Krylov complexity and chaos in quantum mechanics

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based on arXiv:2305.16669 [hep-th]

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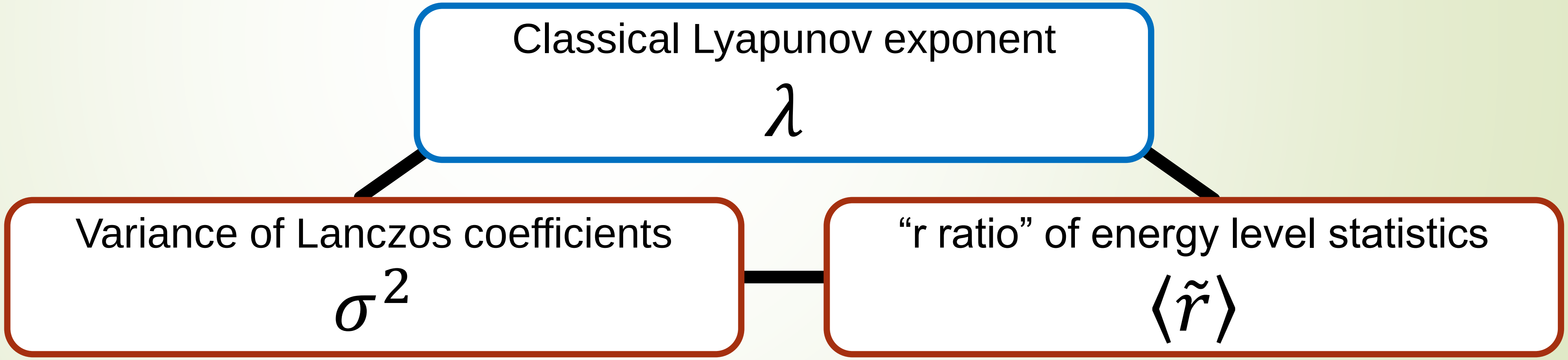
Krylov complexity and chaos in quantum mechanics

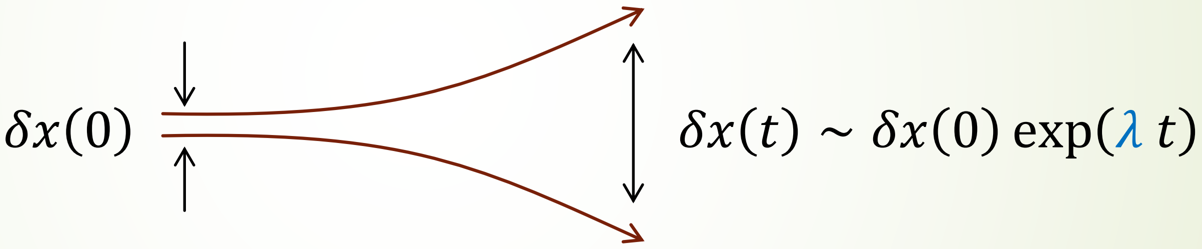
- **Classical** & **quantum chaos** in **billiard systems**
- We find significant correlations between **indicators of chaos**



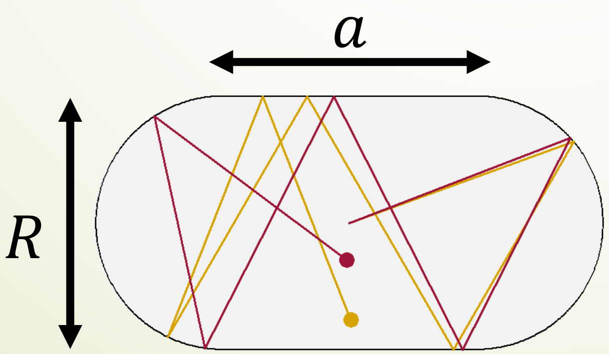
- These correlations are universal for any billiards (stadium, Sinai, …)
- Subtlety in Krylov complexity value?

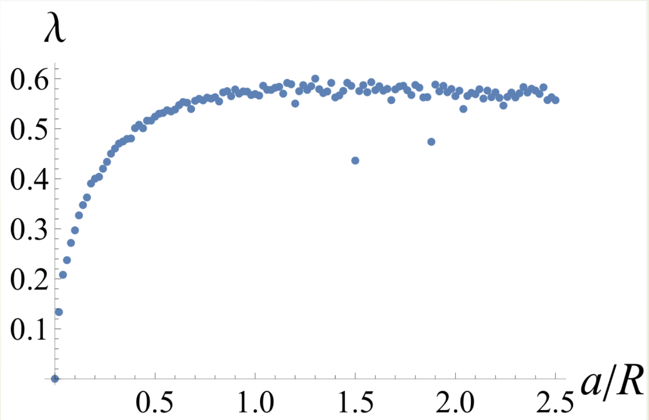
Classical chaos in billiard system

- Classical chaos in non-linear deterministic dynamical system
  - Orbits sensitive to initial conditions → "information loss"
  - Lyapunov exponent $\lambda > 0 \Rightarrow$ chaotic



  - Example: Stadium billiard [Bunimovich '74]



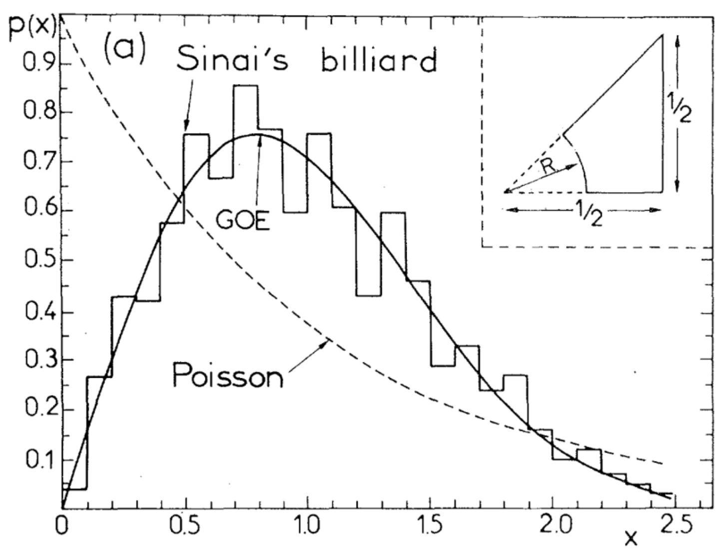


Quantum chaos in billiard system

- Schrodinger eq. = **linear** equation of wave function
  Quantum effect washes out small scale ($\Delta x\,\Delta p \gtrsim \hbar$)

  → No chaos in quantum regime?

- Probes of chaos in quantum billiard
  - Energy level statistics



Statistics of adjacent energy level spacing

$$s_n = E_{n+1} - E_n$$

- Poisson statistics (→ non-chaotic)

$$P(s) \sim e^{-s}$$

- Wigner-Dyson statistics (→ chaotic)

$$P(s) \sim s^{\beta} \exp(-s^2)$$

  - Out-of-time-order correlators (OTOC) in quantum billiard [Hashimoto+ '17]

Krylov operator complexity

- Yet another indicator of quantum chaos: **Krylov complexity**

  [Parker+ '18]

  - Krylov operator complexity for $\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt}$
  - Krylov state complexity for $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$
  - Also **variation of Lanczos coefficients** can be an indicator of chaos

    [Rabinovici+ '21]

- Can we use it to probe *quantum chaos* in billiard system?
  - Comparison with **level statistics**

- Relationship with *classical chaos*?
  - Comparison with **classical Lyapunov exponent**

Krylov operator/state complexity

- Krylov *operator* complexity
  - Time evolution of an operator $\mathcal{O}$ for Hamiltonian H

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}^n \mathcal{O}(0) \qquad (\mathcal{L} \equiv [H, \cdot\,])$$

  - Krylov space $\mathcal{H}_{\mathcal{O}} \equiv \{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \ldots\}$
  - Introduce an inner product by $(\mathcal{O}_1|\mathcal{O}_2) \equiv \mathrm{Tr}\left[\mathcal{O}_1^\dagger \mathcal{O}_2\right]$ ← cf. Wightman inner product $(\mathcal{O}_1|\mathcal{O}_2) \simeq \mathrm{Tr}\left[e^{-\frac{\beta H}{2}}\mathcal{O}_1^\dagger e^{-\frac{\beta H}{2}}\mathcal{O}_2\right]$
  - Switch to orthonormal basis $\mathcal{H}_{\mathcal{O}} = \{\mathcal{O}_0 = \mathcal{O}(0), \mathcal{O}_1, \mathcal{O}_2, \ldots\}$ by Lanczos algorithm

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} i^n \varphi_n(t)\mathcal{O}_n$$

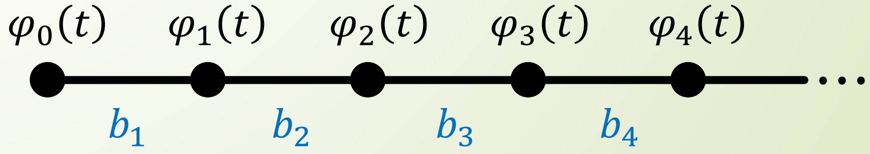
  - Krylov operator complexity $\mathcal{C}_{\mathcal{O}}(t) \equiv \sum_{n=0}^{\infty} n\, |\varphi_n(t)|^2$

Krylov operator/state complexity

- Krylov *operator* complexity
  - Lanczos algorithm → Orthonormal basis $\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots\}$ & Lanczos coeff. b_n

    1. $b_0 \equiv 0\,, \quad \mathcal{O}_{-1} \equiv 0$
    2. $\mathcal{O}_0 \equiv \mathcal{O}/\|\mathcal{O}\|$, where $\|\mathcal{O}\| \equiv \sqrt{(\mathcal{O}|\mathcal{O})}$
    3. For $n \geq 1$: $\mathcal{A}_n = \mathcal{L}\mathcal{O}_{n-1} - b_{n-1}\mathcal{O}_{n-2}$
    4. Set $b_n = \|\mathcal{A}_n\|$
    5. If $b_n = 0$ stop; otherwise set $\mathcal{O}_n = \mathcal{A}_n/b_n$ and go to step 3.

  - $\mathcal{O}(t) = \sum_{n=0}^{\infty} i^n\, \varphi_n(t)\mathcal{O}_n$ → $\varphi_n(t)$ obeys $\dot{\varphi}_n(t) = b_n\, \varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t)$



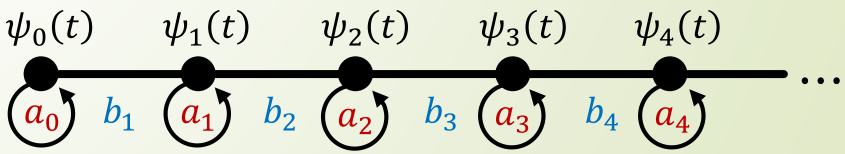
  - Krylov complexity $\mathcal{C}_{\mathcal{O}}(t) \equiv \sum_{n=0}^{\infty} n\, |\varphi_n(t)|^2 \approx$ Distance from initial $\mathcal{O}(0)$

Krylov operator/state complexity

- Krylov *state* complexity
  - Time evolution of a state $|\psi\rangle$ for Hamiltonian H

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} H^n|\psi(0)\rangle$$

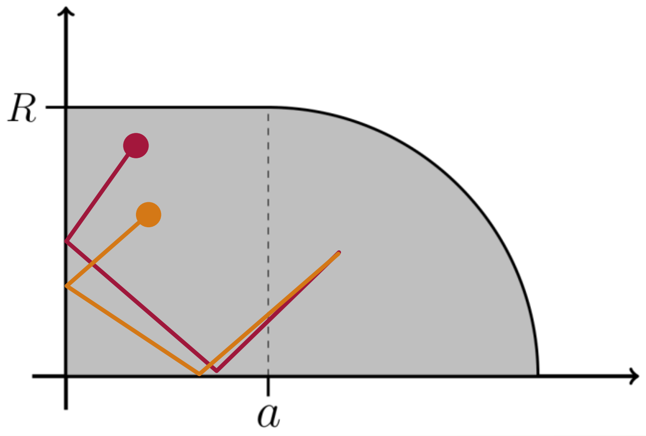
  - Krylov space $\mathcal{H}_{|\psi(t)\rangle} \equiv \{|\psi(0)\rangle, H|\psi(0)\rangle, H^2|\psi(0)\rangle, \ldots\}$
  - Orthonormal basis $\mathcal{H}_{|\psi(t)\rangle} = \{|K_0\rangle = |\psi(0)\rangle, |K_1\rangle, |K_2\rangle, \ldots\}$ by Lanczos algorithm
  - $|\psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n(t)|K_n\rangle \rightarrow \quad i\,\dot{\psi}_n(t) = a_n\,\psi_n(t) + b_n\,\psi_{n-1}(t) + b_{n+1}\psi_{n+1}(t)$



  - Krylov state complexity $\mathcal{C}_{|\psi(t)\rangle}(t) \equiv \sum_{n=0}^{\infty} n\,|\psi_n(t)|^2 \approx$ Distance from initial $|\psi\rangle$

Krylov complexity and chaos in quantum mechanics

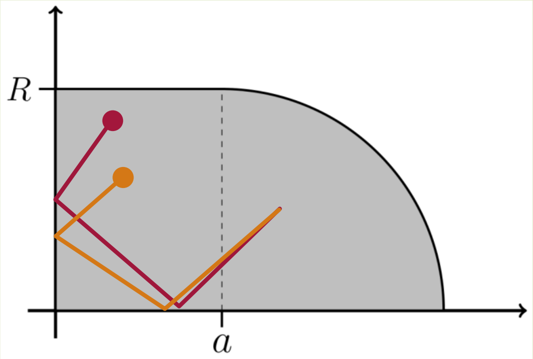
- Setup: stadium billiard (with unit area & velocity)



- Classical billiard → Positive Lyapunov exponent λ
- Quantum billiard
  - Hamiltonian $H = p_x^2 + p_y^2$ with Dirichlet BC at the wall: $\psi(x,y)|_{\text{wall}} = 0$
  - Calculate energy level statistics & Krylov operator/state complexity

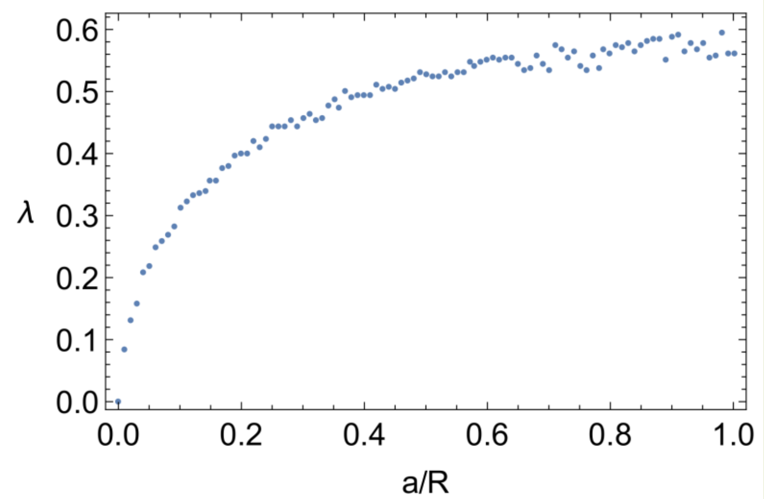
[cf. Finite temperature case: Camargo+ ‘23]

Classical / quantum chaos in stadium billiard



- Classical Lyapunov exponent λ

  λ maximized at $a/R \sim 1$

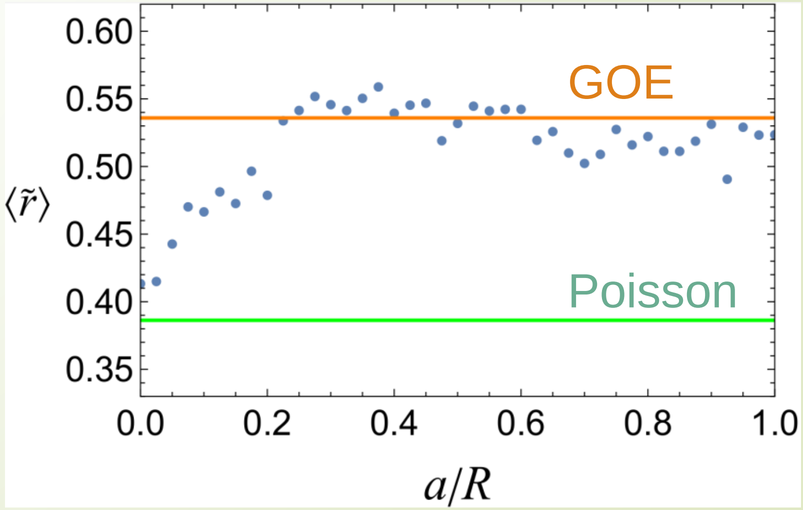


- Energy level statistics

$$\tilde{r}_n \equiv \frac{\min(s_n, s_{n-1})}{\max(s_n, s_{n-1})} \qquad (s_n = E_{n+1} - E_n)$$

$$\langle \tilde{r} \rangle \approx \begin{cases} 0.38629 & \text{(Poisson)} \\ 0.53590 & \text{(GOE)} \end{cases}$$

Poisson (non-chaotic) → GOE (chaotic)



Classical / quantum chaos in stadium billiard

- **Krylov complexity for operator p_x** on stadium billiard
  - Solve Schrödinger eq. to obtain eigenvalues E_n & eigenfunctions $\phi_n(x,y)$

$$H|n\rangle = E_n|n\rangle, \qquad \phi_n(x,y) = \langle x,y|n\rangle \qquad (n = 1,2,\dots)$$

  - Matrix representation of operators x and p_x

$$x_{mn} \equiv \langle m \mid x \mid n\rangle = \int dx\, dy\, \phi_m^*(x,y)\, x\, \phi_n(x,y)$$

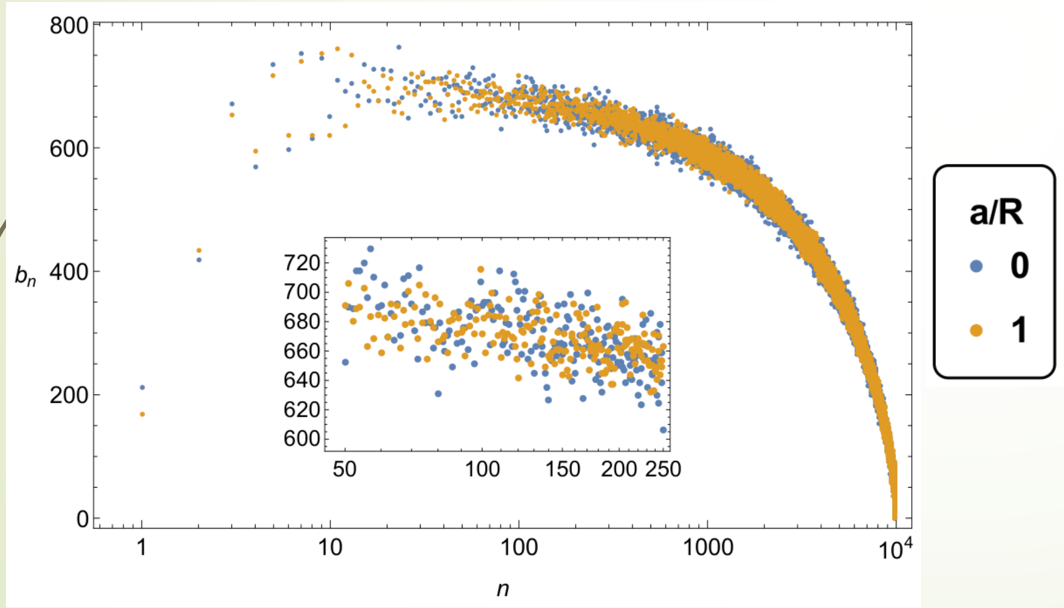
$$P_{mn} \equiv \langle m \mid p_x \mid n\rangle = \left\langle m \left| \frac{i}{2}[H,x] \right| n \right\rangle = \frac{i}{2}(E_m - E_n)x_{mn}$$

  - For numerical calculation, we truncate the energy levels at $n = N_{\max} = 100$
  - Lanczos algorithm → Lanczos coefficients b_n & Krylov complexity $C_{p_x}(t)$

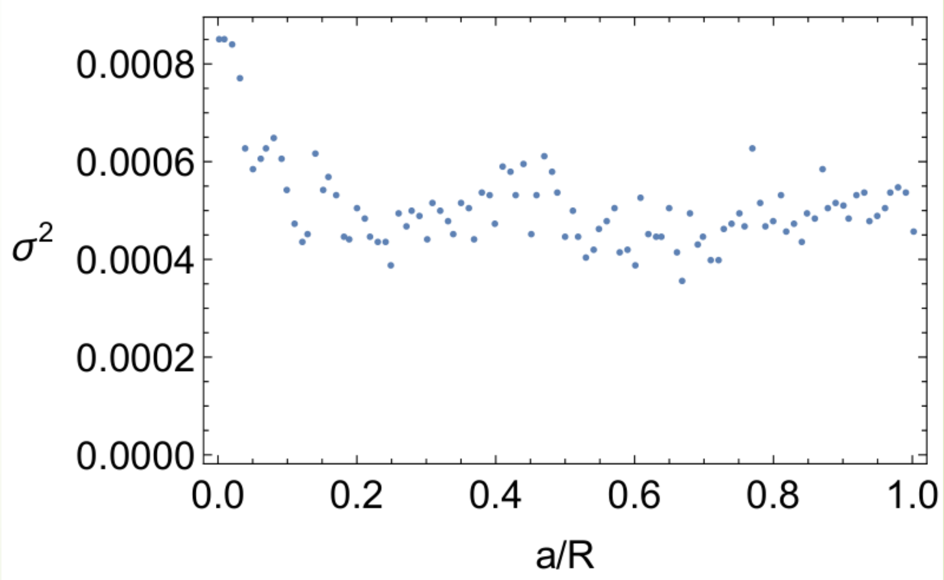
Classical / quantum chaos in stadium billiard

- **Krylov complexity for operator p_x** on stadium billiard

Lanczos coefficient b_n $(0 \le n \le N_{\max}^2 = 10^4)$



Variance σ^2 of b_n $\left(\sigma^2 \equiv \mathrm{Var}\left(\ln\left|\frac{b_{2i-1}}{b_{2i}}\right|\right)\right)$



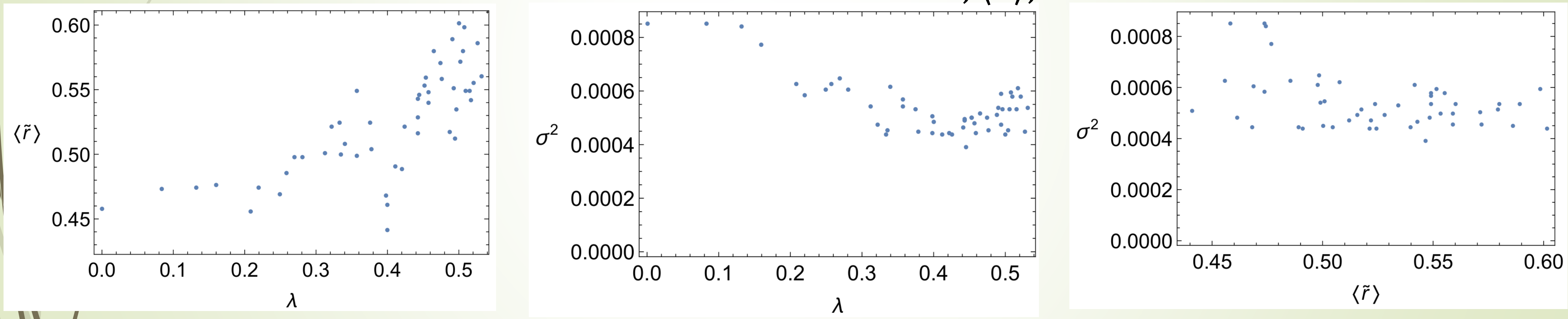
- Non-chaotic → larger σ^2
  Chaotic → smaller σ^2

- Linear growth of b_n at low n not obvious
  (Finite temperature → $b_n \lesssim \pi T \times n$ [Camargo+ '23])

Classical / quantum chaos in stadium billiard

- **Krylov complexity for operator $\boldsymbol{p_x}$** on stadium billiard

Correlations between $\lambda, \langle \tilde{r} \rangle, \sigma^2$



$$\frac{\langle (A-\langle A\rangle)(B-\langle B\rangle)\rangle}{\sqrt{\langle (A-\langle A\rangle)^2\rangle \langle (B-\langle B\rangle)^2\rangle}}$$

Cross correlations

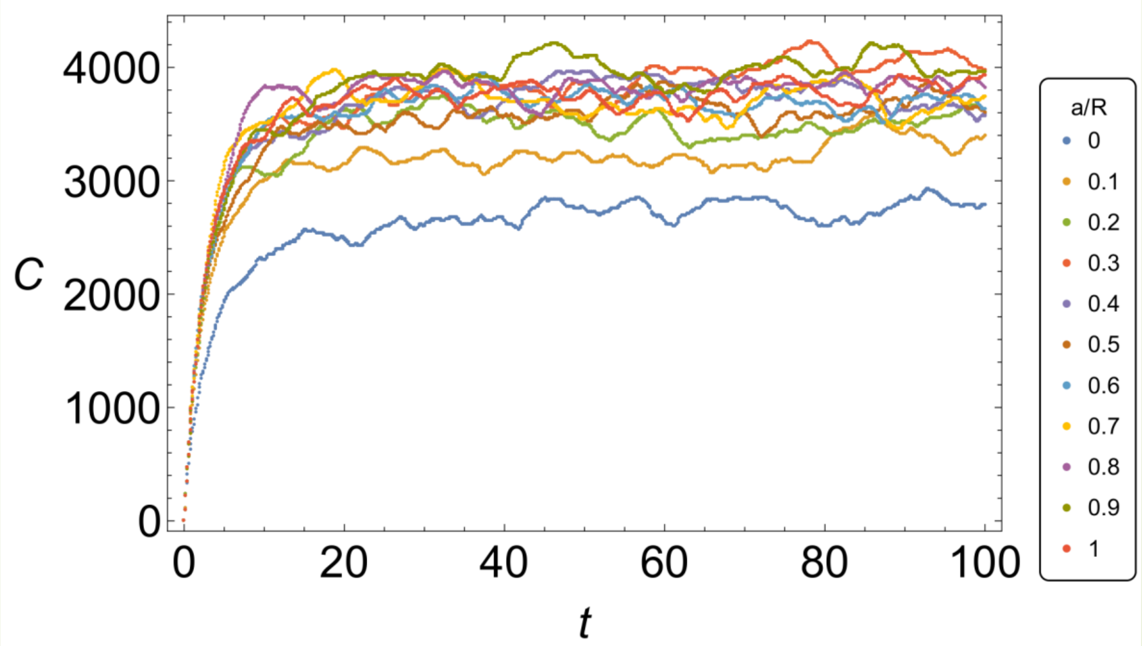
λ vs σ^2	-0.720372
$\langle \tilde{r} \rangle$ vs σ^2	-0.391709
λ vs $\langle \tilde{r} \rangle$	0.741396

- Significant correlations between $\lambda, \langle \tilde{r} \rangle, \sigma^2$
  - σ^2 can be a robust indicator of quantum chaos
  - Correspondence between classical & quantum chaos
- Universal for other billiards / for *state* complexity

Classical / quantum chaos in stadium billiard

- **Krylov complexity for operator p_x** on stadium billiard

Krylov complexity $C_{p_x}(t)$

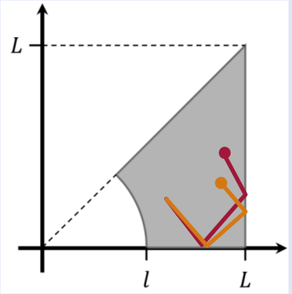


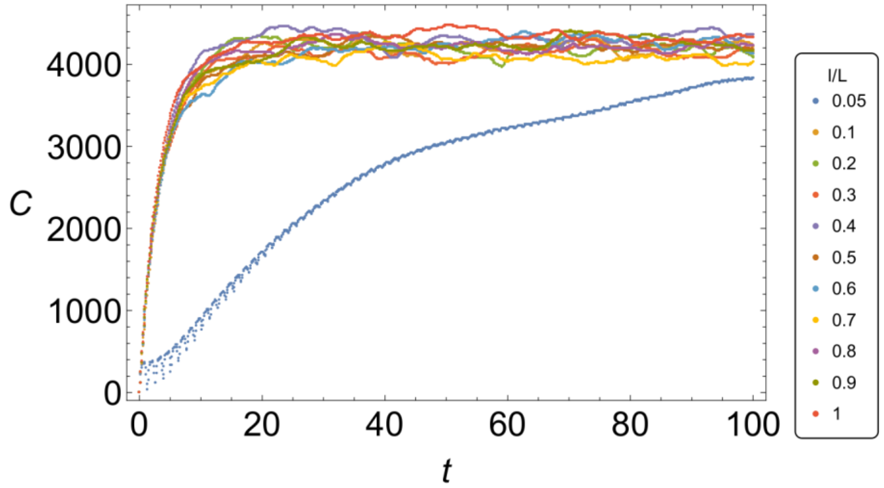
- No exponential growth of $C_{p_x}(t)$ at early time
- Late-time value correlated with classical λ ? (turn out to be system dependent)

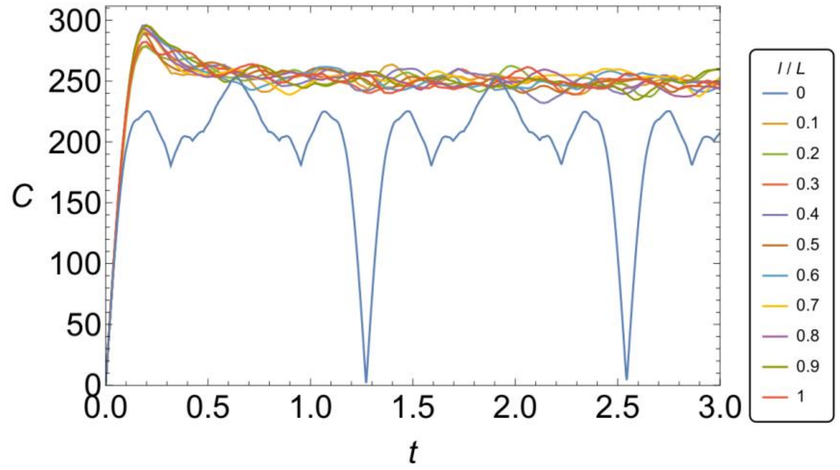
Krylov *operator* complexity C_{p_x}

Krylov *state* complexity $C_{|\psi\rangle}(t)$

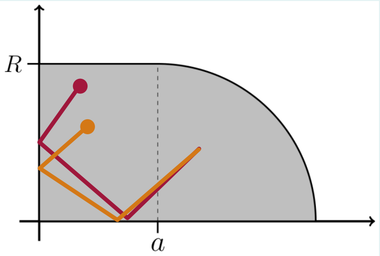
Sinai billiard

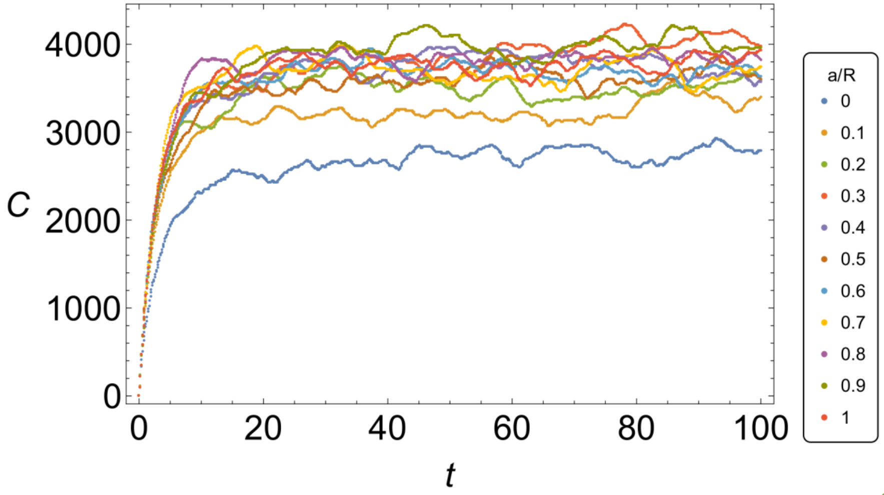


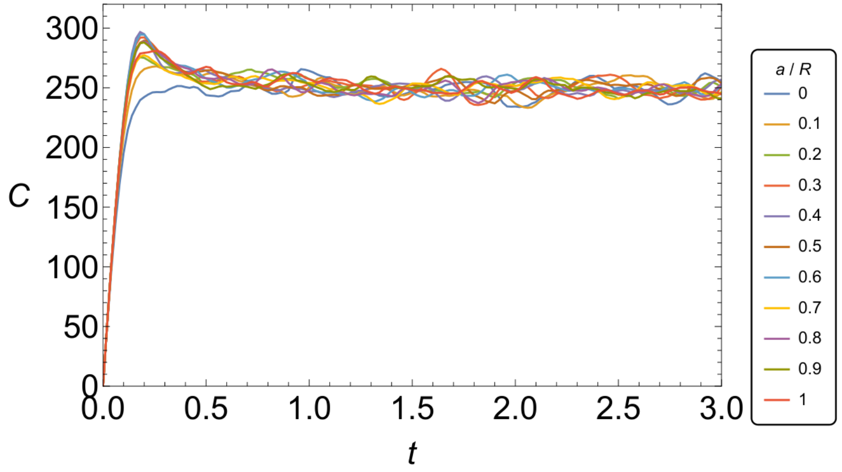




Stadium billiard





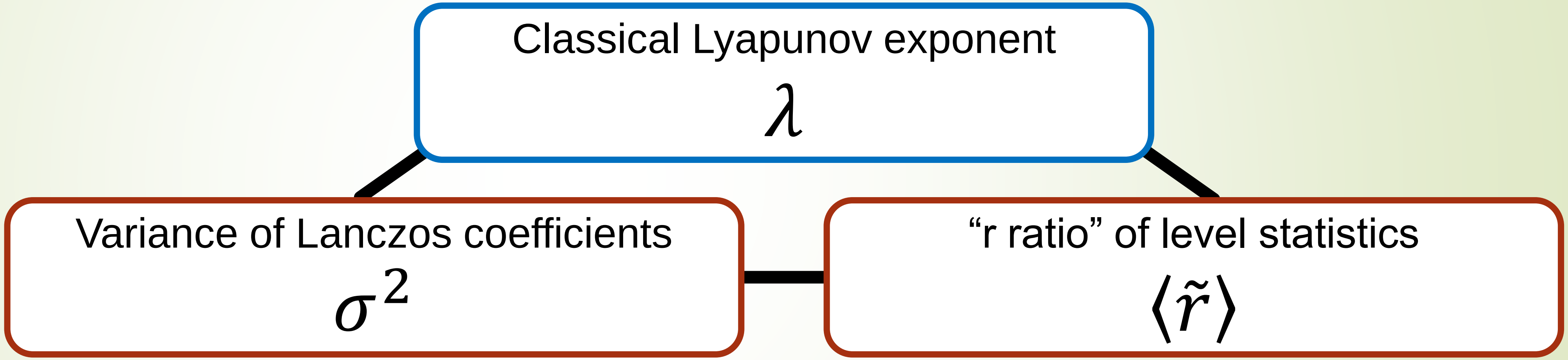


- No exponential growth
- Late time value insensitive to λ
- Peak of $C_{|\psi\rangle}$ correlated with λ

Summary

Krylov complexity and chaos in quantum mechanics

- **Classical** & **quantum chaos** in billiard systems
- We find significant correlations between **indicators of chaos**



- These correlations are universal for any billiards (stadium, Sinai, …), for both Krylov operator & state complexity
- Non-universality in time dependence of Krylov complexity?
- Other quantum mechanical systems? [Camargo+ '23, …]
- Implication to holography? [Rabinovici+ '23, …]