Quantum Error Correction Threshold and von Neumann Algebra Types

Yixu Wang

Institute for Advanced Study, Tsinghua University

September 18, 2023

清華大學 高等研究院
Institute for Advanced Study, Tsinghua University

Threshold of Quantum Error Correction Codes

- Errors occur during the storage, computation, error correction and decoding processes.
- Only codes with error threshold could be used for large scale and long time quantum computing. [Aharonov, Ben-Or 99, Knill; Laflamme, Zurek, 98; Kitaev 97]
- $$\lim_{n\to\infty} \Pr(\text{decoding success}) = \begin{cases} 1, & p < p_0 \\ 1/D, & p > p_0 \end{cases}$$

清華大學 高等研究院
Institute for Advanced Study, Tsinghua University

Thresholds are difficult to extract

- Concatenation of 3-repetition code [Aliferis, Gottesman, Preskill 05]
  Recursion relation between logical errors on different layers of concatenation.
- Toric code or planar code [Dennis, Kitaev, Landahl, Preskill 01; Chubb, Flammia 19]
  Map to a random bond Ising model and study its phase transition.
- Oscillator to oscillator codes do not have a threshold. [Hänggli, König 22]
- Can we try to have a better understanding of the underlying structure of the existence of the threshold of codes?

We provide a plausible conjecture that the existence of threshold is related to von Neumann algebra change.

- von Neumann algebra $\mathcal{A}$ is a set of $*-$algebra of bounded operators acting on a Hilbert space and closed under WOT. $\mathcal{A} = \mathcal{A}^{``}$.
- Relevant to our purpose: hyperfinite algebra

$$\mathcal{A}_\Gamma = \overline{\cup_{\Lambda_0 \in \Gamma} \mathcal{A}_\Lambda}, \quad \mathcal{A}_{\Lambda_0} = \otimes_{x \in \Lambda_0} \mathcal{B}(H_X).$$

- Γ: the whole lattice; Λ_0: a finite region; C: an infinite cone-like subregion
- The type depends on the state used when taking the closure and encodes the entanglement structure of the states in the Hilbert space.
- The type shows the properties of the projectors in the algebra.

More motivations

- von Neumann algebra is the natural mathematical language of systems of infinite degrees of freedom.
- There are examples in which algebra type can change depending on one parameter of the state. [Witten 22]

$$\Psi_{\mathrm{TFD}} = \frac{1}{\sqrt{Z(\beta)}} \bigotimes_{n=1}^{\infty} (|0_L 0_R\rangle + n^{-\beta} |1_L 1_R\rangle), \; Z(\beta) = \prod_{n=1}^{\infty} (1+n^{-\beta}).$$

When $\beta > 1$, $Z(\beta) < \infty$, $\mathcal{A}_L$ is type I_∞; when $\beta < 1$, $Z(\beta) > \infty$, $\mathcal{A}_L$ is type III_1.

- On ferromagnetic and antiferromagnetic phases of Ising model on a Bethe tree, the algebra types are different. [Mukhamedov 00]

More motivations from gravity

- There exists a Hawking-Page transition from thermal AdS space-time to AdS black hole.
- The types of algebras in thermal AdS and AdS black hole are conjectured to be Type I and Type III respectively. [Leutheusser, Liu 22]
- This transition is dual to the confinement/deconfinement phase transition in the dual large N gauge theory. [Witten 98]
- The holographic error correction code is microscopic model for AdS/CFT duality of quantum gravity. [Pastawski, Yoshida, Harlow, Preskill 15]
- In the holographic error correction code, this transition is conjectured to dual to the threshold transition . [Bao, Cao, Zhu 22]
- These facts suggests that a good candidate of holographic code should have threshold and the type change should in agreement with the type change in the gravity.

Which state? Which algebra?

States in the logical subspace?

- Stabilizer formalism gives logical subspace as ground states of a gapped Hamiltonian and they are pure states.
- If consider the algebra of operators on the full lattice constructed on this pure state, it is always a type I factor.
- How about some infinite subregion on the lattice?
- Trivial example: for repetition code, it is again type I;
- For any codes that can be realized by a 1D gapped Hamiltonian ground state, the algebra of its infinite half line subregion is type I. [Matsui 13]
- Non-trivial example: the algebra of a cone-shaped region of planar code ground state is type II_∞.[Ogata 22]

Which state? Which algebra?

States in the logical subspace dressed by error channel?

- Let the error channel on one lattice site x be $\varepsilon_x(\cdot) = \sum_i K_i(\cdot)K_i^\dagger$ and the total error channel be $\varepsilon_\Gamma(\cdot) = \bigotimes_{x\in\Gamma} \varepsilon_x(\cdot)$
- Consider the state $\omega \circ \varepsilon_\Gamma(\cdot)$, where ω is a pure state in the logical subspace.
- For example, when the error model is the bit flip error $\varepsilon_x(\cdot) = (1-p)I(\cdot) + pX(\cdot)X$, $0 < p \leq 1/2$, the algebra $\mathcal{A}_\Gamma$ is type III_λ with $\lambda = \frac{p}{1-p}$. If $p = 1/2$, $\mathcal{A}_\Gamma$ is type II_1. [Araki, Woods 68]
- How about considering $\mathcal{A}_C$ with state $\omega_C \circ \varepsilon_C(\cdot)$?
- For 1D, it is essentially the same case as $\omega \circ \varepsilon_\Gamma(\cdot)$. For 2D planar code, we are not sure but suspect it is type III for generic p.

Which state? Which algebra?

The thermal state in the different phases of the dual statistical model? [Dennis, Kitaev, Landahl, Preskill 01; Chubb, Flammia 19]

- For each error configuration, there is an Hamiltonian H_E such that its partition function corresponds to the probability of the error configuration class conjugated by stabilizers. $Z_E = \Pr(\bar{E})$.
- For toric code, this is the random bond Ising model.
- The threshold corresponds to a ordered/disordered phase transition in this model.
- An order parameter is the quenched free energy difference between error E and EL_m, L_m is a non-trivial logical operator.
- The free energy difference diverges below the threshold. When above threshold, the configurations where it diverges become negligible.

Which state? Which algebra?

We may take a thermal state of a given H_E, then averaging over different error configurations

- $\sum_E \Pr(E)\mathrm{tr}(e^{-\beta H_E}(\cdot)) = \langle\langle(\cdot)\rangle_\beta\rangle_p$.
- Hint that this might be the correct choice:
- In the disordered phase,

$$\lim_{r\to\infty} \langle\langle \sigma_o \sigma_r \rangle_\beta\rangle_p = 0.$$

- This is a necessary condition of the split property of the state in on two different regions. If the global state were pure, this ensures that the algebra on the cone-like subregion is type I.
- Still far from a complete proof!

Summary and outlook

- We provide a conjecture that the existence of the threshold of code implies the type of algebra changes along a one parameter family of states related to the code.
- We give some plausible arguments supporting this conjecture.
- We provide some trial states constructions.
- We hope this new probe of von Neumann algebra type could help to characterize different types of threshold transition.
- If the conjecture is (partially) true, we hope this may guide the search of realistic holographic codes.

Thank you!