Grant-in-Aid for Transformative Research Areas (A)
Extreme Universe

Annual Meeting Sep.11, 2023

Aspects of Holographic Pseudo Entropy (C01 group)

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Organization of C01 Group

PI: Tadashi Takayanagi (YITP, Kyoto U)

Quantum gravity from holography and quantum information

CoI: Yasuaki Hikida (YITP, Kyoto U)

Higher Spin Holography, Mathematics in 2d CFTs, dS/CFT

CoI: Kazumi Okuyama (Shinshu U)

2D Quantum Gravity (JT) and its connection to Matrix Models

CoI: Yasuhiro Sekino (Takushoku U)

Holography for general spaces, Scrambling

CoI: Shigeki Sugimoto (Kyoto U)

Gauge theory, Holographic QCD, D-branes

ExU post-doc fellow: Jonathan Harper (YITP, Kyoto U)

Holography and Entanglement, Bits threads

[International Research Collaborators]

Pawel Caputa (Warsaw U.): AdS/CFT and Quantum Info., Complexity
Shinsei Ryu (Princeton U.) : Quantum Entanglement and Cond-mat
Beni Yoshida (Perimeter Institute): Quantum Information and BH

[Domestic Research Collaborators]

Kanato Goto (Princeton/YITP) : Island Formula and 2d CFTs
Tomotaka Kitamura (Rikkyo U.) : Holography at weak coupling
Masamichi Miyaji (Nagoya U.) : AdS/CFT and quantum info.
Shoichiro Miyashita (Waseda U.): Holography at weak coupling
Kazuhiro Sakai (Meiji Gakuin U.) : Integrable models and JT gravity
Kenta Suzuki (Rikkyo U.): SYK and JT, collective field theory
Takahiro Uetoko (Kushiro College) :Higher spin gravity

[Post-doc Fellows]

Ali Mollabashi (YITP, Kyoto): Field theory and quantum entanglement
Shan-Ming Ruan (YITP, Kyoto) : EE and Complexity in Holography
Takato Mori (Perimeter/YITP): Holography and Quantun Information

[Graduate Students (YITP, Kyoto)]

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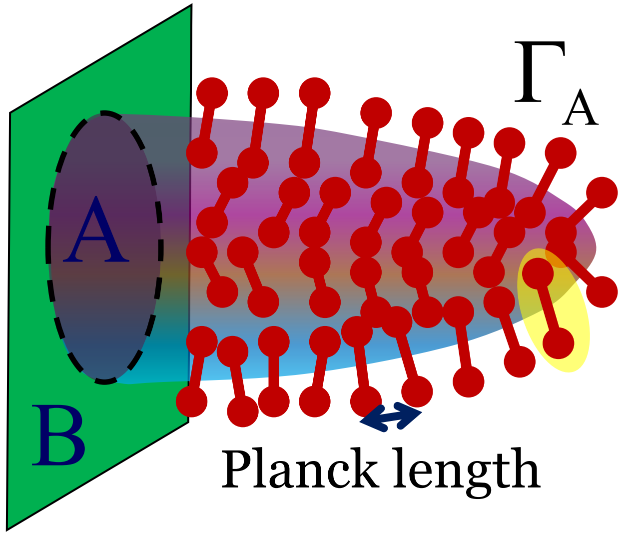
① Motivation (C01 project)

Final Goal = Developing quantum gravity (QG) to answer
"How was the Universe created from the big bang ?"

➡ Explore Holography (AdS/CFT) from quantum info. viewpoints !
[Emergent space from quantum entanglement]
How about emergent time ?



Study non AdS Universe using Holography and Quantum info. !
e.g. de Sitter space
→ Do we need to generalize entanglement entropy ?

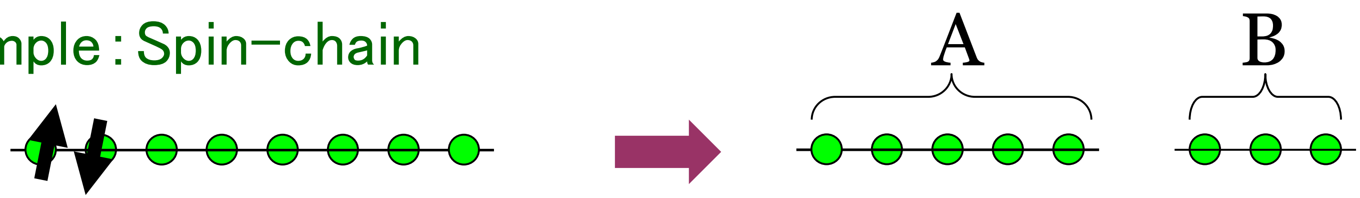
Contents

② Holographic Pseudo Entropy (HPE)

(2-1) Entanglement Entropy (EE) and Holography

We decompose the Hilbert space: $H_{tot} = H_A \otimes H_B$.

Example : Spin-chain



Introduce the reduced density matrix $\rho_A = \mathrm{Tr}_B\left[|\Psi_{tot}\rangle\langle\Psi_{tot}|\right]$.

The entanglement entropy (EE) S_A is defined by

$$S_A = -\mathrm{Tr}[\rho_A \log \rho_A]$$

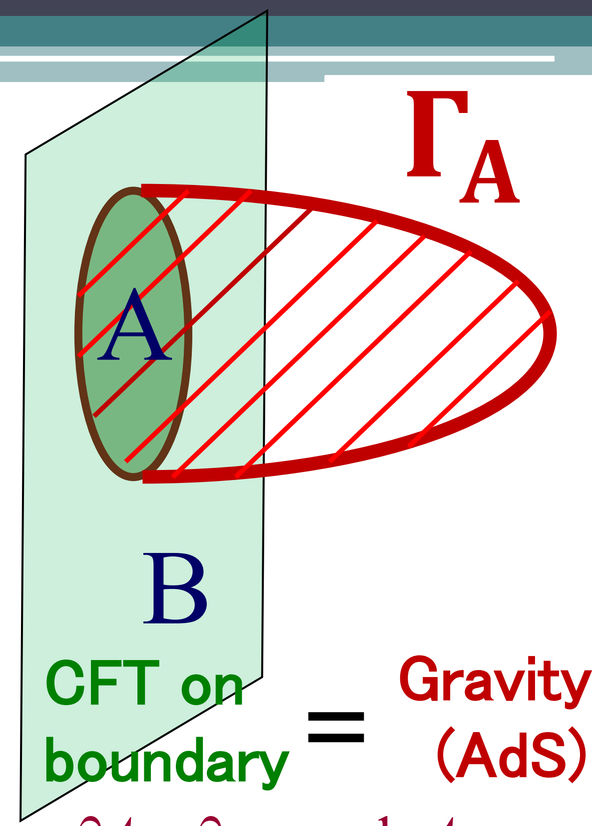
$\propto$ # of Bell Pairs between A and B

Holographic Entanglement Entropy

[Case 1: Static States] [Ryu-TT 06]

In AdS/CFT, SA can be computed from the minimal area surface ΓA:

$$S_A = \underset{\Gamma_A}{\mathrm{Min}}\left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N}\right]$$



$\partial A = \partial\gamma_A$ and $A \sim \gamma_A$.

[Case 2: Time-dependent States] [Hubeny-Rangamani-TT 07]

SA can be computed from the area of extremal surface ΓA:

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|]$$

$$\Downarrow$$

$$S_A(t)$$

$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

(2-2) Holographic Pseudo Entropy

Question

Minimal areas in ***Euclidean time dependent*** asymptotically AdS spaces

= What kind of QI quantity (Entropy ?) in CFT ?

→ **Pseudo Entropy !**

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

Definition of Pseudo Entropy

For two quantum states $|\psi\rangle$ and $|\varphi\rangle$, define the **transition matrix**:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

Decomposing the Hilbert space as $H_{tot} = H_A \otimes H_B$, we introduce

the **reduced transition matrix**: $\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right].$

The **pseudo entropy** is defined by

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

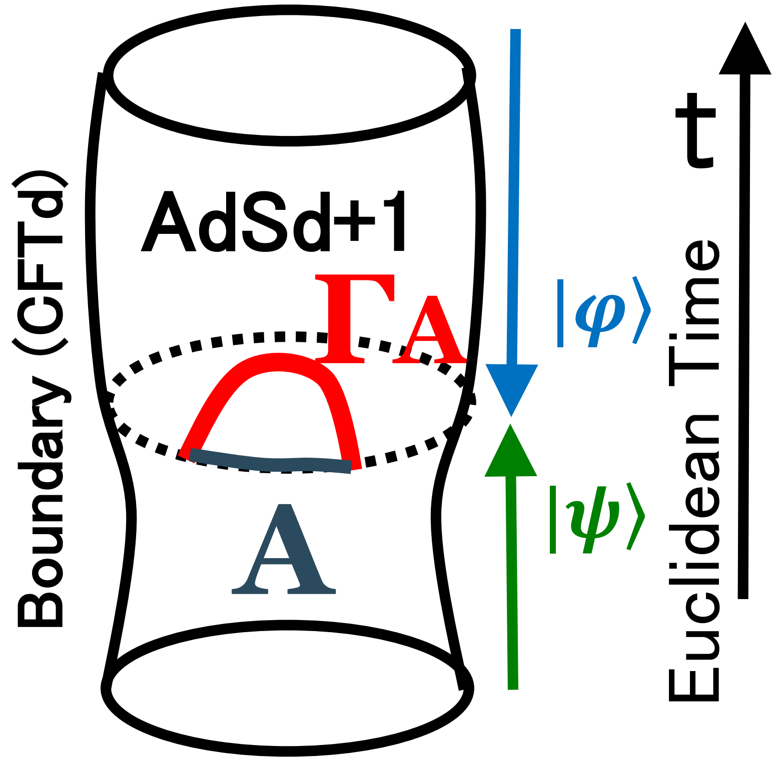
Note: This quantity is **complex valued in general**.

Holographic Pseudo Entropy

In Euclidean time dependent background,
the mininal surface area coincides with
the pseudo entropy.

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

[Nakata–Taki–Tamaoka–Wei–TT, 2020]



Comment

In quantum theory, transition matrices arise when we consider post-selection.

$$\frac{\langle\varphi|O_A|\psi\rangle}{\langle\varphi|\psi\rangle} = \mathrm{Tr}[O_A \tau_A^{\psi|\varphi}]$$

Final state after post-selection

Initial State

This quantity is called **weak value** and is complex valued in general. [Aharanov-Albert-Vaidman 1988,…]

Thus "**Hol. pseudo entropy = weak value of area operator**":

$$S\left(\tau_A^{\psi|\varphi}\right) = \frac{\langle\varphi|A|\psi\rangle}{\langle\varphi|\psi\rangle}.$$

(2-3) Pseudo Entropy and Quantum Phase Transitions

[Mollabashi-Shiba-Tamaoka-Wei-TT 20, 21]

Basic Properties of Pseudo entropy in QFTs

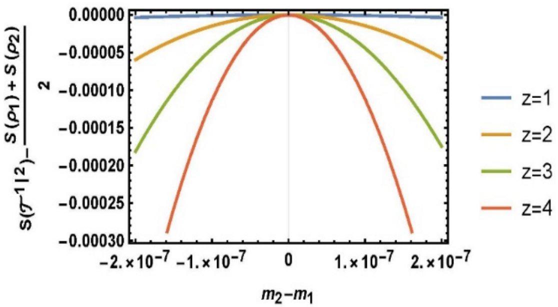
[1] Area law

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2)$$

is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.



PE in a 2 dim. free scalar when we change its mass.

➡ **What happen if they belong to different phases ?**
Can ΔS be positive ?

Example: Quantum Ising spin chain with a transverse magnetic field

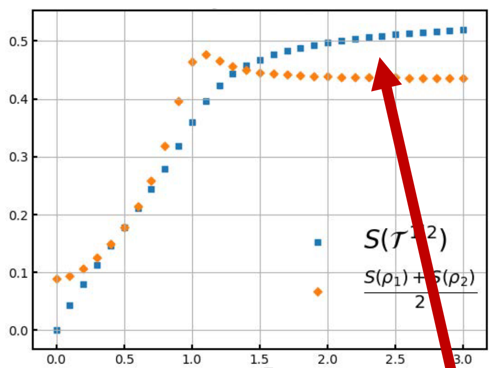
$$H = -J \sum_{i=0}^{N-1} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=0}^{N-1} \sigma_i^x ,$$

Ψ1→ vacuum of H(J1)
Ψ2→ vacuum of H(J2)
(We always set h=1)

J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

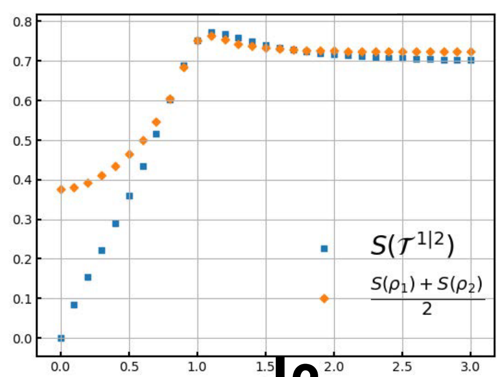
N=16, NA=8

J1=1/2



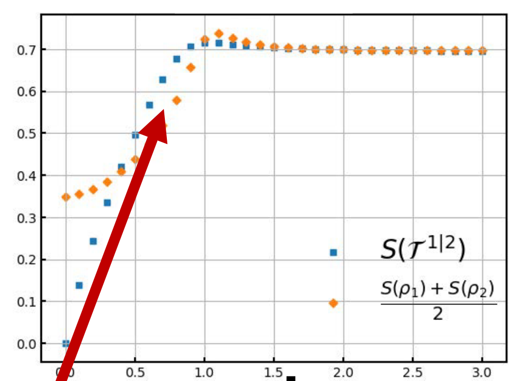
J2

J1=1



J2

J1=2



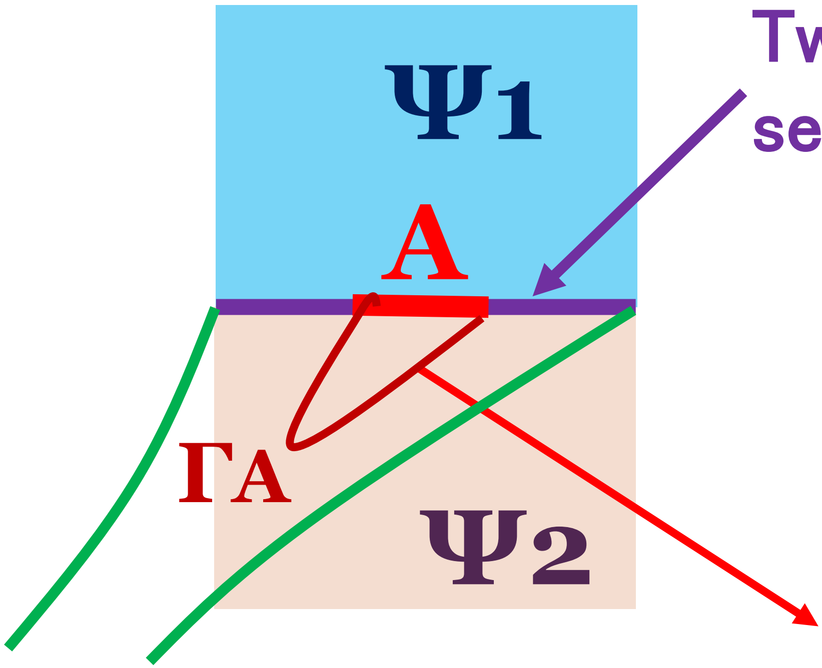
J2

We find

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$$

when Ψ1 and Ψ2 are in different phases !

Heuristic Interpretation



Two gapped phases are separated by a gapless phase.

CFT !

PE is enhanced ! $\Delta S > 0$

AdS Dual of Gapless Interface

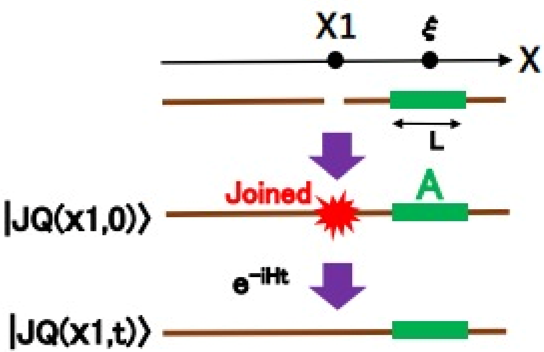
The gapless interface (edge state) also occurs in topological orders.

➔Topological pseudo entropy [Nishioka-Taki-TT 2021].

Holographic(Chaotic) CFT vs Free CFTs in PE

Ex. Calculation of ΔS under local joining quantum quenches

[Shinmyo-Tasuki-TT, in preparation]



Always $\Delta S \leqq 0$ for free fermion CFTs.

$\Delta S > 0$ is possible for holographic CFTs !

➔Multi-partite entanglement in hol CFTs

Tasuki's Poster tomorrow !

③ Pseudo Entropy and dS/CFT

De Sitter Space (Λ>0)➔ Very important in cosmology !

A Sketch of dS/CFT

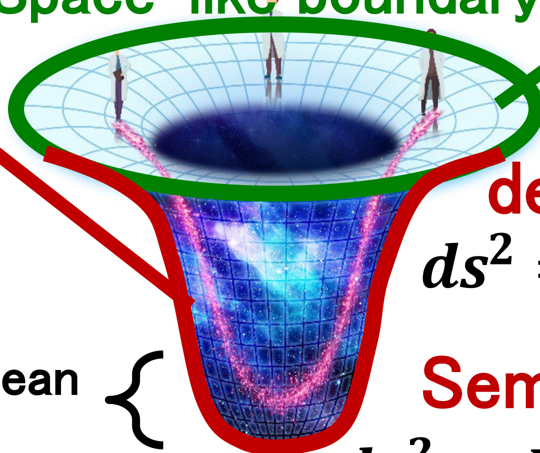
[Strominger 2001, Witten 2001, Maldacena 2002,⋯.]

d+1 dim. Lorentzian de-Sitter spacetime

Dual ⟷

Euclidean d. dim CFT on S^d

Space-like boundary



$t=\infty$

Time

Lorentzian time

Euclidean time

de Sitter

$$ds^2 = L_{dS}^2(-dt^2 + \mathrm{Cosh}^2 t\, d\Omega^2)$$

Semi sphere

$$ds^2 = L_{dS}^2(d\theta^2 + \mathrm{Sin}^2\theta d\Omega^2)$$

$$\theta = it + \frac{\pi}{2}$$

Time emerges from Euclidean CFT !

Ψ [dS gravity]=Z [CFT]

AdS/CFT: Gravity in 3D AdS = 2D (unitary) CFT

dS/CFT: Gravity in 3D dS = 2D (non-unitary) CFT

CFT dual of dS in Einstein gravity

[Hikida-Nishioka-Taki-TT, 2022]

Large c limit of SU(2)k WZW model (a 2dim. CFT)

= **Einstein Gravity** on 3 dim. de Sitter (radius L_{dS})

Level

$$k \approx -2 + \frac{4iG_N}{L_{dS}} \rightarrow \Delta \approx iL_{dS} \cdot E_{dS}$$

Conformal dim. Energy in dS

Central charge

$$c = \frac{3k}{k+2} \approx i\frac{3L_{dS}}{2G_N}$$

$$Z[S^3, R_j] = |S_j^0|^2 \approx e^{\frac{\pi L_{dS}}{2G_N}\sqrt{1-8G_N E}}$$

CFT partition function De Sitter Entropy

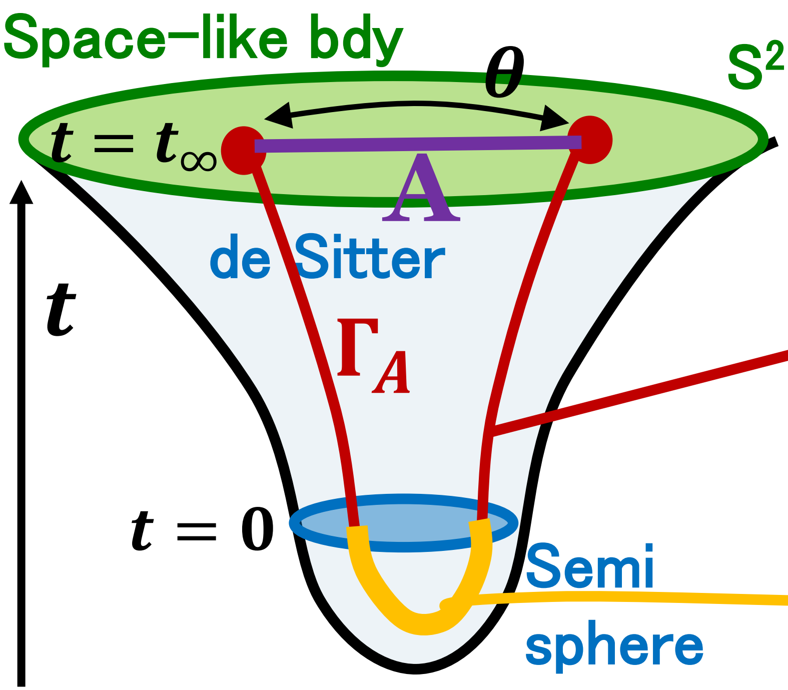
[cf. Triality: Gaberdiel-Gopakumar, 2012]

"Holographic entanglement entropy" in dS3/CFT2 leads to

$$S_A = \frac{\mathrm{L}(\Gamma_A)}{4G_N} = i\frac{c_{ds}}{3}\log\left(\frac{2}{\epsilon}\mathrm{Sin}\frac{\theta}{2}\right) + \underbrace{\frac{c_{dS}}{6}\pi}_{S_{dS}/2}.$$

SdS/2

$$ds^2 = L_{ds}^2(-dt^2 + \mathrm{Cosh}^2 t\,(d\theta^2 + Sin^2\theta d\varphi^2))$$



$$\mathrm{L}(\Gamma_A) = 2it_\infty + i\log(\mathrm{Sin}^2\frac{\theta}{2}) + \pi$$

Length of time-like geodesics
➔imaginary value

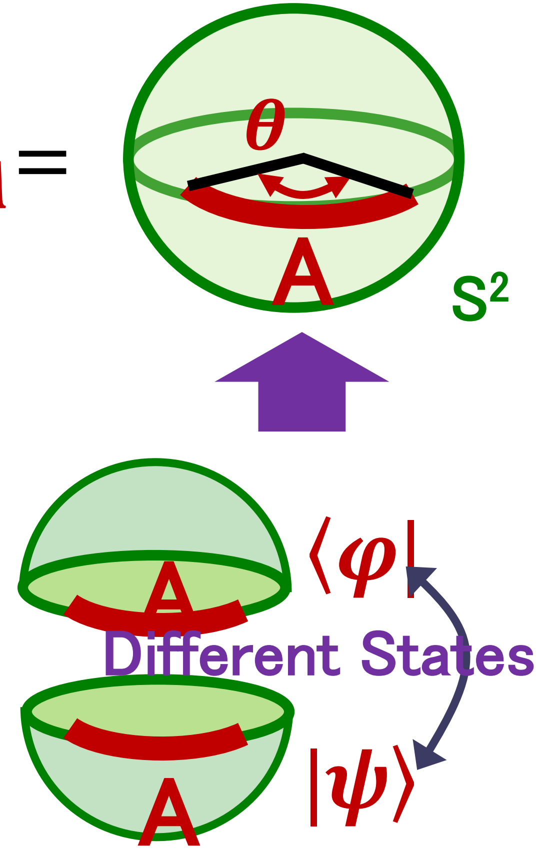
Space-like geodesic (Semi circle)

This nicely reproduces the 2d CFT result as follows:

$$S_A = \frac{c_{CFT}}{6} \log \left[\frac{\mathrm{Sin}^2 \frac{\theta}{2}}{\tilde{\varepsilon}^2} \right],$$ by setting

$$c_{CFT} = i c_{dS} \text{ and } \tilde{\varepsilon} = i\varepsilon = i e^{-t_\infty}.$$

$\rho_A =$



Why is the EE complex valued ?

➔ **It should be regarded as pseudo entropy**

because ρ_A **is not Hermitian !**

[Doi–Harper–Mollabashi–Taki–TT 2022, 2023]

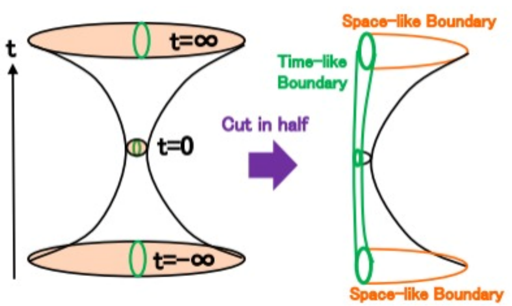
Entanglement entropy ➔ Emergent space in AdS
Imaginary part of Pseudo entropy ➔ Emergent time in dS !

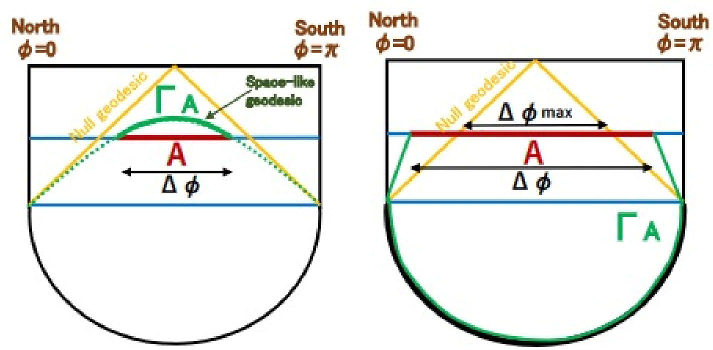
Another approach to holography for de Sitter space

[Kawamoto–Ruan–Suzuki–TT 2023]

We want a time-like boundary !➔ A half de Sitter space

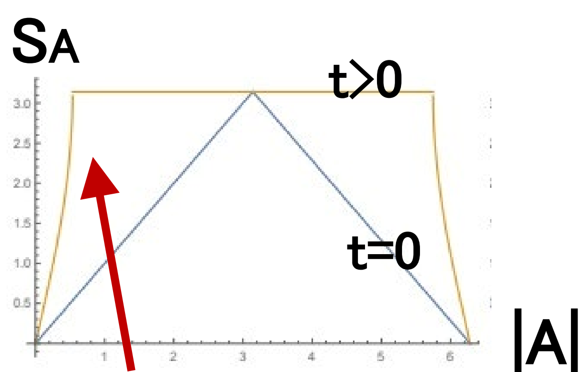
Non-local QFT on dS_d = Gravity on a half dS_{d+1}





Ruan's talk on Friday 5pm

HEE for dS



Violation of strong subadditivity !

Does space in dS emerge from quantum entanglement ?

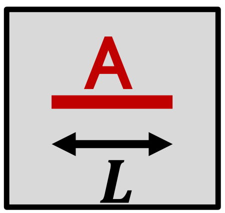
④ Other Applications of Pseudo Entropy

(4-1) Time-like Entanglement Entropy

[Liu-Chen-Lian 2022, Doi-Harper-Mollabashi-Taki-TT 2022, Li-Xiao-Yang 2022]

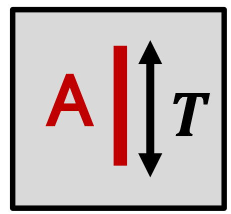
A special class of pseudo entropy is time-like version of entanglement entropy by rotating subsystem A into a time-like one:

Ex. CFT on an infinite line



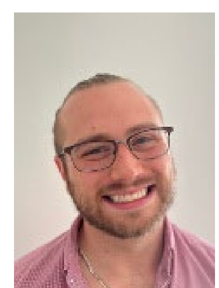
$$S_A = \frac{c_{CFT}}{3}\log\left[\frac{L}{\varepsilon}\right]$$

$L \rightarrow iT$

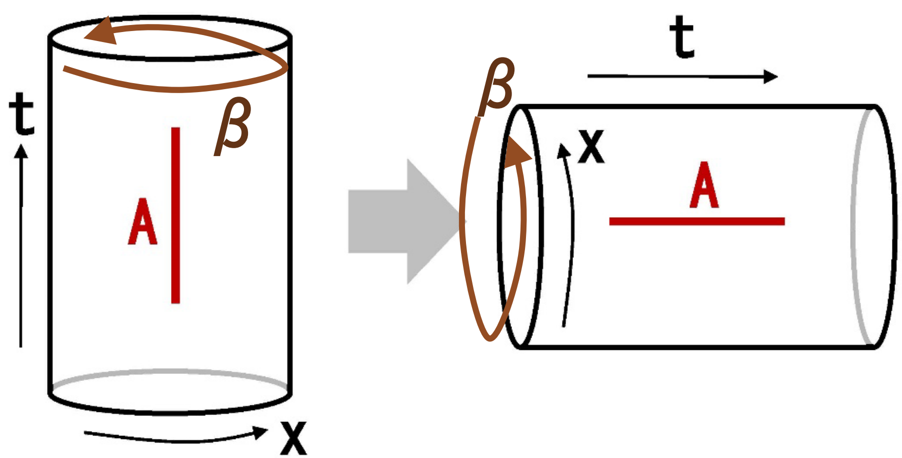


$$S_A = \frac{c_{CFT}}{3}\log\left[\frac{T}{\varepsilon}\right] + \frac{\pi}{6} i c_{CFT}$$

For a holographic dual of TEE,
See Harper's talk on Friday 11:30 !



What does the time-like EE compute ?



Consider 2d CFT on a cylinder.
If we regard t as a space coordinate and x as a Euclidean time, then then the Hamiltonian looks like

$$H_{time-like} = iH_{CFT}$$

If we trace out a part of t-axis, the reduced density matrix reads

$$\rho_A = Tr_B[e^{i\beta H_{CFT}}]$$ ➡ Non-Hermitian !

We can interpret this as pseudo entropy by doubling Hilbert space:

$$|\psi_{TFD}\rangle \propto \sum_n e^{i\beta E_n/2}|n\rangle_1 |n\rangle_2$$

$$|\varphi_{TFD}\rangle \propto \sum_n e^{-i\beta E_n/2}|n\rangle_1 |n\rangle_2$$

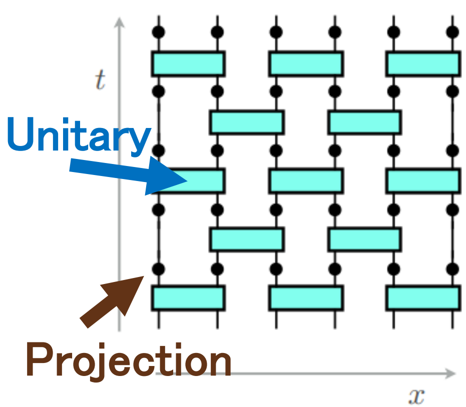
➡

$$\rho_A = Tr_B\left[\frac{|\psi_{TFD}\rangle\langle\varphi_{TFD}|}{\langle\varphi_{TFD}|\psi_{TFD}\rangle}\right]$$

(4-2) Entanglement Phase Transition

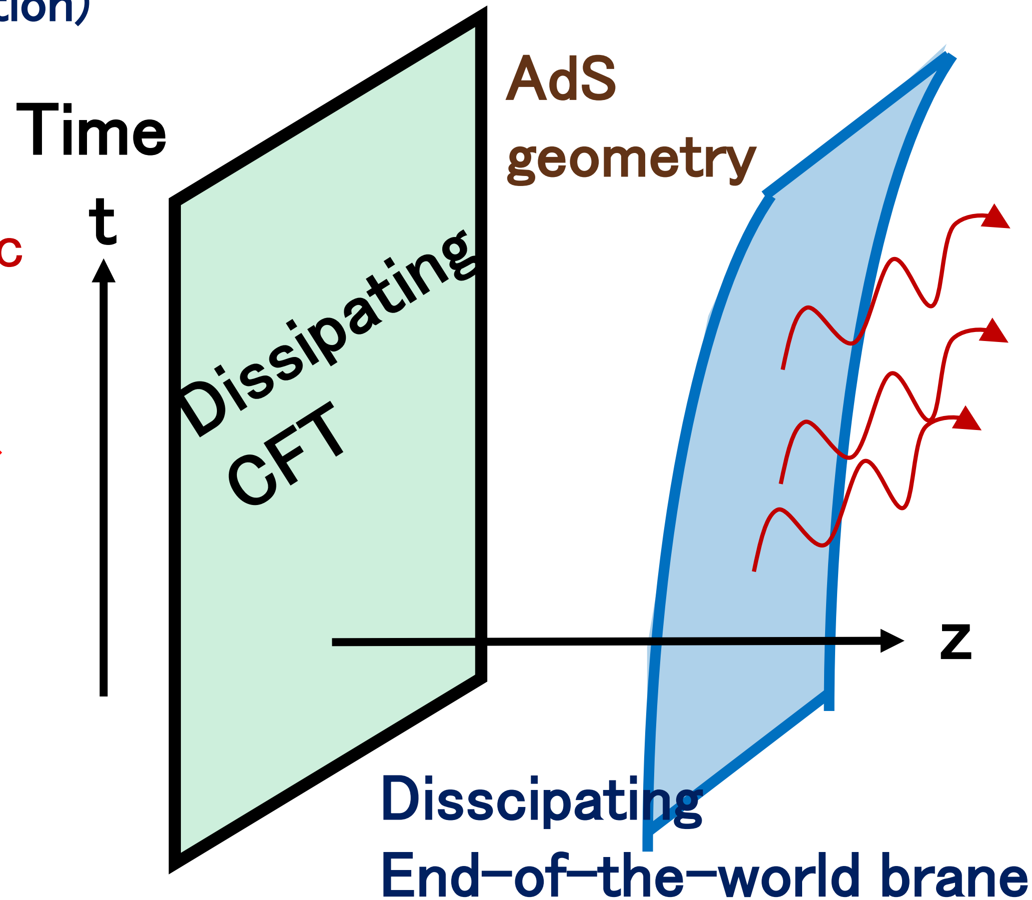
[Kanda-Sato-Suzuki-Wei-TT 2023 + in preparation]

Entanglement Phase Transition [Skinner-Ruhman-Nahum, Li-Chen-Fisher 2018]
(Measurement Induced Phase Transition)

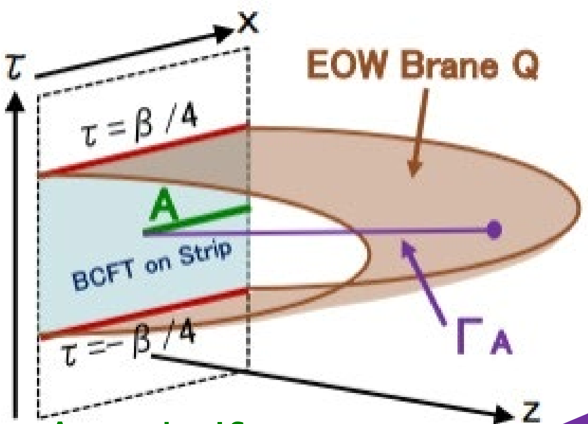


$$(i)\quad p < p_* : \quad S_A \propto t,$$
$$(ii)\quad p = p_* : \quad S_A \propto \log t,$$
$$(iii)\quad p > p_* : \quad S_A = \text{finite},$$

Holographic Model



Double Wick Rotation and Time Evolution



A= a half space

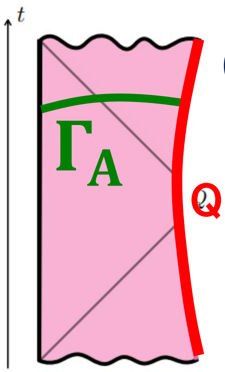
τ =it

Imaginary valued scalar field on EOW brane ➔ Disscipation

$$\rho(t) = e^{-\left(\frac{\beta}{4}+it\right)H} \left|B(\varphi_0+\Delta\varphi)\right\rangle\left\langle B(\varphi_0)\right| e^{-\left(\frac{\beta}{4}-it\right)H}$$

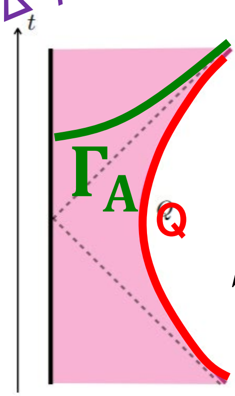
Boundary state

$\Delta\phi<\Delta\phi_*$ $\qquad$ $\Delta\phi=\Delta\phi_*$ $\qquad$ $\Delta\phi>\Delta\phi_*$



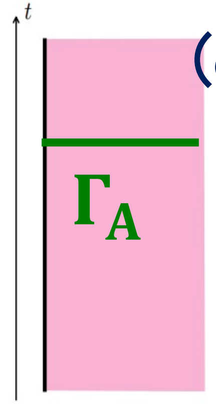
(a) BTZ phase

$$S_A \approx \frac{c}{6}\cdot\frac{\pi}{2\alpha}\cdot t$$



(b) Poincare phase

$$S_A \approx \frac{c}{3}\cdot\log t.$$



(c) Thermal AdS phase

$$S_A \approx \text{const.}$$

➡ For more details,
see Kanda's poster tomorrow !

⑤ SVD entropy [Parzygnat-Taki-Wei-TT 2023]

Idea: Improve PE so that **(i)** it takes *real and non-negative values* and **(ii)** it has *a quant info. interpretation.*

➡ SVD entropy

$$S_{SVD}\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[|\tau_A^{\psi|\varphi}|\cdot \log|\tau_A^{\psi|\varphi}|\right].$$

$$\text{here, } |\tau_A^{\psi|\varphi}| \equiv \sqrt{\tau_A^{\dagger\psi|\varphi}\tau_A^{\psi|\varphi}}$$

- This is always non-negative and is bounded by log dim H_A.

- From quantum information theoretic viewpoint, this is the number of Bell pairs distilled from the intermediate state:

$$\tau_A^{\psi|\varphi} = U \cdot \Lambda \cdot V, \qquad \frac{\langle\varphi|V^\dagger \sum_k |\mathrm{EPR}_k\rangle\langle \mathrm{EPR}_k| U^\dagger|\psi\rangle}{\langle\varphi|V^\dagger U^\dagger|\psi\rangle} = \sum_k p_k = 1$$

➡ $$S_{SVD} \approx \sum_k p_k \cdot \#\text{of Bell Pairs in } |\mathrm{EPR}_k\rangle$$

- This entropy also shows an enhancement similar to PE for two difference states in different quantum orders.

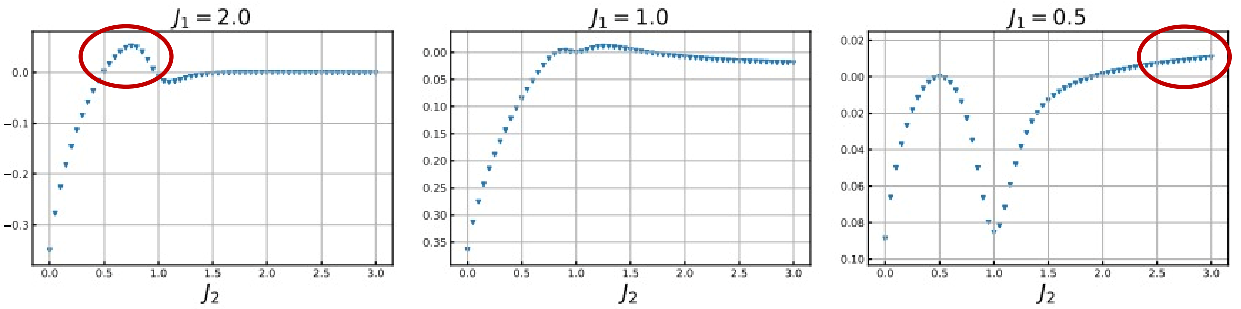
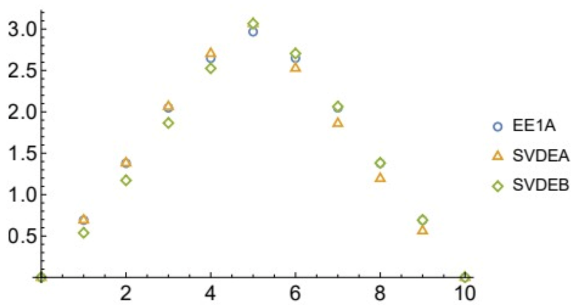


Figure 23: Plots of $S(\rho_A^{1|2}) - \left(S(\rho_A^1) + S(\rho_A^2)\right)/2$ in different cases. $S(\rho_A^{1|2}) - \left(S(\rho_A^1) + S(\rho_A^2)\right)/2 \leq 0$ when the two states are in the same quantum phase, and can be violated when the two states are in different quantum phases.

- This SVD entropy also shows the Page curve like behavior.



- However, we have $S_A \neq S_B$, as opposed to pseudo entropy !

(This suggest the gravity dual will be very complicated….)

⑥ Conclusions

- Pseudo entropy (PE) generalizes EE to post-selected states.
- In AdS/CFT, HPE = minimal area in Euclidean time-dep. AdS.
- In dS/CFT, HPE naturally arises and may explain emergent time.
- Application of PE ➔ time-like EE and entanglement transition
- Modifying PE into non-negative entropy leads to SVD entropy

Does Spacetime from Quantum Entanglement (or PE) ?

	AdS	(global) dS
Space coordinate	Yes	No ?
Time coordinate	Yes ?	Yes

Thank you very much !