Random Matrices and Hydrodynamics

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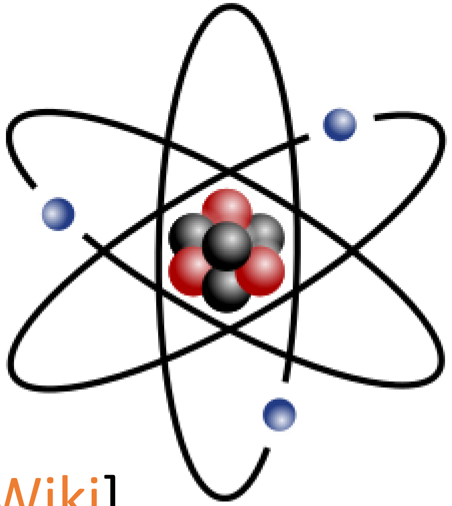
Extreme Universe Colloquium

October 2, 2023

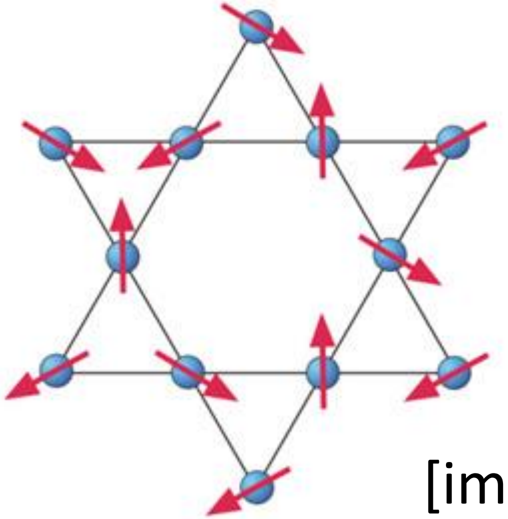
It from Qubit
Simons Collaboration on
Quantum Fields, Gravity and Information

What do these have in common?

Atoms/Nuclei



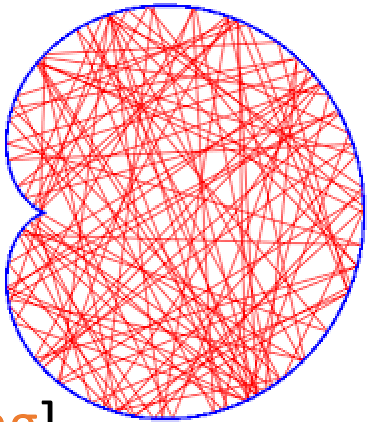
[image: Wiki]



Quantum spins

[image: R. Aguiar]

Billiards



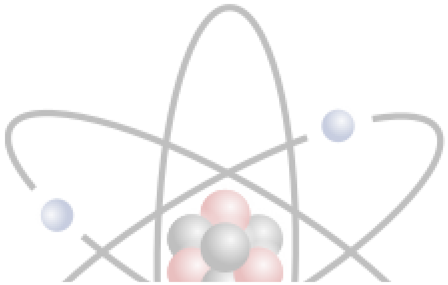
[image: E. Yeung]

Black holes

[image: NASA]

Random matrices

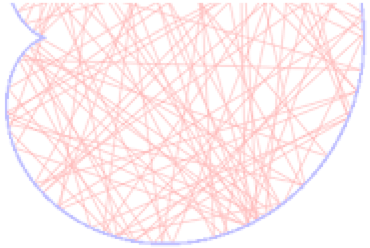
Atoms/Nuclei

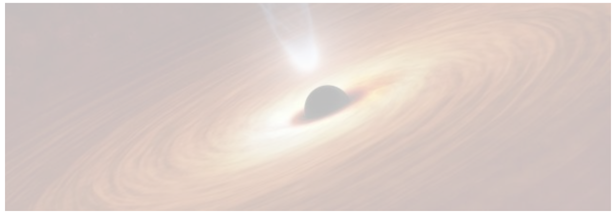


Quantum spins

$$
\begin{pmatrix}
0.8404 & -0.7442 & -1.0191 & -0.2102 & 1.6058 \\
-0.7442 & 0.4900 & -0.0501 & 1.5743 & 0.3155 \\
-1.0191 & -0.0501 & 1.3546 & -1.5165 & 1.1700 \\
-0.2102 & 1.5743 & -1.5165 & -0.1977 & -1.1330 \\
1.6058 & 0.3155 & 1.1700 & -1.1330 & -0.4686
\end{pmatrix}
$$

Billiards





Black holes

Hydrodynamics, construed broadly

Atoms/Nuclei

Quantum spins

Billiards

Black holes

[image: MIT ME]

Are these two widespread phenomena related?

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\end{pmatrix}
$$

Spectral form factor (SFF)

YES!

Total return probability (TRP)

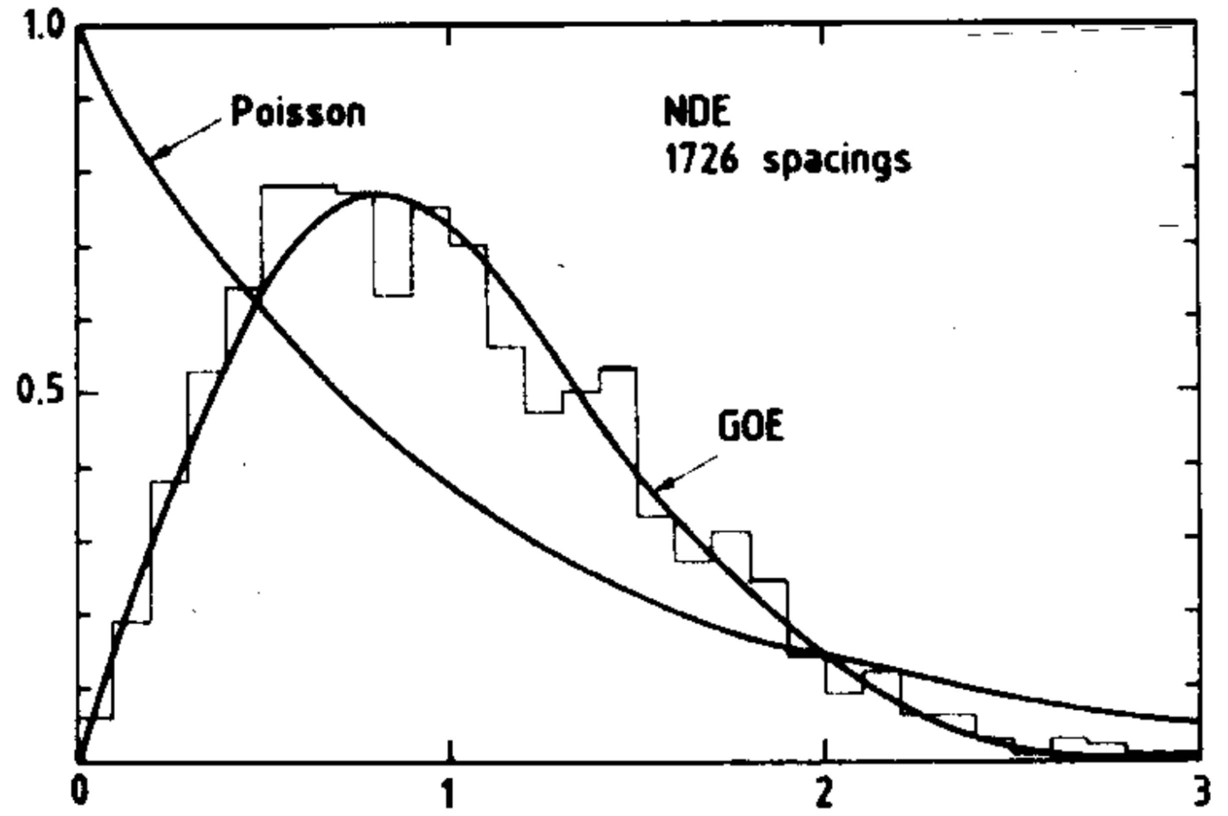
$$\mathrm{SFF}(T) = \mathrm{SFF}_{\mathrm{RMT}}(T) \times \mathrm{TRP}(T)$$

$$\begin{pmatrix} 0.8404 & -0.7442 & -1.0191 & -0.2102 & 1.6058 \\ -0.7442 & 0.4900 & -0.0501 & 1.5743 & 0.3155 \\ -1.0191 & -0.0501 & 1.3546 & -1.5165 & 1.1700 \\ -0.2102 & 1.5743 & -1.5165 & -0.1977 & -1.1330 \\ 1.6058 & 0.3155 & 1.1700 & -1.1330 & -0.4686 \end{pmatrix}$$

Random matrix theory (RMT) in physics

- Many complex quantum systems have an “unstructured” energy spectrum, especially far away from the ground state
- Wigner’s idea: we can model the spectrum of such “unstructured” systems using random* Hermitian matrices [Wigner, Dyson, Mehta, …]

Level spacings of different nuclei with the same spin/parity:



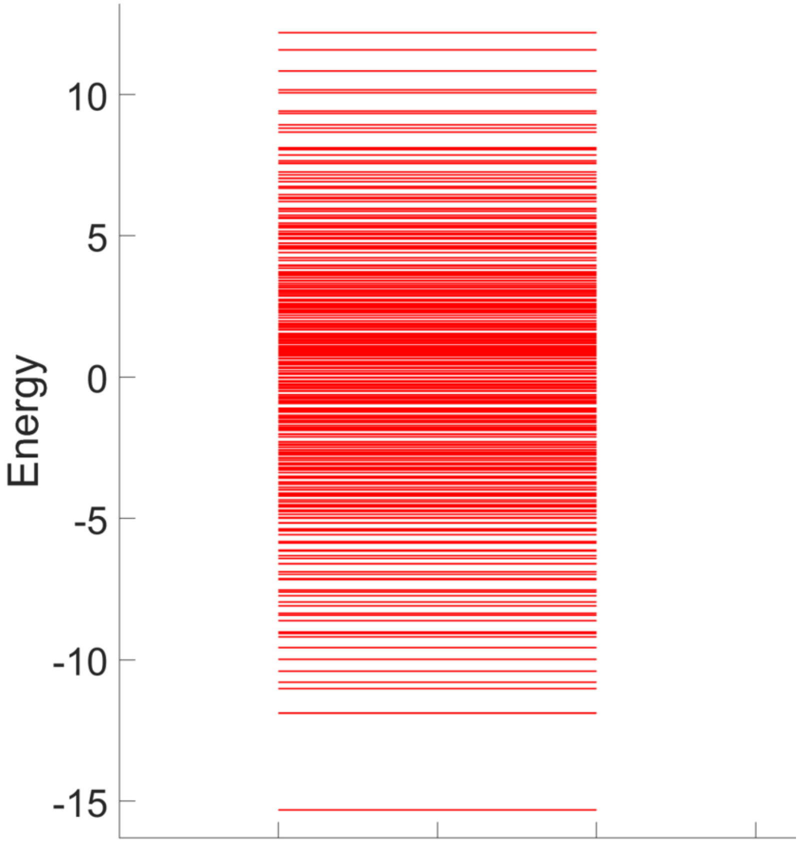
[Bohigas-Haq-Pandey]

*Many types of random matrices, but we’ll consider primarily Gaussian ensembles

Spectrum and statistics

$$H = -J\sum_r \sigma_r^z\sigma_{r+1}^z - h_x\sum_r \sigma_r^x - h_z\sum_r \sigma_r^z$$

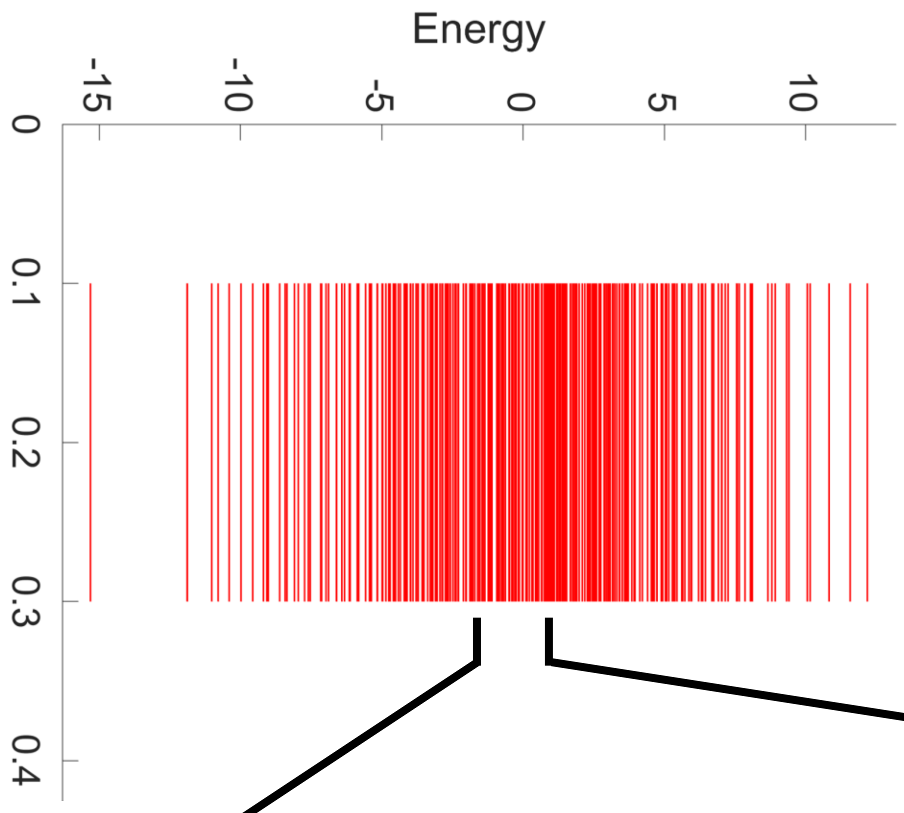
n=8 spins, 256 energy levels



Ensemble of Hamiltonians with random fields:

$$\Delta H = \sum_r \delta h_r \sigma_r^z$$

[chaos-RMT conjecture: Bohigas-Giannoni-Schmit]

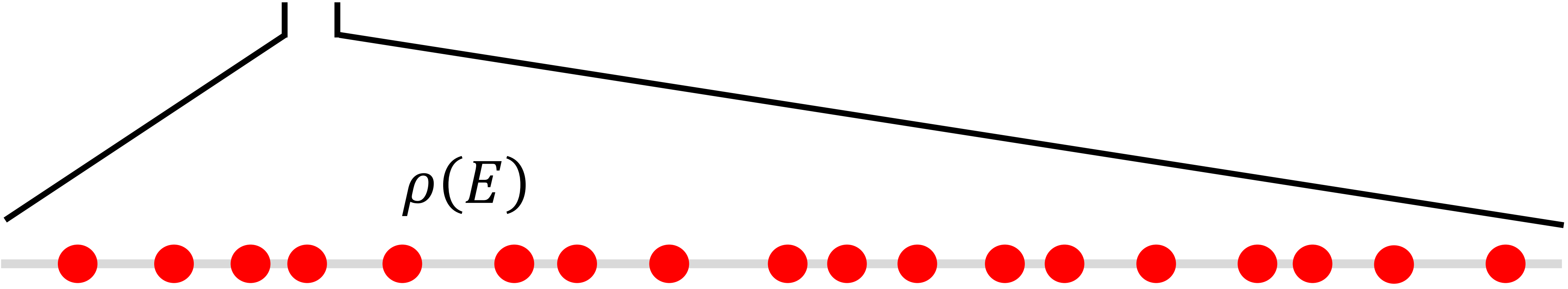


In a conventional liquid, we probe the particle positions using scattering, e.g. of neutrons

POSITION / MOMENTUM

In our eigenvalue liquid, we can also probe the energies by “scattering” (Fourier transform)

ENERGY / TIME



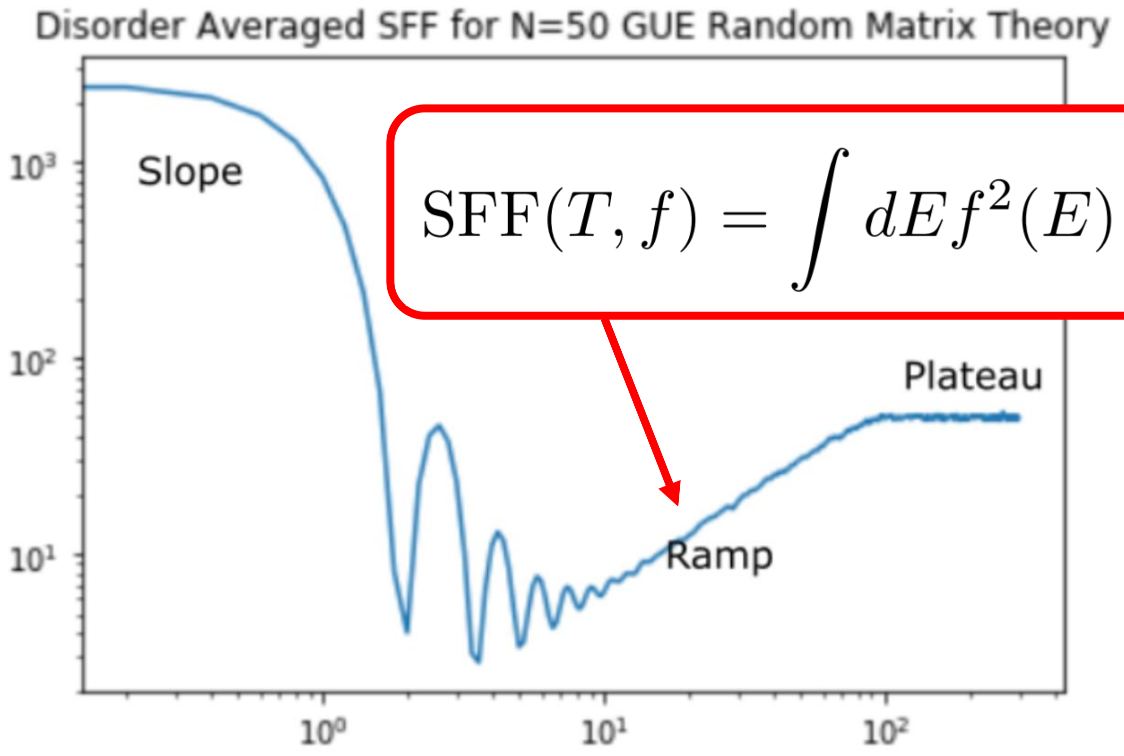
$$SFF(T) = \int dE \int d\epsilon\, e^{-i\epsilon T/\hbar} \overline{\rho\left(E+\frac{\epsilon}{2}\right)\rho\left(E-\frac{\epsilon}{2}\right)}$$

RMT and SFF

$$dP \propto \prod_{ij} dH_{ij} \exp\left(-\mathrm{Tr}[V(H)]\right)$$

$$dP = \frac{1}{\mathcal{Z}} \prod_{i<j} |E_i - E_j|^{\mathfrak{b}} \prod_i e^{-V(E_i)}$$

data: Dyson index and potential



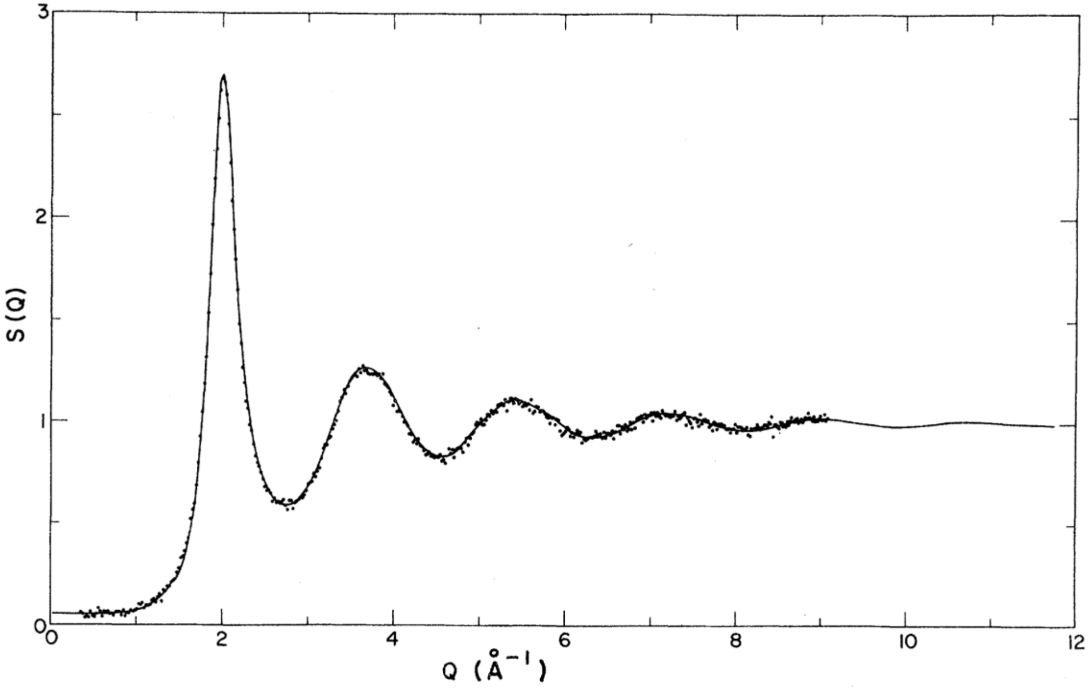
$$\mathrm{SFF}(T,f) = \int dE\, f^2(E) \frac{T}{\pi \mathfrak{b}}$$

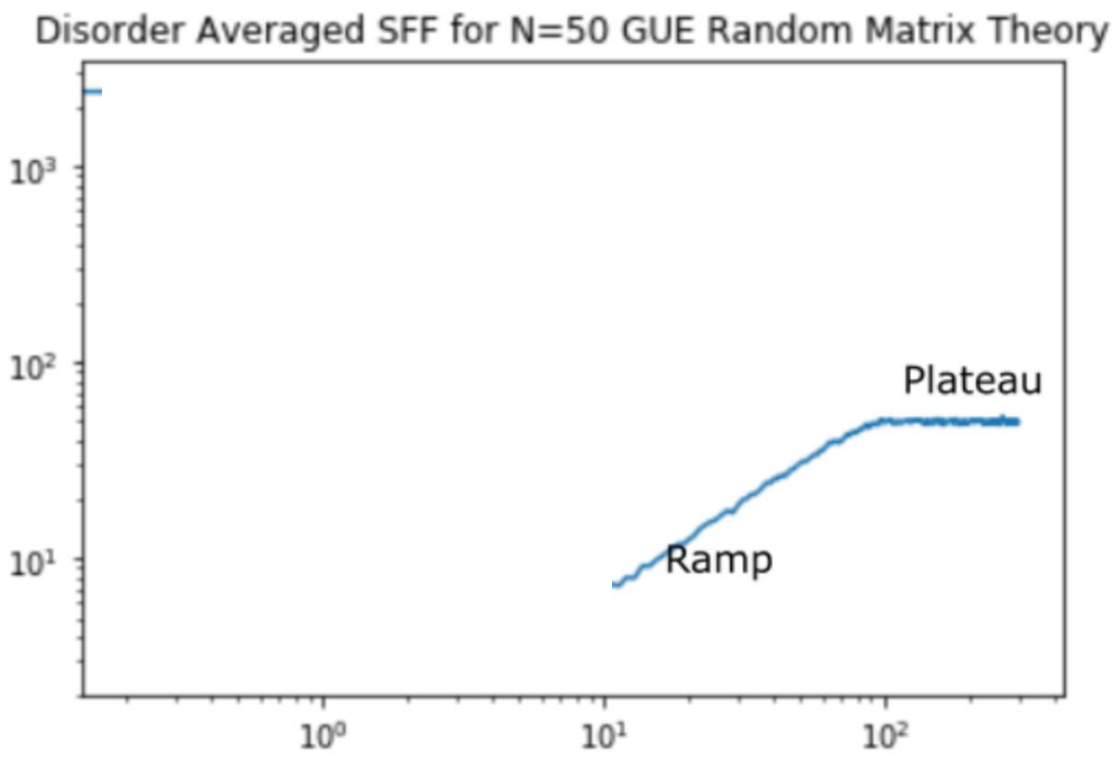
$$\mathrm{SFF}(T,f) = \overline{|\mathrm{Tr}[f(H)e^{iHT}]|^2} = \overline{\sum_{i,j} f(E_i) f(E_j) e^{i(E_i - E_j)T}}$$

f = filter function

[review: Haake]

For fun … liquid Argon vs liquid eigenvalues





[Yarnell-Katz-Wenzel-Koenig]

$$\mathrm{SFF}(T,f) = \overline{|\mathrm{Tr}[f(H)e^{iHT}]|^2} = \overline{\sum_{i,j} f(E_i)f(E_j)e^{i(E_i-E_j)T}}$$

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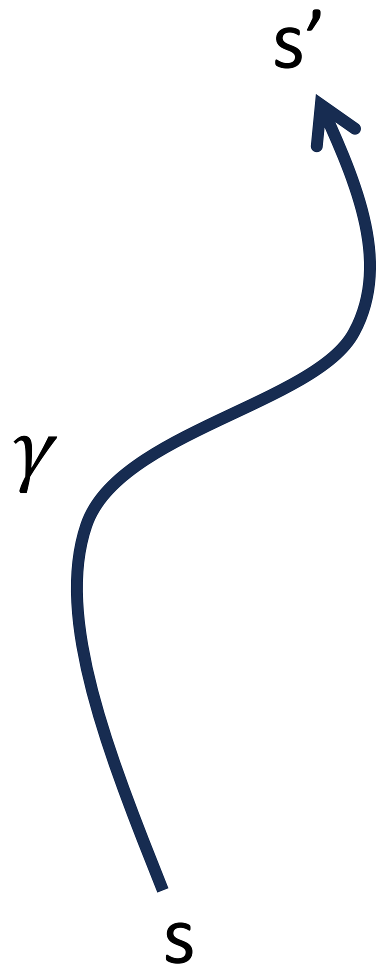
Quantum rules



The SFF can be written as a sum over return amplitudes, so let us recall the quantum rules for such amplitudes

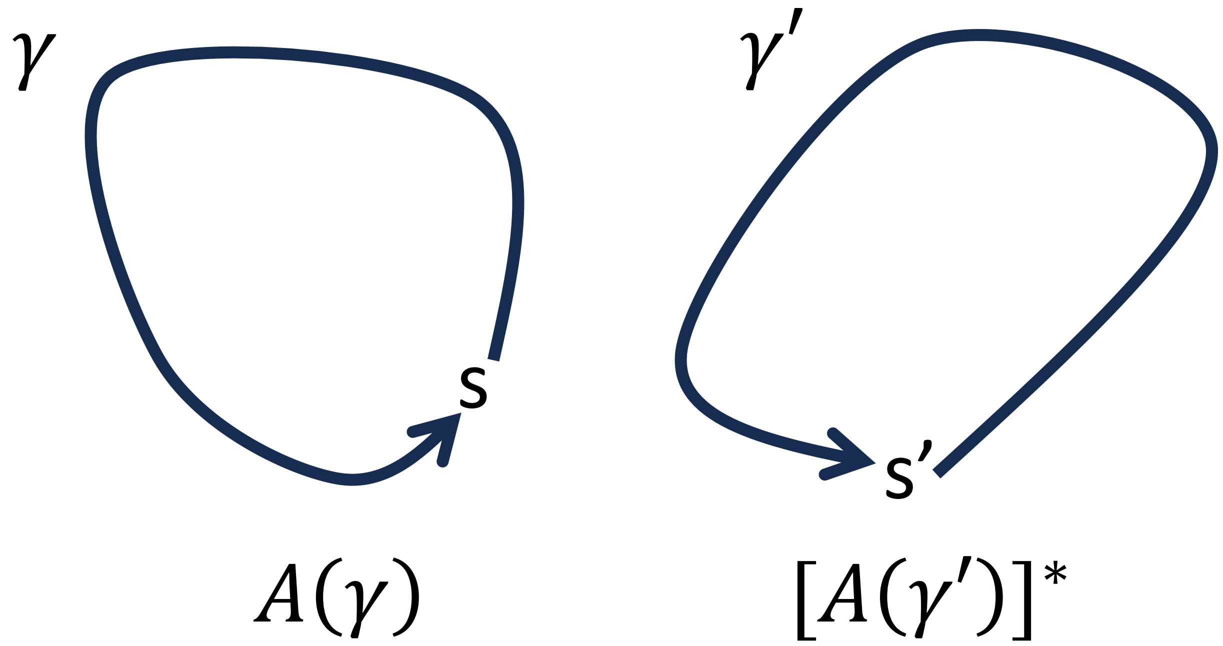
$$\mathrm{A}(\mathrm{s} \to \mathrm{s'}) = \mathrm{A}(\gamma) \subset \langle s|\exp(-iHt/\hbar)|s'\rangle$$

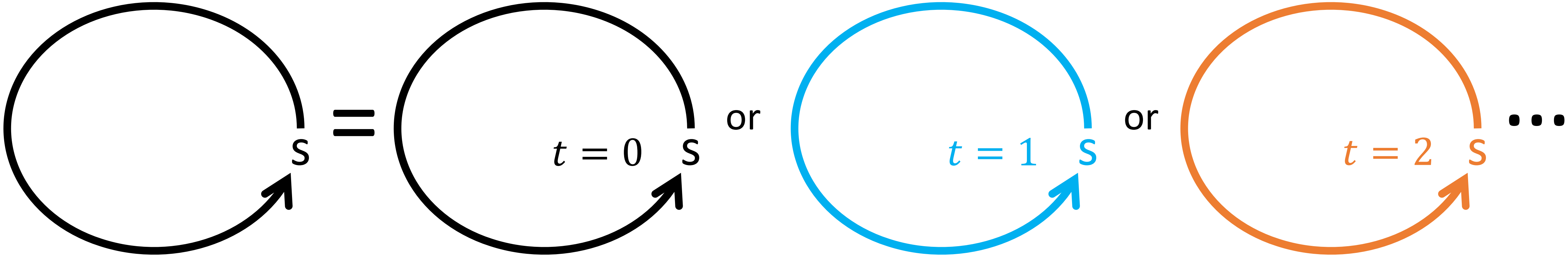
$$\mathrm{P}(\mathrm{s} \to \mathrm{s'}) = |\mathrm{A}(\mathrm{s} \to \mathrm{s'})|^2$$

Probabilities are squares of amplitudes

$$SFF = \sum_{\gamma,\gamma'} A(\gamma)[A(\gamma')]^*$$

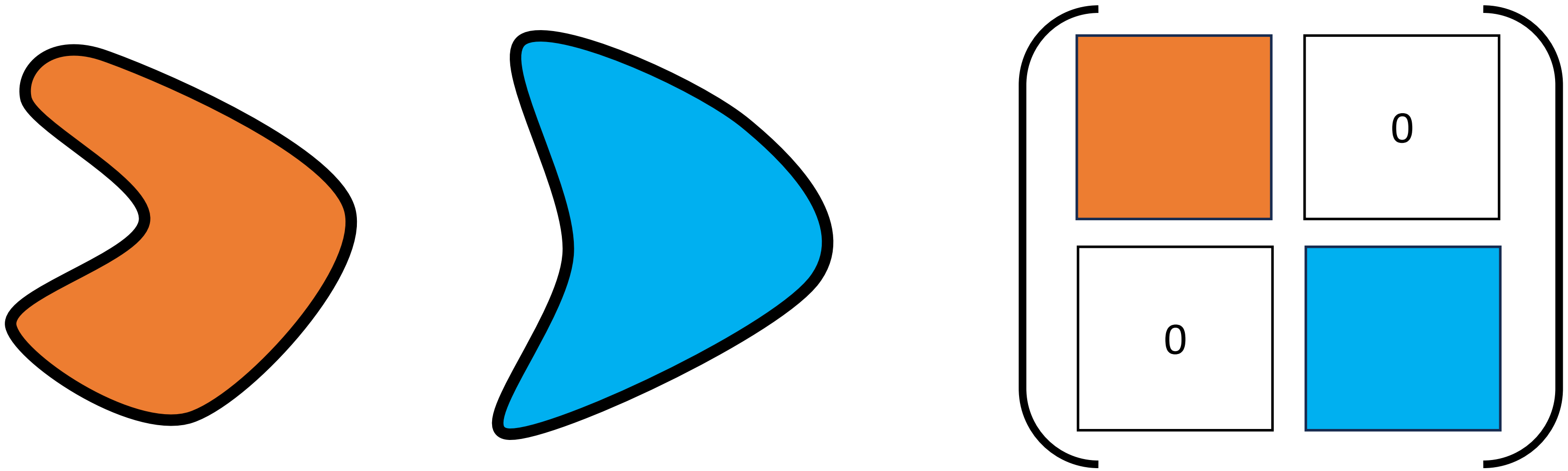
Terms in the SFF sum tend to destructively interfere unless the paths γ, γ' are the same



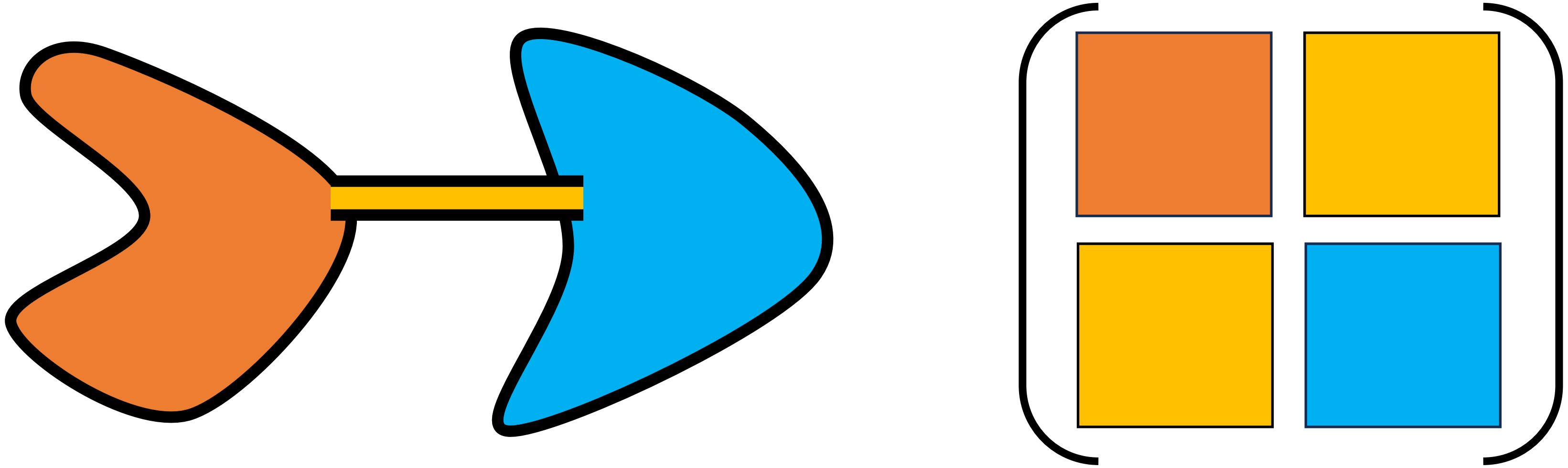


This residual freedom is responsible for the linear-in-T ramp

Toy example: coupled billiards



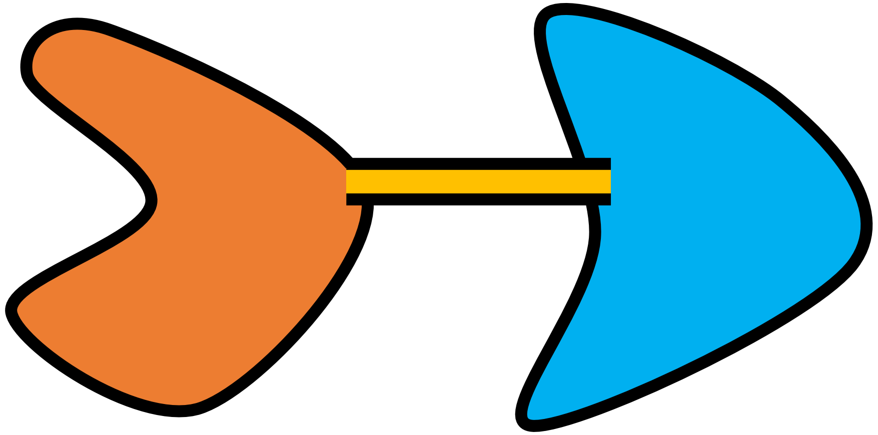
SFF = 2 x RMT

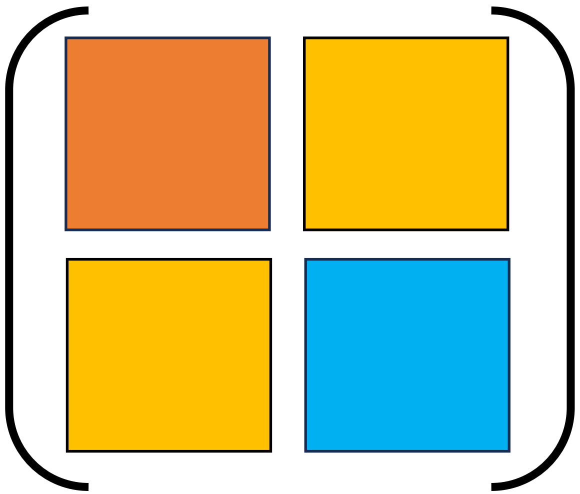


Short time:
SFF = 2 x RMT

Long time:
SFF = RMT

Toy example: coupled billiards





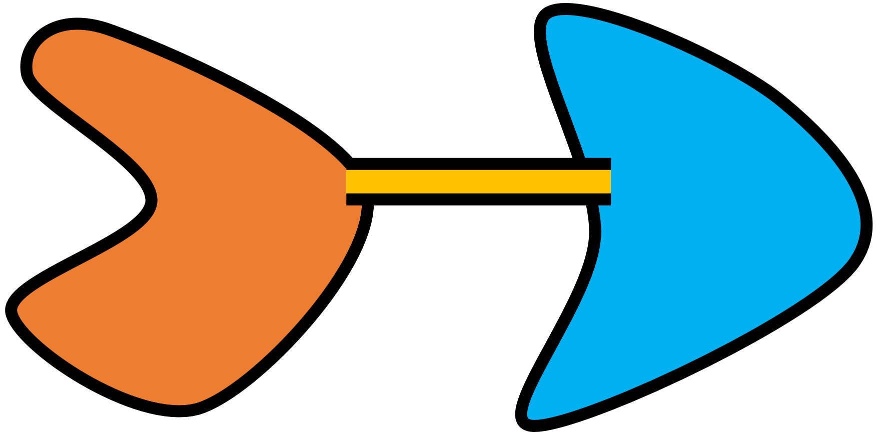
Short time:
SFF = 2 x RMT

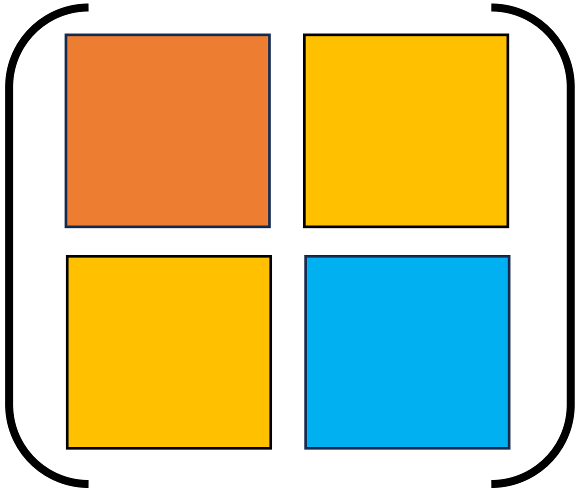
Long time:
SFF = RMT

$$\mathrm{SFF}(T,f) = \sum_{\alpha} f(E_\alpha)^2 p_{\alpha\to\alpha}(T)\mathrm{SFF}_\alpha(T)$$

[Winer-S]

Toy example: coupled billiards





Short time:
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Long time:
SFF = RMT

$$\mathrm{SFF}(T,f) = \underbrace{\sum_{\alpha} \qquad\qquad p_{\alpha\to\alpha}(T)}_{\mathrm{TRP}(T)} \; \mathrm{SFF}_{\mathrm{RMT}}(T)$$

[Winer-S]

Energy diffusion

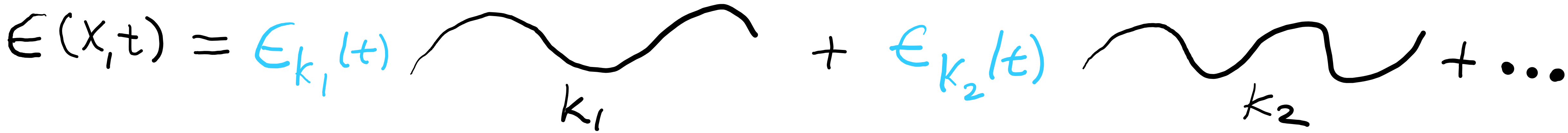
$$\partial_t \epsilon = D\nabla^2 \epsilon + \xi$$

diffusion + "noise" with appropriate correlations

- Imagine breaking all other symmetries: all that remains is energy diffusion → minimal slow dynamics in a local Hamiltonian system
- At time T, there are an extensive number of almost conserved modes:

$$k_T \sim \frac{1}{\sqrt{DT}} \qquad N_T \sim \sum_k \theta(k_T - |k|) \sim V \int \frac{d^d k}{(2\pi)^d} = \frac{V S_d}{(2\pi)^d} \frac{k_T^d}{d}$$

- If each sector is random matrix like, then the SFF should correspond to a sum of many almost-independent ramps → sectors are labelled by amplitudes of nearly-conserved energy fluctuations



Linear diffusion

$$p(\epsilon_{k,\mathrm{final}}, T) = \frac{\exp\left(-\frac{(\epsilon_{k,\mathrm{final}} - e^{-\gamma_k T}\epsilon_k)^2}{2\sigma^2(T)}\right)}{\sqrt{2\pi\sigma^2(T)}}$$

$$\partial_t \epsilon = D\nabla^2 \epsilon + \xi$$

$$\int d\epsilon_k p(\epsilon_{k,\mathrm{final}} = \epsilon_k, T) = \frac{1}{1 - e^{-\gamma_k T}}$$

$$\sum_\alpha p_{\alpha\to\alpha}(T) = \prod_k \frac{1}{1 - e^{-Dk^2 T}} = \exp\left(V\left(\frac{1}{4\pi D T}\right)^{d/2} \zeta(1 + d/2)\right)$$

exclude zero mode, quasi-continuous wavevector regime

TRP(T)

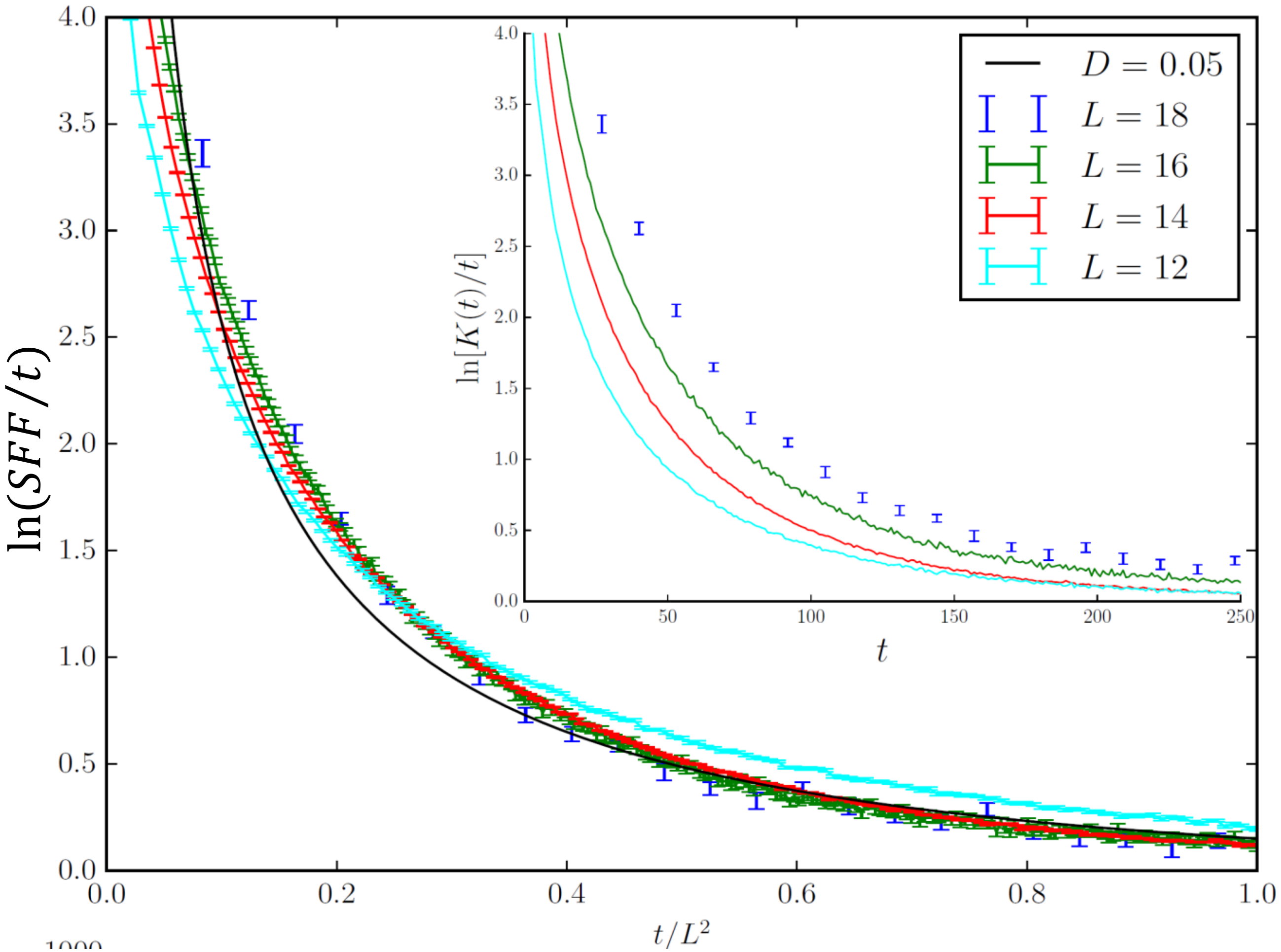
[Winer-S, large-q d=1 Floquet model Friedman et al. '19]

$$T = t_{\mathrm{Th}} = \frac{L^2 \log \frac{1}{\epsilon}}{(2\pi)^2 D} \longrightarrow \sum_\alpha p_{\alpha\to\alpha}(T) = 1 + 2d\epsilon + O(\epsilon^2)$$

periodic box

Comparison with numerical data

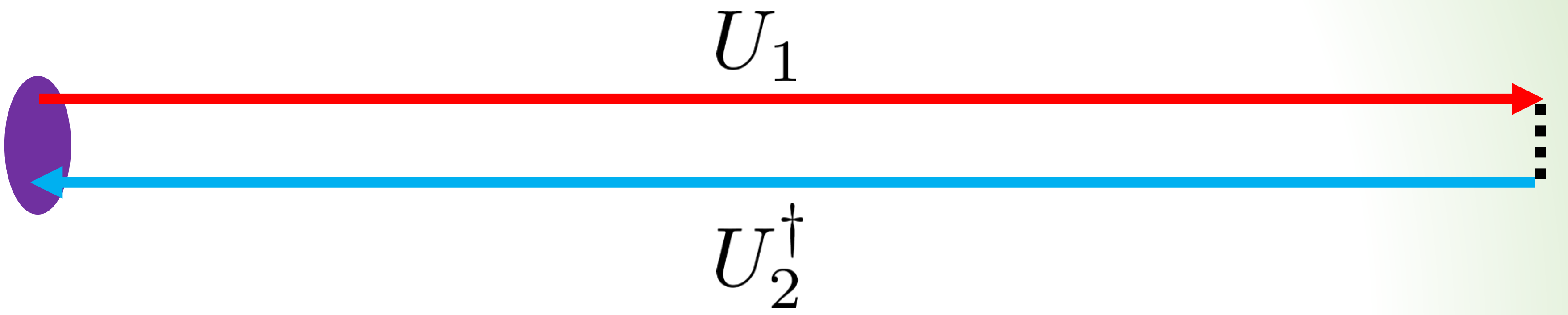
- Consistent with numerical data from [Friedman et al.], which derives the previous formula (in the context of U(1) conservation) in d=1 with large onsite dimension
- We show that it arises generally from linearized diffusion; and we can compute corrections



[data from Friedman et al. 1906.07736]

Theorist's corner

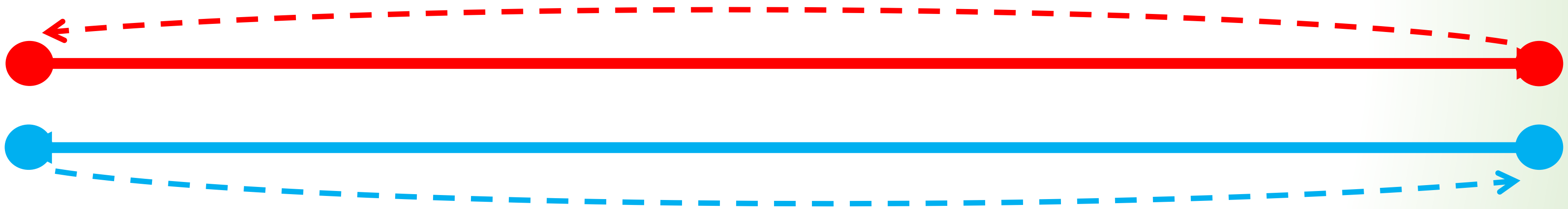
$\mathrm{Tr}[U_1(T)\rho U_2(T)^\dagger]$



VS

$\mathrm{Tr}[U(T)]$

$\mathrm{Tr}[U(T)]^*$



SFF effective theory should be related to an effective theory on a Schwinger-Keldysh contour → hydro!

Theorist's corner

$$\mathrm{SFF} = \int \mathcal{D}\epsilon \mathcal{D}\phi_a \exp(iS_{\mathrm{hydro}})$$

$$\mathcal{D}\epsilon\mathcal{D}\phi_a = \prod_x \prod_{\ell=0}^{T/\Delta t - 1} \frac{d\epsilon(x, t=\ell\Delta t)\, d\phi_a(x, t=\ell\Delta t)}{2\pi}$$

$$S_{\mathrm{hydro}} = \int dV\, dt \left(-\phi_a(\partial_t - D\Delta)\epsilon + i\beta^{-2}\kappa(\nabla\phi_a)^2 \right)$$

eigenvalues of $dt\partial_t$: $T/\Delta t$ complex numbers $i\omega$ obeying $(i\omega + 1)^{T/\Delta t} = 1$

$$\mathrm{SFF} = \prod_k \prod_\omega \frac{1}{i\omega - \lambda_k \Delta t} = \prod_k \frac{1}{1 - e^{\lambda_k T}} \qquad \lambda_k = -Dk^2$$

exactly reproduces prior calculation

[Winer-S]

Are these two widespread phenomena related?

$$\begin{pmatrix} 0.8404 & -0.7442 & -1.0191 & -0.2102 & 1.6058 \\ -0.7442 & 0.4900 & -0.0501 & 1.5743 & 0.3155 \\ -1.0191 & -0.0501 & 1.3546 & -1.5165 & 1.1700 \\ -0.2102 & 1.5743 & -1.5165 & -0.1977 & -1.1330 \\ 1.6058 & 0.3155 & 1.1700 & -1.1330 & -0.4686 \end{pmatrix}$$

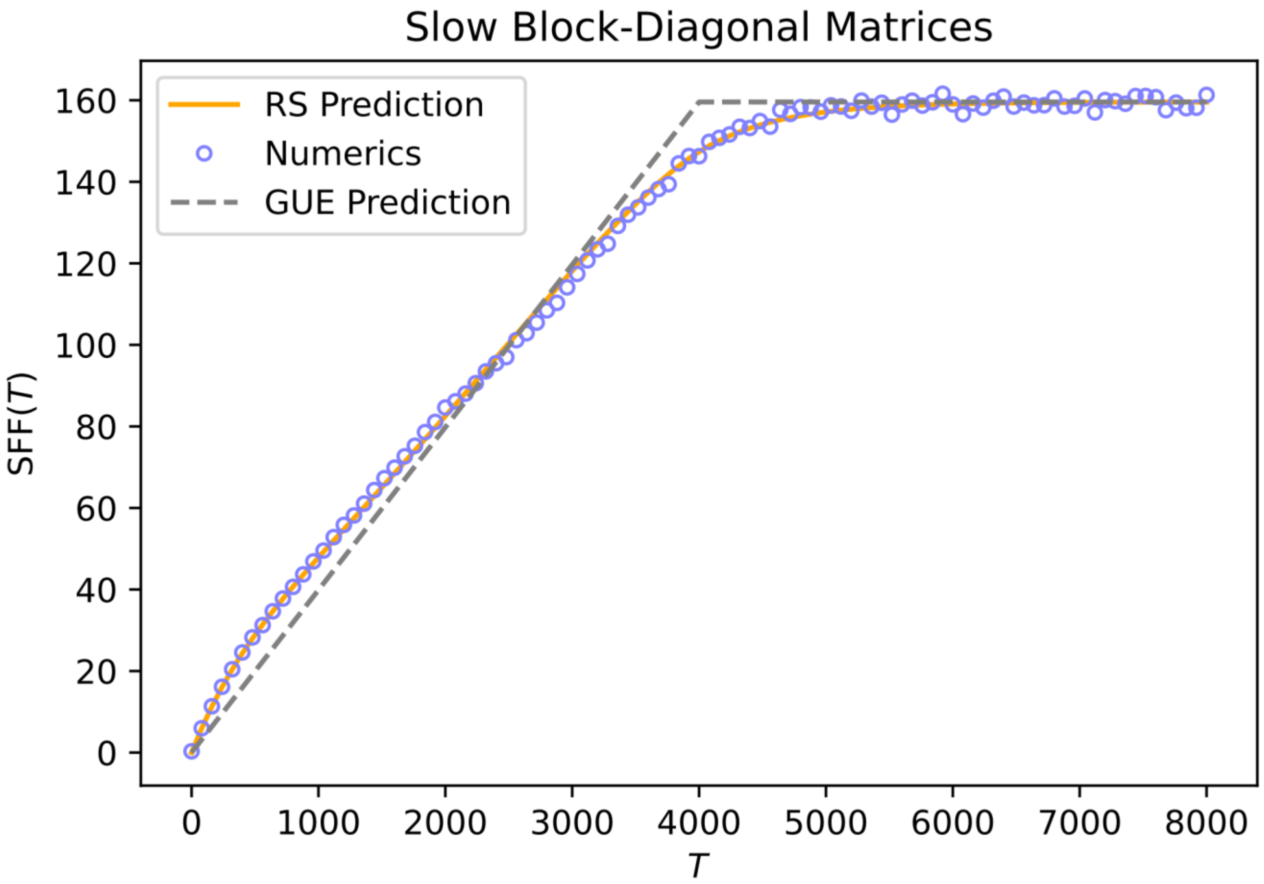
YES!

Spectral form factor (SFF)

Total return probability (TRP)

$$\mathrm{SFF}(T) = \mathrm{SFF}_{\mathrm{RMT}}(T) \times \mathrm{TRP}(T)$$

But there is a twist …



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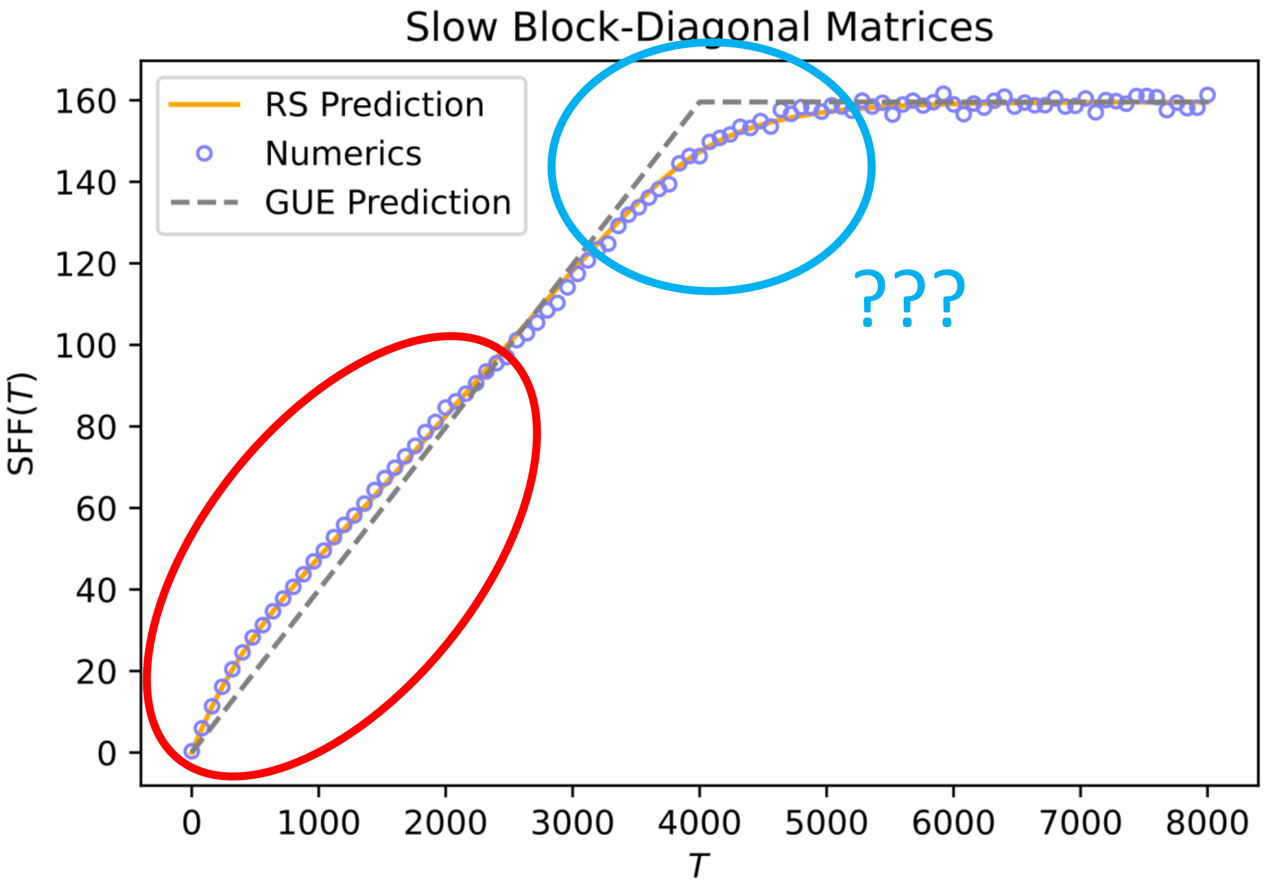


$$\mathrm{SFF}(T) = \mathrm{SFF}_{\mathrm{RMT}}(T) \times \mathrm{TRP}(T)$$

We first observed this suppression in SFFs in a "block Rosenzweig-Porter" model [Barney-Winer-Baldwin-Galitski-S]

Detour: Zeta zeros

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

- Riemann zeta has zeros on the "critical" line $s = \frac{1}{2} + it$; amazingly, these critical line zeros are distributed like the eigenvalues of a random Hermitian matrix! [Montgomery, Odlyzko…]
- Hilbert, Polya, Berry, Keating, and others' inspiring idea: what if the zeta zeros are secretly the energies of a quantum chaotic system?
- Still incomplete ... but one fruit is the Riemann-Siegel lookalike formula – a resummation formula for the Gutzwiller trace formula inspired by the Riemann-Siegel formula for zeta, a relationship between the contributions of short and long periodic orbits

SFF "sum rule"

$$\int_{-\infty}^{\infty} [\mathrm{SFF}(T) - L]\, dT = 0$$

- Valid for any system with enough level repulsion → early time enhancements must be "paid for" with a late time suppression
- We conjecture a specific formula for GUE-type problems (derivation in special cases from the Riemann-Siegel lookalike [Berry-Keating])

$$\mathrm{SFF}(T) = \mathrm{SFF}_{\text{short time}}(T) + \mathrm{SFF}_{\text{long time}}(T)$$

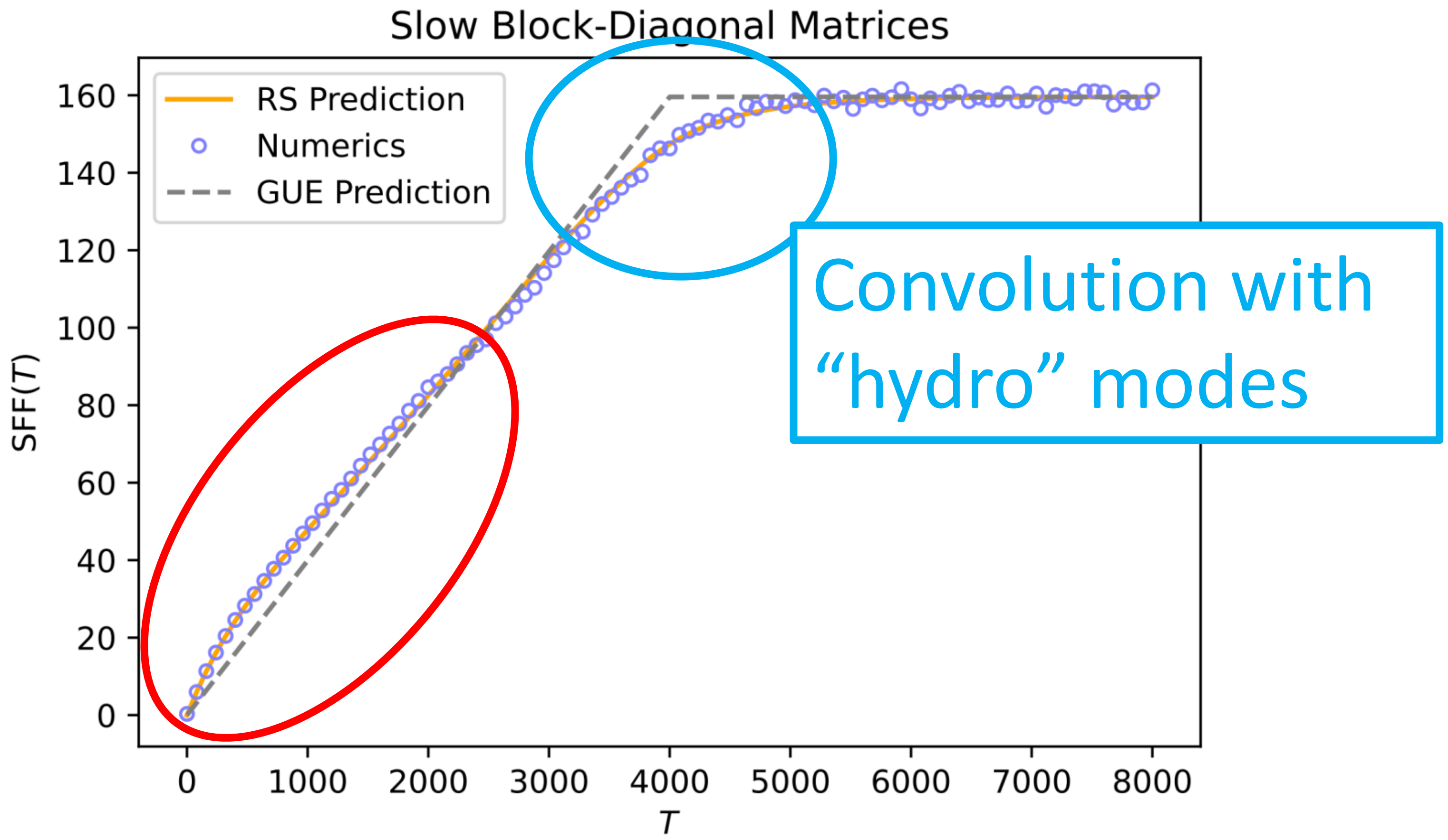
$$\mathrm{SFF}_{\text{short time}}(T) = \frac{|T|}{2\pi}\left(1 + \sum_n e^{-\lambda_n |T|}\right)$$

$$\mathrm{SFF}_{\text{long time}}(T) = \frac{\lambda_1 e^{-\lambda_1 |T|}}{2} * \frac{\lambda_2 e^{-\lambda_2 |T|}}{2} * \frac{\lambda_3 e^{-\lambda_3 |T|}}{2} \ldots * \mathrm{SFF}^0_{\text{long time}}(T)$$

$$\mathrm{SFF}^0_{\text{long time}}(T) = \begin{cases} 0 & \text{if } |T| \leq 2\pi\hat{\rho} \\ \hat{\rho} - \frac{|T|}{2\pi} & \text{if } |T| > 2\pi\hat{\rho} \end{cases}$$

[Winer-S]

But there is a twist …

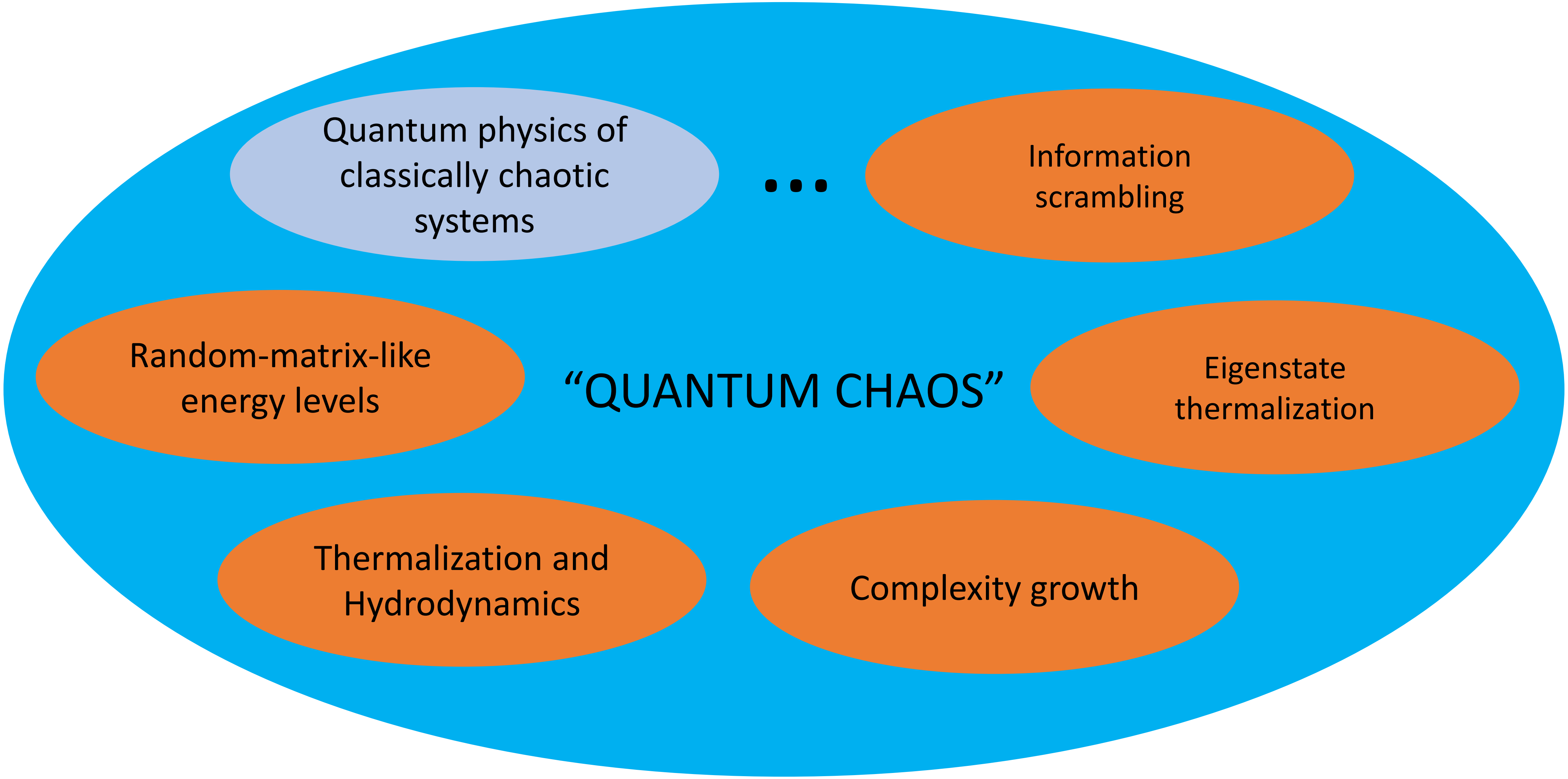


$SFF(T) = SFF_{RMT}(T) \times TRP(T)$

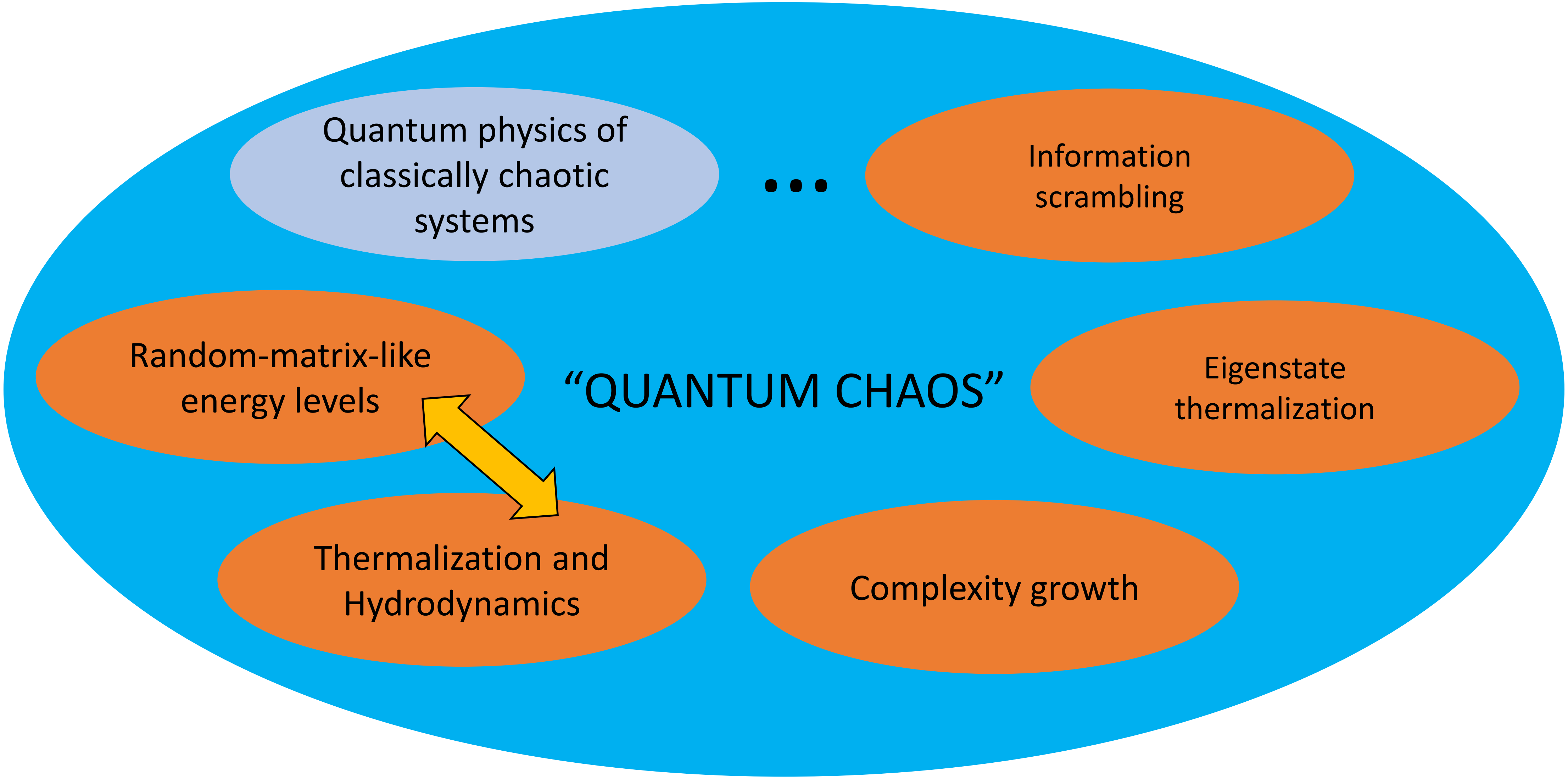
Summary

- Random-matrix-like energies are common to many quantum systems
- But real systems are not literally random matrices, and deviations from RMT are controlled by hydro, construed broadly
- We didn’t emphasize it, but there is even a sense in which this hydro-based “effective theory of spectral correlations” yields the ramp itself
- So far, we’ve applied this theory to spin chains and simple block models; what about nuclei and elsewhere?
- Quantum information has provided many new inspirations, e.g. the growth of complexity, scrambling and thermalization, and more …

Outlook – chaos and quantum information



Today – linking hydro and RMT



Thanks and references

- Mostly based on results with my student Mike Winer
  - *Hydrodynamic theory of the connected spectral form factor*, 2012.01436
  - *Reappearance of Thermalization Dynamics in the Late-Time Spectral Form Factor*, 2307.14415
- Quantum chaos, e.g. Altshuler-Shklovskii '86, D'Alessio-Kafri-Polkovnikov-Rigol, …
- Analytic results: Bertini-Kos-Prosen, Dubertrand-Muller, Chan-De Luca-Chalker, Saad-Shenker-Stanford, Garcia-Garcia-Verbaarschot, Altland-Sonner, …
- RMT Onset: Schiulaz-Torres-Herrera-Santos, Gharibyan-Hanada-Shenker-Tezuka, Friedman-Chan-De Luca-Chalker, Altland-Bagrets, …
- Fluctuating hydro: Dubovsky-Hui-Nicolis-Son, Grozdanov-Polonyi, Haehl-Loganayagam-Rangamani, Crossley-Glorioso-Liu, Jensen-Pinkani-Fokeeva-Yarom, Chen-Lin-Delacretaz-Hartnoll, …