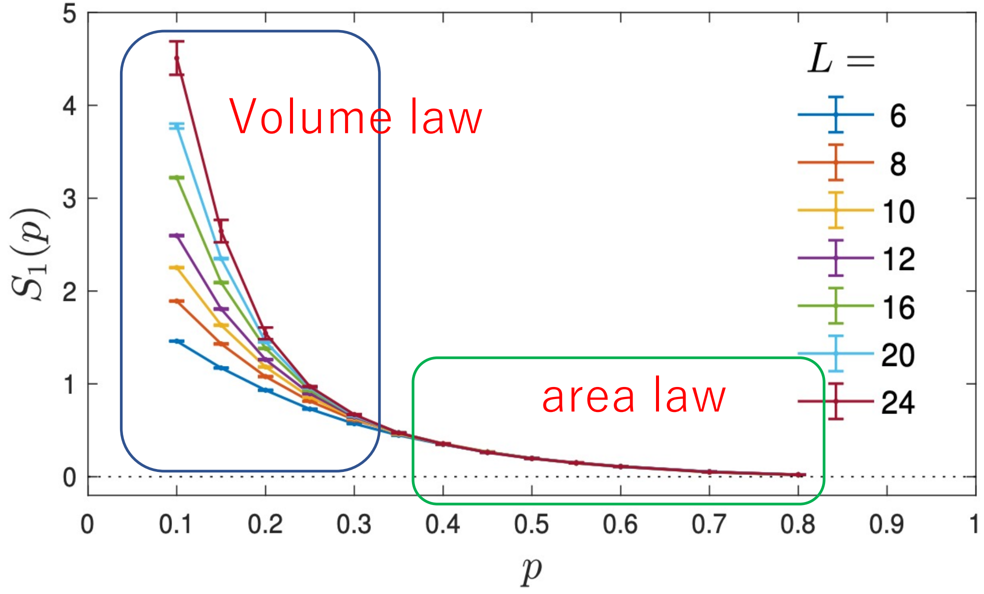
A Entanglement entropy of the subsystem?

After applying unitary gates, measure individual qubit at a probability p.

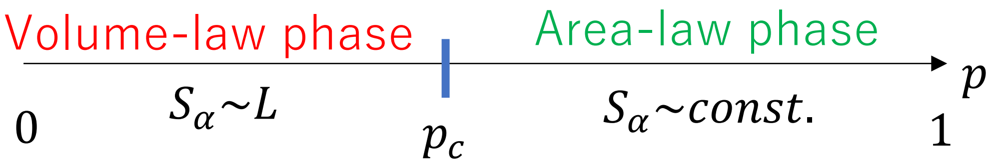
Probability of getting specific measurement outcomes follows the born rule: $\mathrm{p_M} = \langle\psi|\psi\rangle$

Monitored quantum circuits

Measurement-induced phase transition



$$S_\alpha(A) = \frac{1}{1-n} \log_2 \mathrm{Tr}(\rho_A^\alpha)$$



For Haar random gates, $\alpha = 0;\ p_c = 0.5$
$\alpha > 0;\ p_c \in [0.2, 0.35]$

[Li-Chen-Fisher; Skinner-Ruhman-Nahum; Bao-Choi-Altoman]

How about dynamics of quantum complexity in monitored circuit?

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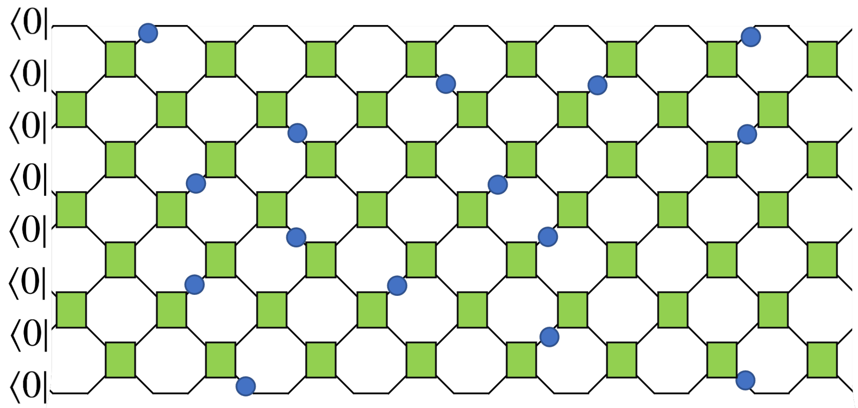
Our setting – Complexity dynamics of MRCs

Monitored random circuit

$|\psi\rangle =$



: 2-qubit unitary gate (Haar random)

: 1-qubit projector in the computational basis $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$

C_0-complexity

Min # of two-qubit gates to prepare $|\psi\rangle$ exactly by a **unitary circuit**.

C_m-complexity

Min # of two-qubit gates to prepare $|\psi\rangle$ exactly by a **post-selected quantum circuit**.

In generall, $C_0 > C_m$.

Q. What is the complexities of the output state, depending on t, p?

Results: Complexity phase transitions

Complex phase

$p < p_c = 0.5$:

$C_0(|\psi\rangle) \geq \Omega(t)$
$C_m(|\psi\rangle) = \Theta(t)$

until saturating to $\exp(n)$

with probability $1 - e^{-\Omega(n)}$.

Uncomplex phase

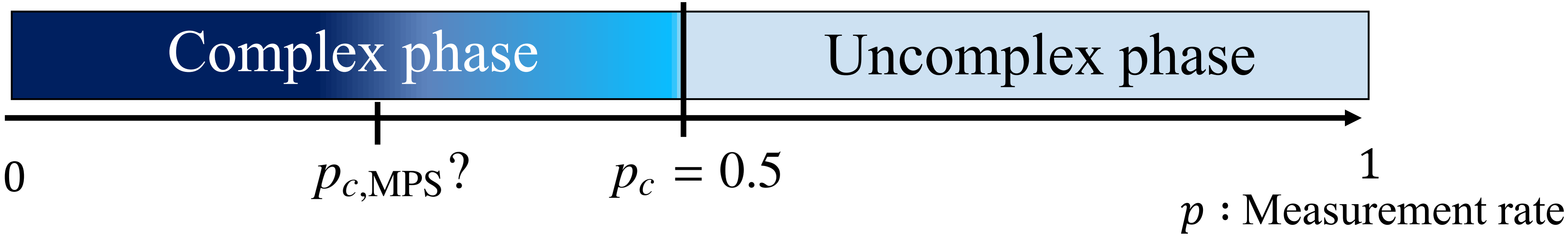
$p > p_c$:

$C_0(|\psi\rangle) \leq poly(n)$
$C_m(|\psi\rangle) \leq O(n\log n)$

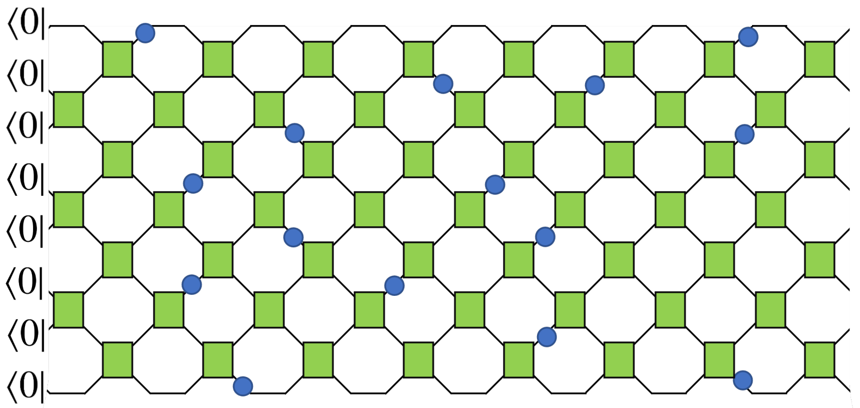
saturating in $\mathrm{t} < \mathrm{O}(\log(n))$

with probability $1 - e^{-\Omega(n)}$.

n: number of qubits



Remark and Implication

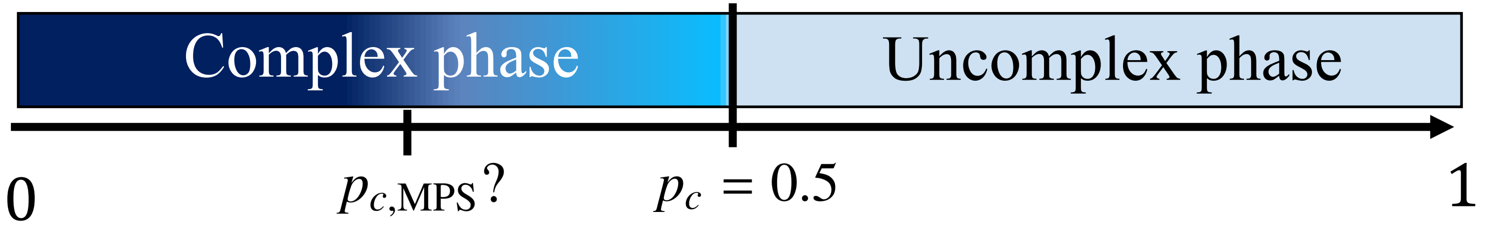


With a fixed measurement configuration,
the complexity is maximum over all unitary gates
except for a measure zero set.

⇒ Typical monitored dynamics undergo the complexity phase transition.

Previous numerical results show that above $p \approx 0.3$, output states satisfy area law.
[Bao-Choi-Altoman (2018)]

⇒ Described by MPS efficiently, and hence $\mathrm{poly}(n)$ approximate complexity.



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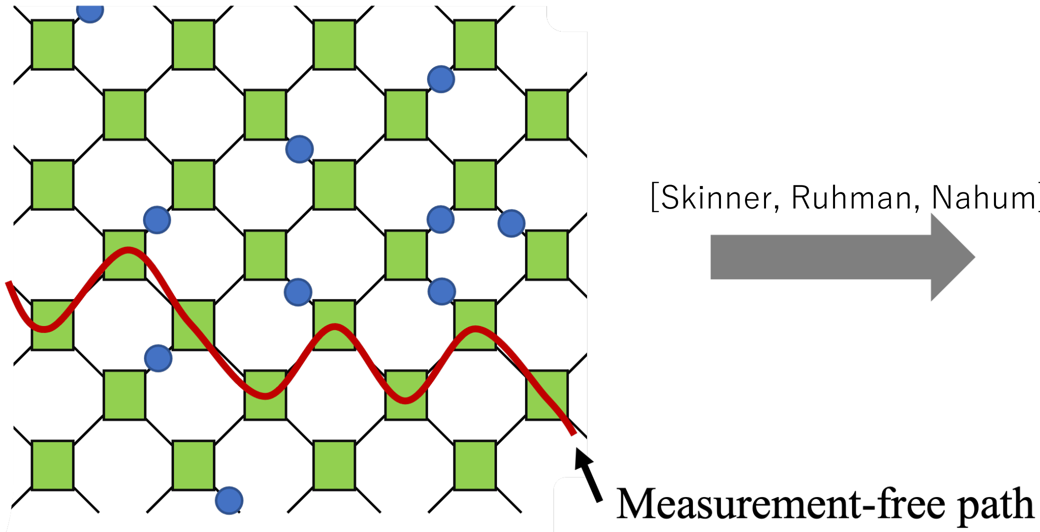
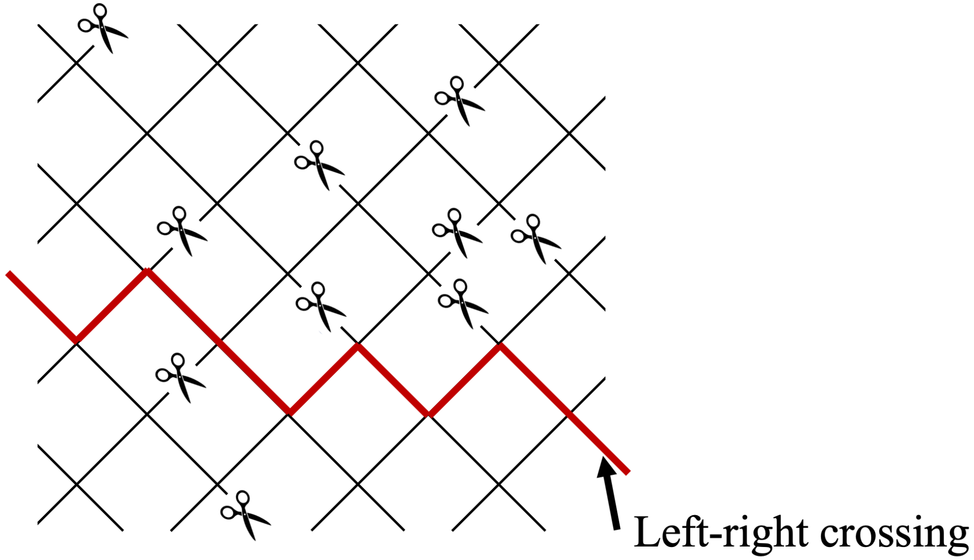
Mapping to percolation model



Monitored random circuit

[Skinner, Ruhman, Nahum]

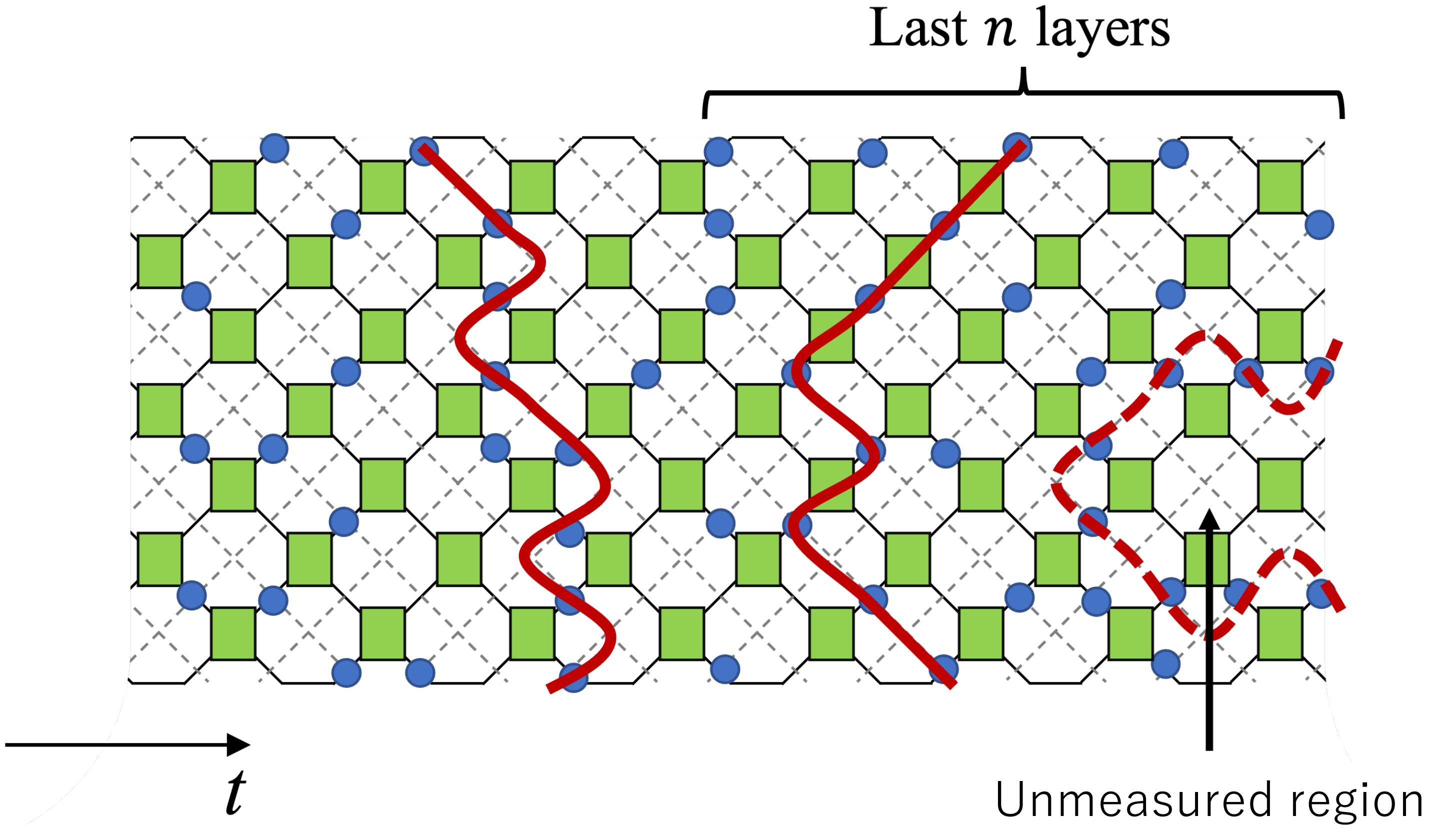


Percolation model

$p > p_c$: No measurement-free paths

$p < p_c$: Embedding a unitary circuit to monitored circuit

Proof sketch: Uncomplex phase $p > p_c$



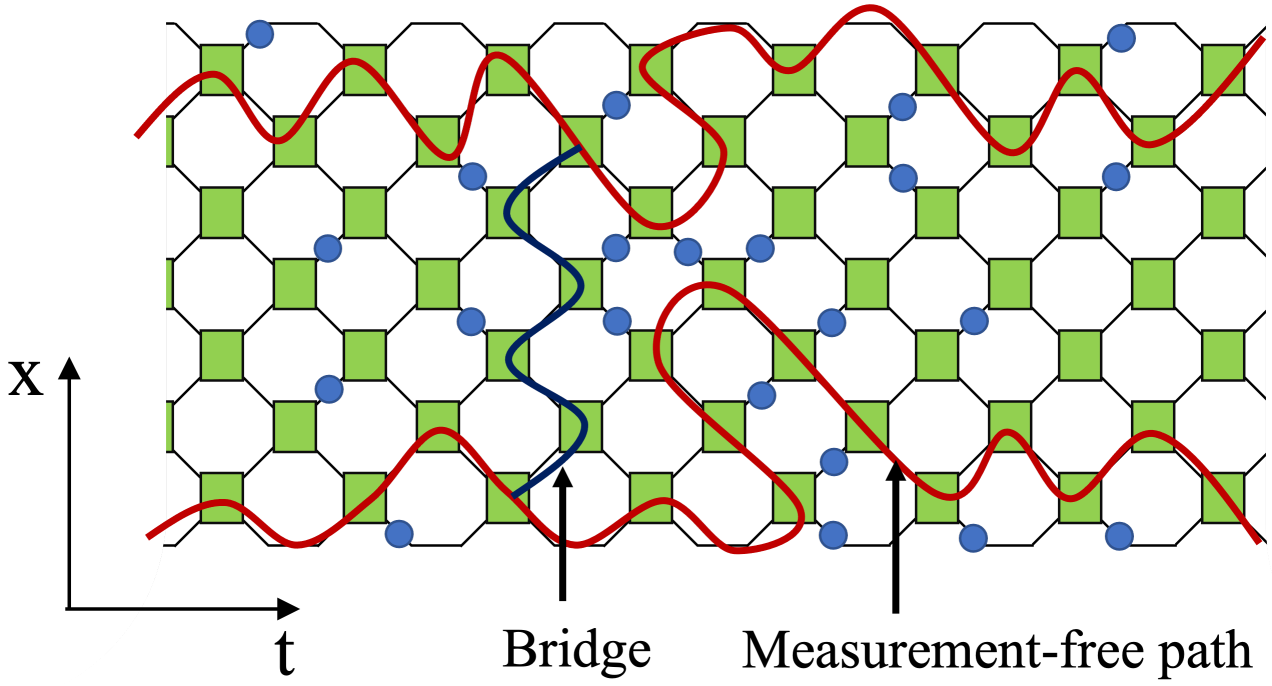
Measurements reset to $|0^n\rangle$.

Bound a size of an unmeasured region by $\log(n)$

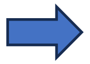
$C_0(|\psi\rangle) \le poly(n)$

$C_m(|\psi\rangle) \le O(n\log n)$

Proof sketch: Complex phase $p < p_c$



$\Theta(n)$ measurement-free paths



Unitary circuit
with $\Theta(n)$ qubits

Strategy:

- Embed a unitary circuit to a monitored circuit

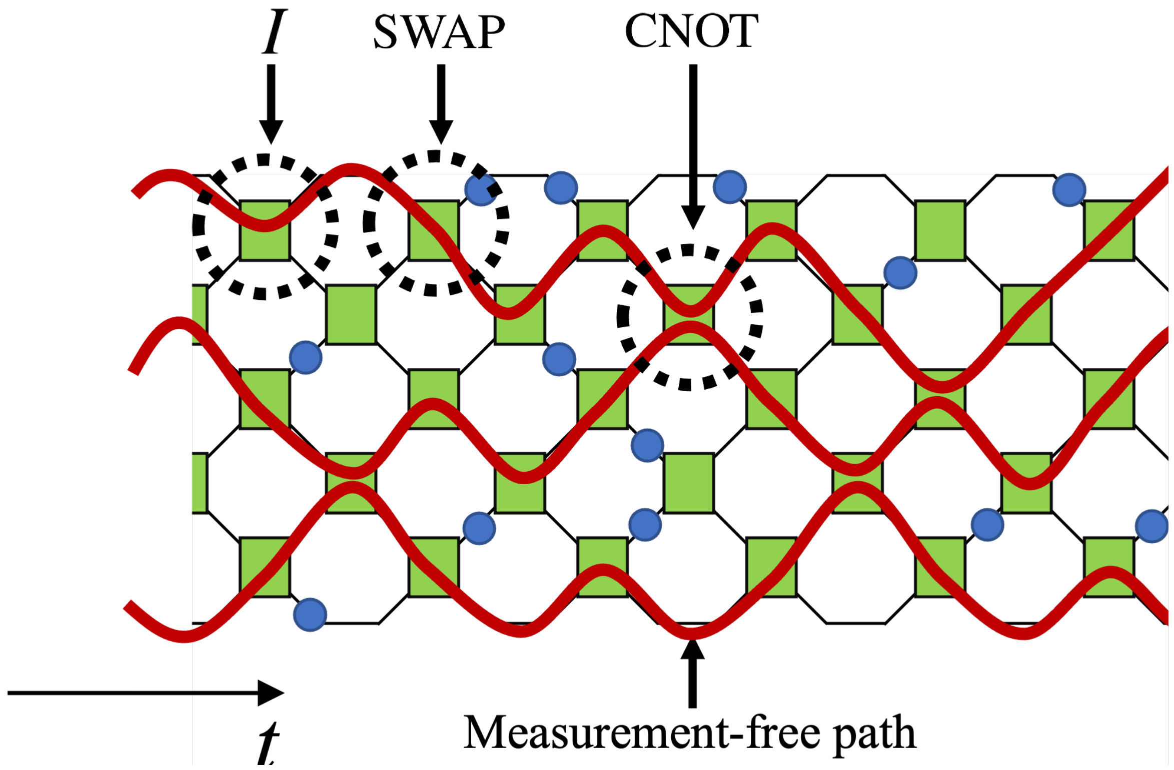
(cf. non-dynamical case: [Browne et. al. 2008])

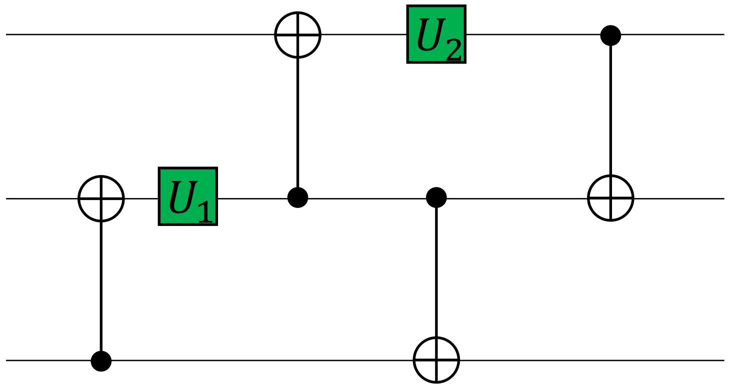
- Lower-bound the complexity by dimension counting arguments

[Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern, 2022]

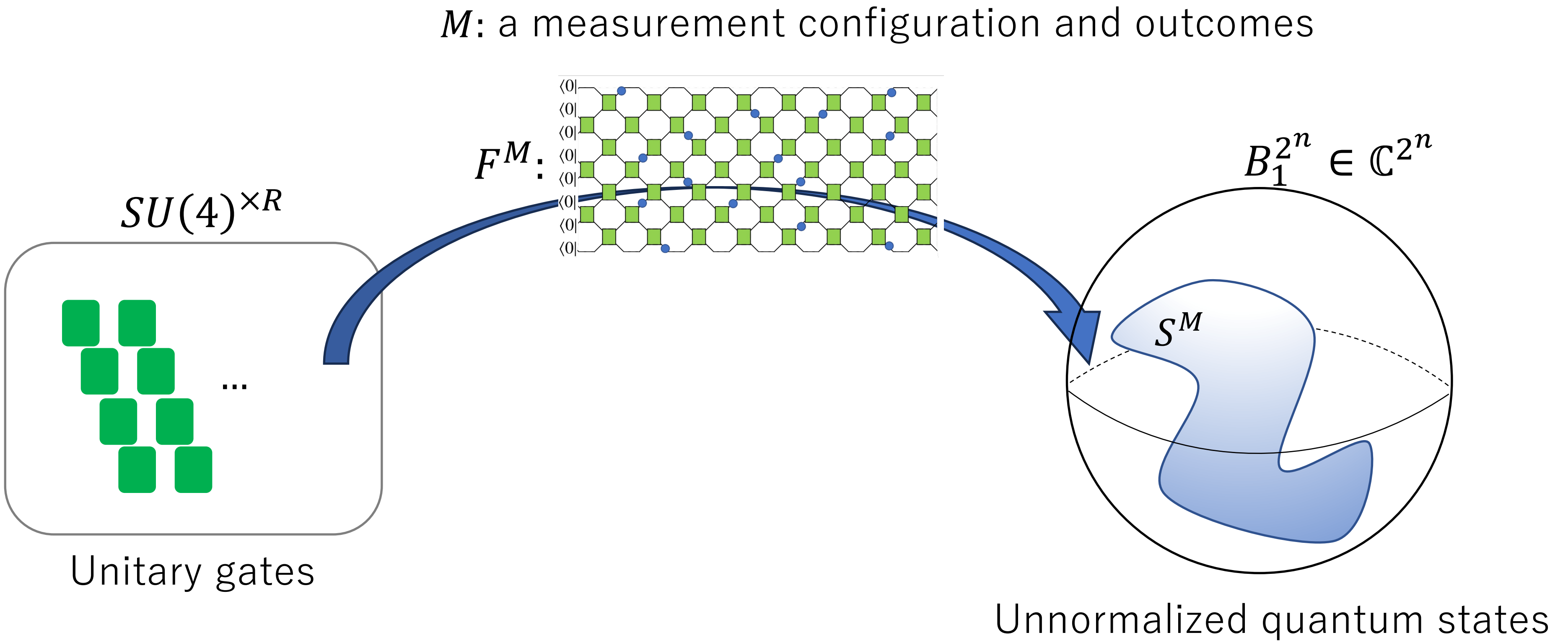
Embedding a unitary circuit: simple case

- Measurement-free paths are “causal”, i.e. not changing the time direction
- Paths share unitary gates





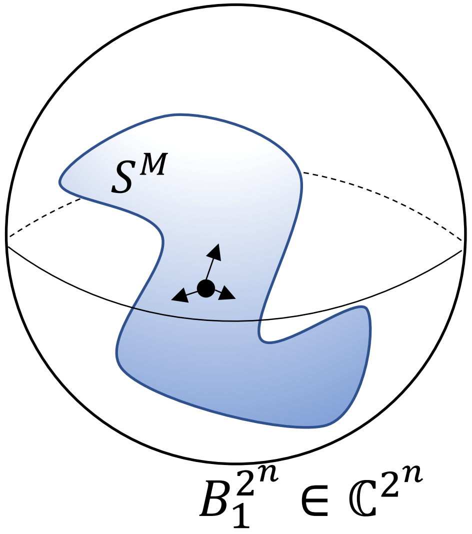
Complexity by dimension



The larger the set is, the more complex the states are.

Unitary circuit case [Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern, 2022]

Complexity by dimension



Lem. Lower bound on complexity

For a quantum state $|\psi\rangle \in S^M$,

$C_m(|\psi\rangle) \geq \Omega(\dim S^M)$ with unit probability.

$\dim(S^M)$: Accessible dimension

Lowerbound on the accessible dimension in unitary case

$$\dim(S^0) \geq \Omega(t)$$

[Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern, 2022]

+ Embedding unitary scheme ⇒

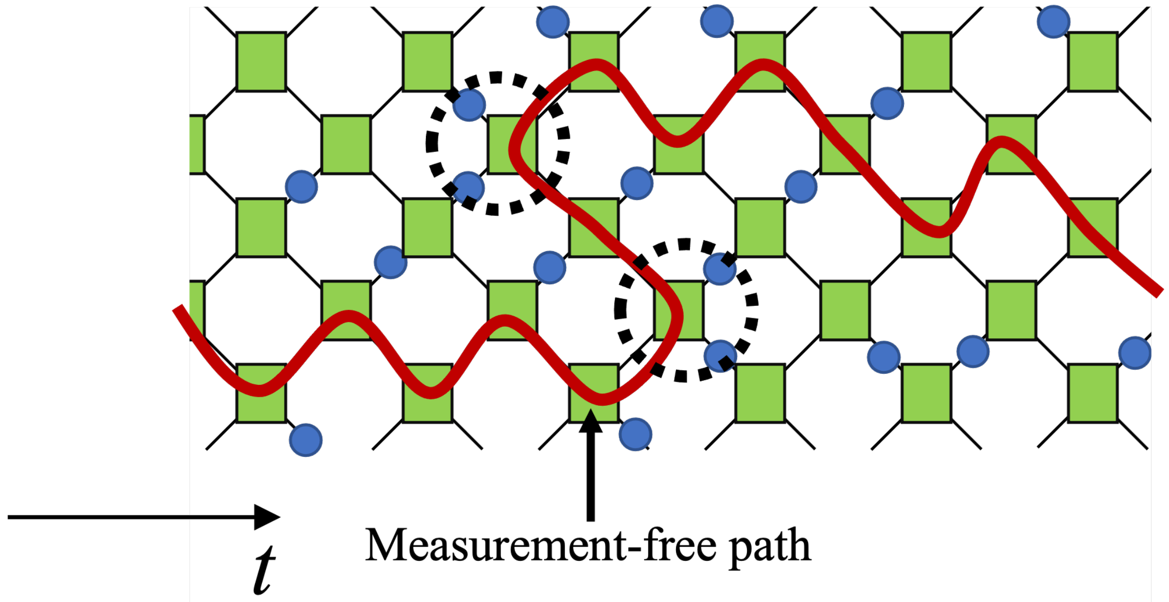
$$\dim(S^M) \geq \Omega(t)$$

$$C_m(|\psi\rangle) \geq \Omega(t)$$

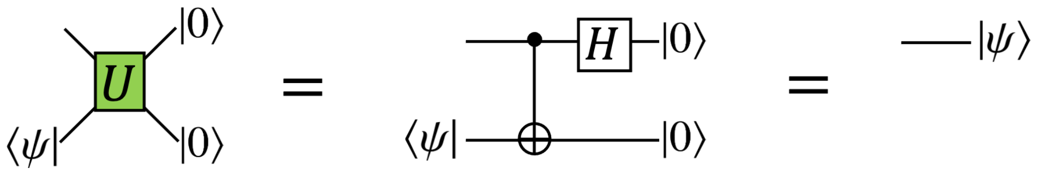
Embedding a unitary circuit: general case

- Measurement-free paths are not “causal”

Assume there are measurements at which the path changes the time direction



Entanglement teleportation-like scheme helps the embedding:



Adding measurements scheme

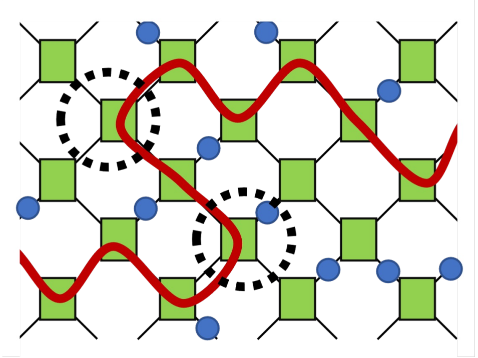
What if there are no measurements at the desired locations…?

→ We can add measurements there!

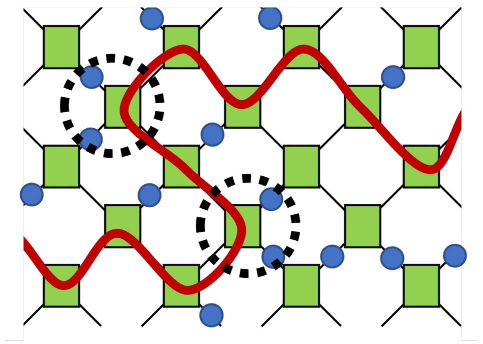
Lemma (Measurement cannot increase the accessible dimension)

M' is a measurement configuration by adding measurements to M.

$$\dim S^{M} \geq \dim S^{M'}$$



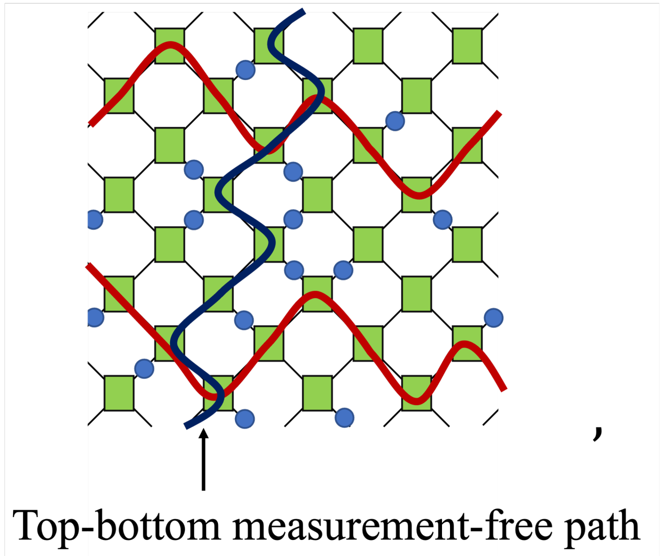
Measurement configuration M

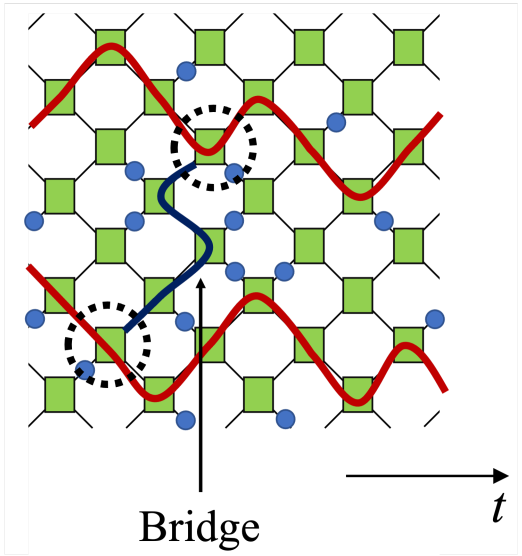


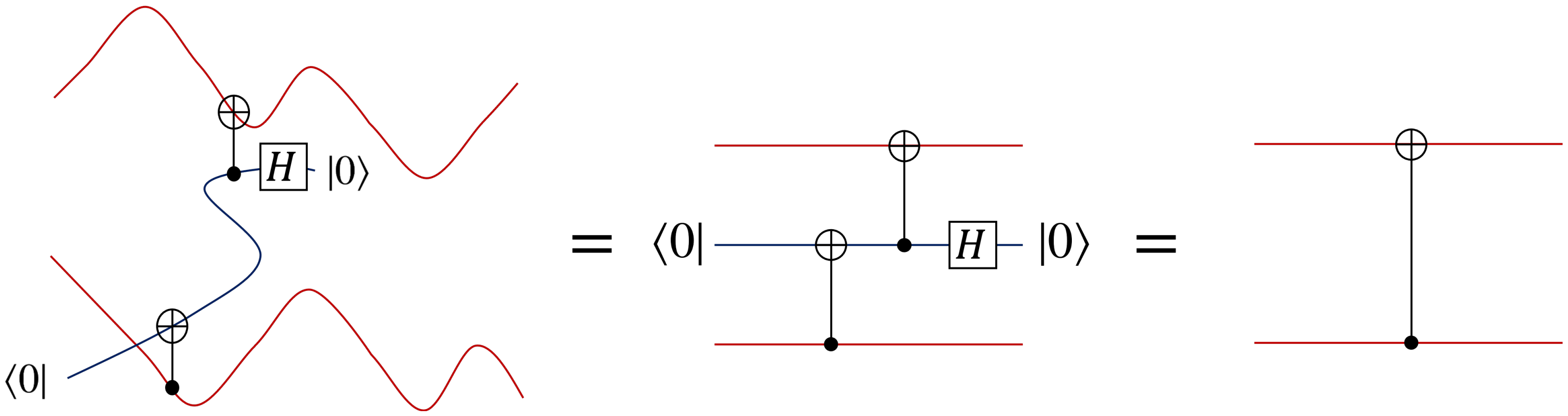
Measurement configuration M'

Sufficient to lower-bound $\dim S^{M'}$ instead of $\dim S^{M}$.

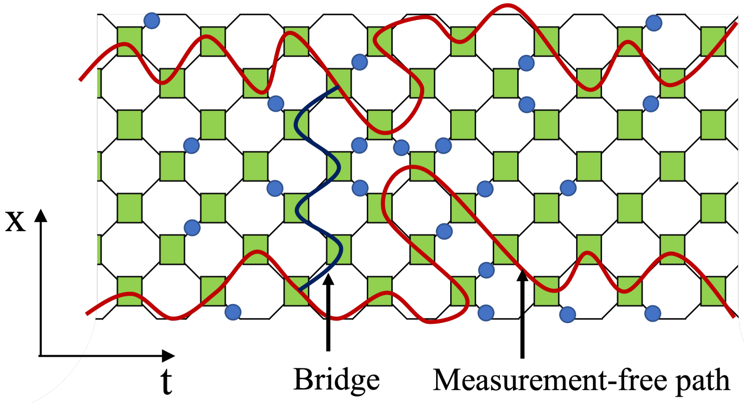
Effective two-qubit gate between paths



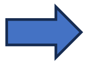




Proof sketch: Complex phase $p < p_c$



$\Theta(n)$ measurement-free paths



Unitary circuit
with $\Theta(n)$ qubits

✓ Lower-bound the complexity by dimension counting arguments

$C_m \geq \dim S^M \geq \Omega(t)$

✓ Embed a unitary circuit to a monitored circuit

$$C_0(|\psi\rangle) \geq \Omega(t)$$
$$C_m(|\psi\rangle) = \Theta(t)$$

until saturating to $\exp(n)$

Summary

- State complexity of Monitored random circuit undergoes a phase transition

- Phase transition of a more robust state complexity?
- Critical phenomenon?
- Super linear growth of complexity in monitored circuit?
- Applications of the accessible dimension to quantum computing?