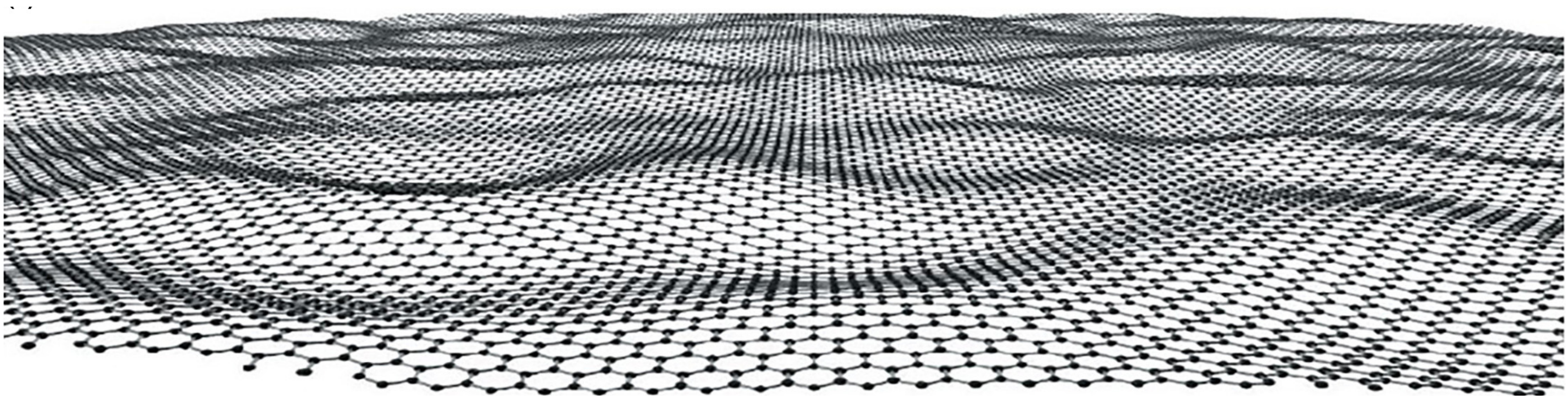
Analog gravity and the effective theory of graphene

Based on arXiv:2112.09144, 2305.08897



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YITP ‘23

Plan

Goal: understand graphene-like materials when deformed

- Graphene lattice model and Dirac equation
- Deforming graphene — early claims of curved space Dirac theory
- What goes wrong with naive effective theory
- How a consistent effective theory works
- An effective theory for `real' graphene?

Graphene

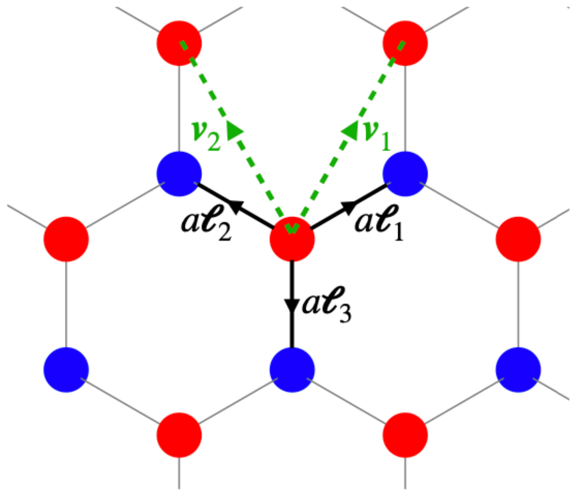
Lattice model and Dirac description

- Typical to model graphene as a nearest neighbour tight-binding model that accounts for the σ-bonds between carbon atoms.

- Hexagonal lattice of spins with two triangular sub-lattices (A & B);

Lattice spacing is a

$$\vec{\ell}_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \ , \quad \vec{\ell}_2 = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \ , \quad \vec{\ell}_3 = -\vec{\ell}_1 - \vec{\ell}_2 = (0, -1)$$



- Tight-binding Hamiltonian;

$$H_{undeformed} = T \sum_{n, \vec{x}_A} \left(a^\dagger_{\vec{x}_A} b_{\vec{x}_A + a\vec{\ell}_n} + \text{h.c.} \right)$$

Graphene

Lattice model and Dirac description

$$H_{undeformed} = T \sum_{n,\vec{x}_A} \left(a^\dagger_{\vec{x}_A} b_{\vec{x}_A + a\vec{\ell}_n} + \text{h.c.} \right)$$

- We are interested in the band structure given by the one particle states;

$$|\Psi(t)\rangle = \left(\sum_{\vec{x}_A} A_{\vec{x}_A}(t) a^\dagger_{\vec{x}_A} + \sum_{\vec{x}_B} B_{\vec{x}_B}(t) b^\dagger_{\vec{x}_B} \right) |0\rangle$$

- Write in terms of smooth wavefunctions, $\psi_{1,2}$, and rapidly varying wave vector K;

$$A_{\vec{x}_A}(t) = \psi_1(t, \vec{x}_A) e^{-\frac{i\pi}{4}} e^{+i\vec{K}\cdot\vec{x}_A} \;, \quad B_{\vec{x}_B}(t) = \psi_2(t, \vec{x}_B) e^{+\frac{i\pi}{4}} e^{+i\vec{K}\cdot\vec{x}_B}$$

- Then Schrödinger eq;

$$0 = i\hbar\partial_t \begin{pmatrix} \psi_1 \\ -\psi_2 \end{pmatrix} - iT \sum_n \begin{pmatrix} 0 & +e^{+ia\vec{K}\cdot\vec{\ell}_n} \\ -e^{-ia\vec{K}\cdot\vec{\ell}_n} & 0 \end{pmatrix} a\vec{\ell}_n \cdot \vec{\partial} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + O(a^2)$$

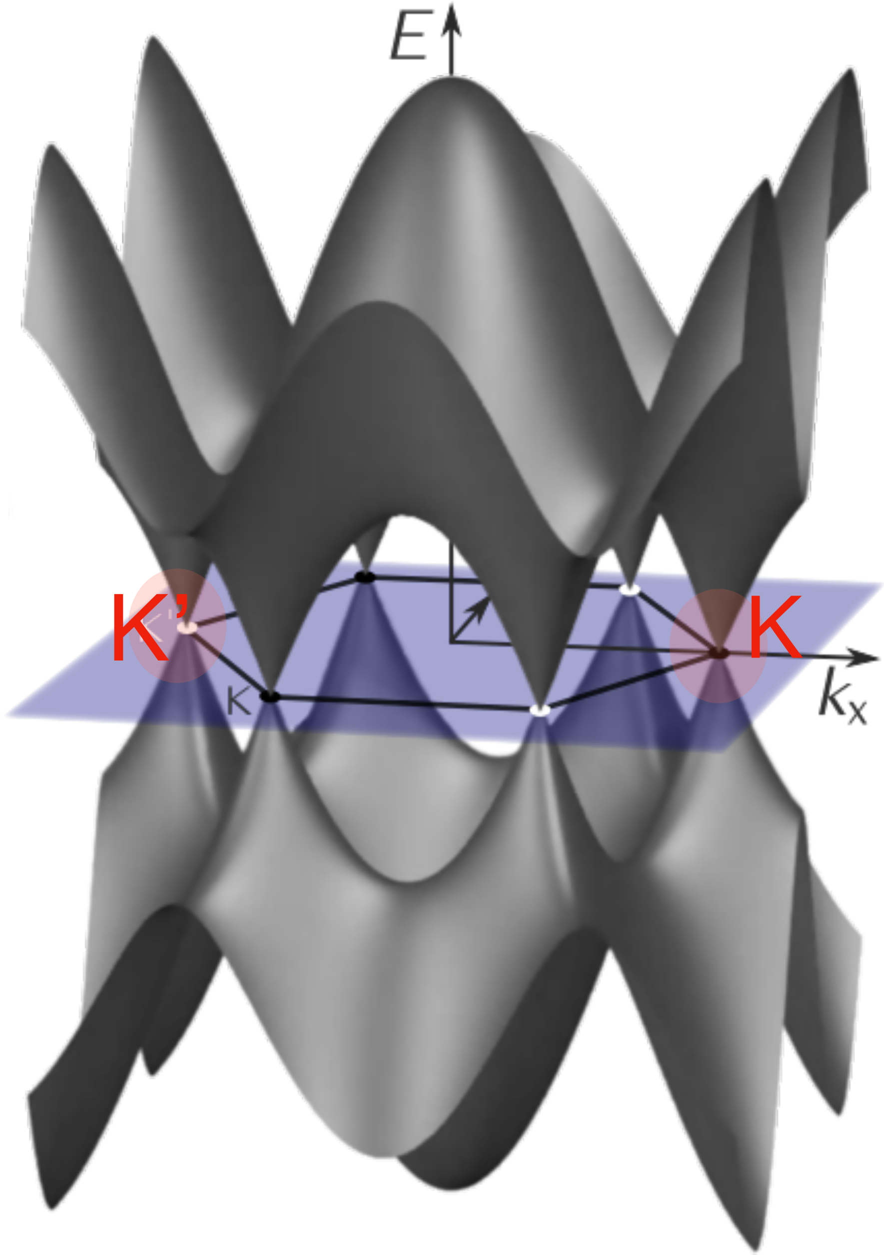
- And the `Dirac points' K, K' defined by;

$$\sum_n e^{ia\vec{K}\cdot\vec{\ell}_n} = 0 \quad \longrightarrow \quad \vec{K} = \frac{1}{a}\left(-\frac{4\pi}{3\sqrt{3}}, 0 \right)$$

$$\vec{K}' = -\vec{K}$$

Graphene

Lattice model and Dirac description



Graphene

Lattice model and Dirac description

- This is simply the 2+1 Dirac equation;

$$0 = e^{\mu}{}_{A}\gamma^{A}\partial_{\mu}\Psi + O(a)\ , \qquad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\gamma^{A} = \left(\gamma^{0}, \gamma^{I}\right) = \left(-i\sigma^{3}, \sigma^{1}, \sigma^{2}\right)$$

$$e^{\mu}{}_{A} = \begin{pmatrix} \frac{1}{c_{eff}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \quad g_{\mu\nu} = \begin{pmatrix} -c_{eff}^{2} & 0 \\ 0 & \delta_{ij} \end{pmatrix}$$

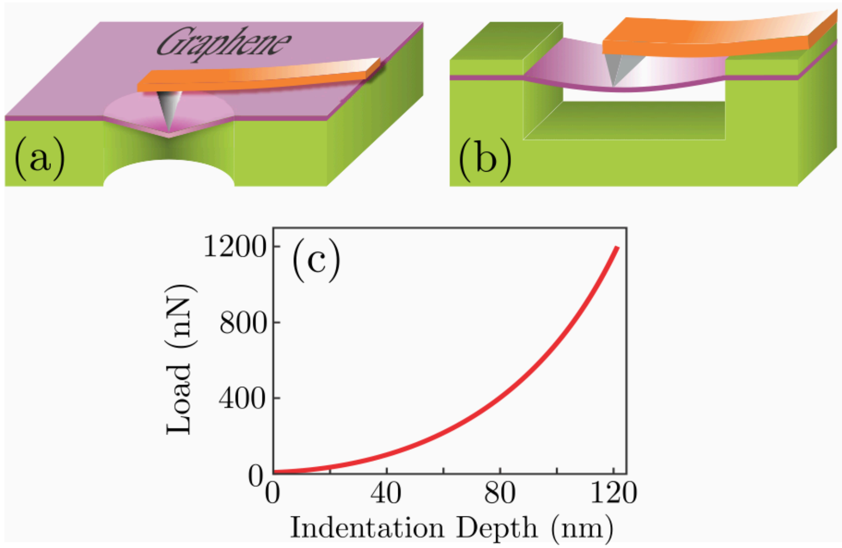
$$c_{eff} = \frac{3aT}{2\hbar}$$ ~ 1/100 speed of light

Deformed Graphene

Deformed lattice model

- Graphene is a flexible material - can sustain strains of ~25%;

[Naumis, Barraza-Lopez, Oliva-Leyva, Terrrones ’17]



- What happens to the Dirac equation?

- A natural conjecture is it becomes the curved space Dirac equation — an interesting case of `*analog gravity*’

Deformed Graphene

Deformed lattice model

- Keeping the nearest neighbour model then for `weak' bending, ie. on scales much larger than lattice scale;

$$H_{undeformed} = T \sum_{n,\vec{x}_A} \left(a^\dagger_{\vec{x}_A} b_{\vec{x}_A + a\vec{\ell}_n} + \text{h.c.} \right)$$

Write in terms of smooth '*hopping*' functions

$$H_{deformed} = \sum_{n,\vec{x}_A} T_{n,A} \left(a^\dagger_{\vec{x}_A} b_{\vec{x}_A + a\vec{\ell}_n} + \text{h.c.} \right) \;=\; T \sum_{n,\vec{x}_A} t_n(\vec{x}_A + \frac{a}{2}\vec{\ell}_n) \left(a^\dagger_{\vec{x}_A} b_{\vec{x}_A + a\vec{\ell}_n} + \text{h.c.} \right)$$

- Take units so that variations of $t_n(\vec{x})$ are on scales of $O(1)$, and the lattice scale $a \ll 1$

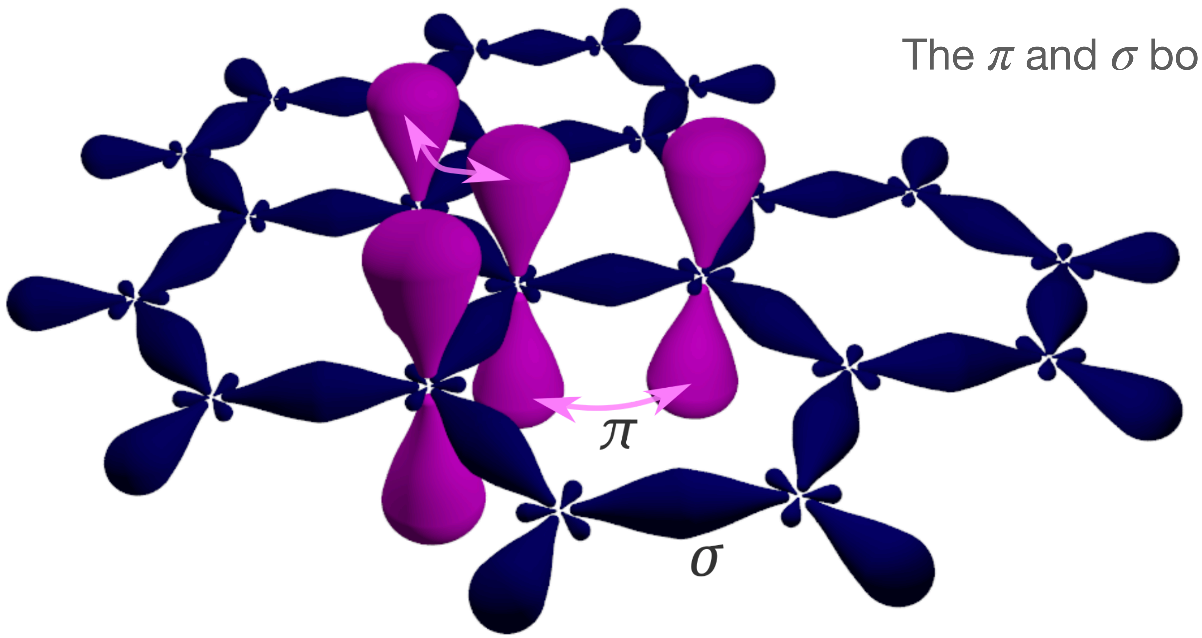
- Then we treat the deformation perturbatively;

$$t_n(\vec{x}) = 1 + \epsilon \delta_1 t_n(\vec{x}) + \epsilon^2 \delta_2 t_n(\vec{x}) + \ldots$$

Deformed Graphene

Deformed lattice model

- Important caveat: very unclear this is good model for ‘out of plane’ deformations of graphene



The π and σ bonds are orthogonal when in plane

- Hence we could focus of ‘in-plane’ deformations

Deformed Graphene

Previous claims

- It was known that the first effect of strain, or deformation, was to produce a magnetic gauge field (called a ‘pseudo’ or ‘strain’ magnetic field);

[eg. Vozmediano, Katsnelson, Guinea, *Phys Reports* ’10]

$$0 = e^{\mu}{}_{A}\gamma^{A}\partial_{\mu}\Psi$$

$$a e^{\mu}{}_{A}\gamma^{A}D_{\mu}\Psi$$

$$D_{\mu}\Psi = \partial_{\mu}\Psi \mp i A_{\mu}\Psi$$

$$A_{\mu} = (0, A^{i}) \qquad A_{i} = \frac{\epsilon}{3a}\begin{pmatrix} \delta t_1 + \delta t_2 - 2\delta t_3 \\ -\sqrt{3}\,(\delta t_1 - \delta t_2) \end{pmatrix}$$

- Note — scales inversely with lattice spacing….

[Zubkov, Volovik ’13]

Deformed Graphene

Previous claims

- Then it was claimed that the full effect of deformation caused curved spacetime (to leading low energy order)

[de Juan, Sturla, Vozmediano, *Phys. Rev. Lett.* '12]

$$0 = a e^{\mu}{}_{A} \gamma^{A} D_{\mu} \Psi \qquad D_{\mu}\Psi = \partial_{\mu}\Psi \mp i A_{\mu}\Psi - \frac{i}{2}\Omega_{\mu AB} S^{AB}\Psi$$

$$D_t \Psi = \partial_t \Psi \quad , \qquad D_i \Psi = \partial_i \Psi \mp i A_i \Psi + \frac{i}{2}\omega_i \sigma^3 \Psi$$

- The frame is perturbed;

$$e^{i}{}_{I} = \delta^{i}{}_{I} + \epsilon \delta e^{i}{}_{I} + \ldots \qquad \delta e^{i}{}_{I} = \begin{pmatrix} \frac{2}{3}\delta t_1 + \frac{2}{3}\delta t_2 - \frac{1}{3}\delta t_3 & \frac{1}{\sqrt{3}}\delta t_1 - \frac{1}{\sqrt{3}}\delta t_2 \\ \frac{1}{\sqrt{3}}\delta t_1 - \frac{1}{\sqrt{3}}\delta t_2 & \delta t_3 \end{pmatrix}$$

- This is an analog gravity model — and very interesting

A subtlety

- First we consider undeformed theory to higher order;

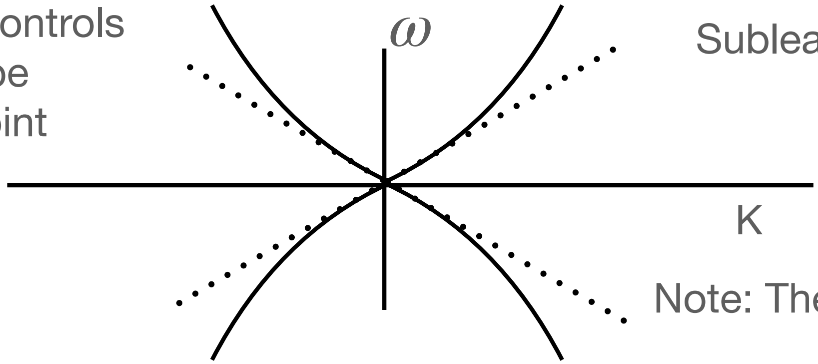
$$|\Psi(t)\rangle = \left(\sum_{\vec{x}_A} A_{\vec{x}_A}(t) a^\dagger_{\vec{x}_A} + \sum_{\vec{x}_B} B_{\vec{x}_B}(t) b^\dagger_{\vec{x}_B} \right) |0\rangle$$

$$A_{\vec{x}_A}(t) = \psi_1(t, \vec{x}_A) e^{-\frac{i\pi}{4}} e^{+i\vec{K}\cdot\vec{x}_A} \; , \quad B_{\vec{x}_B}(t) = \psi_2(t, \vec{x}_B) e^{+\frac{i\pi}{4}} e^{+i\vec{K}\cdot\vec{x}_B}$$

- This yields a low energy expansion;

$$0 = e^\mu{}_A \gamma^A \partial_\mu \Psi + ia\, \eta_{AB} \gamma^A e^B{}_\sigma C^{\sigma\mu\nu} \partial_\mu \partial_\nu \Psi + a^2\, \eta_{AB} \gamma^A e^B{}_\sigma D^{\sigma\mu\nu\rho} \partial_\mu \partial_\nu \partial_\rho \Psi + O(a^3)$$

Leading term controls
linear slope
of Dirac point



Subleading terms control shape

Note: These higher derivative terms break Lorentz symmetry
[Iorio et al ’17]

- Controlled by lattice invariants;

$$C^{ijk} = -\frac{1}{3} \epsilon_{kl} \sum_n \ell^i_n \ell^j_n \ell^l_n \qquad D^{ijkl} = \frac{1}{9} \sum_n \ell^i_n \ell^j_n \ell^k_n \ell^l_n = \frac{1}{24} \left(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + \delta^{ij}\delta^{kl} \right)$$

A subtlety

- Now we identify a gauge and frame symmetry as a freedom in making the wavefunction identification;

$$|\Psi(t)\rangle = \left(\sum_{\vec{x}_A} A_{\vec{x}_A}(t) a^\dagger_{\vec{x}_A} + \sum_{\vec{x}_B} B_{\vec{x}_B}(t) b^\dagger_{\vec{x}_B} \right) |0\rangle$$

$$A_{\vec{x}_A}(t) = \psi_{1,2}(t, \vec{x}_A) e^{\frac{i}{2}\left(-\frac{\pi}{2} \pm \phi(\vec{x}_A)\right)} e^{\pm i(\vec{K}\cdot\vec{x}_A - \lambda(\vec{x}_A))}, \quad B_{\vec{x}_B}(t) = \psi_{2,1}(t, \vec{x}_B)\, e^{-\frac{i}{2}\left(-\frac{\pi}{2} \pm \phi(\vec{x}_B)\right)} e^{\pm i(\vec{K}\cdot\vec{x}_B - \lambda(\vec{x}_B))}$$

Spatial frame rotation

Spatial gauge transformation

- This yields a low energy theory with local symmetry;

$$0 = e^{\mu}{}_{A}\gamma^{A} D_\mu \Psi \pm ia\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} C^{\sigma\mu\nu} D_\mu D_\nu \Psi + a^2\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} D^{\sigma\mu\nu\rho} D_\mu D_\nu D_\rho \Psi + O(a^3)$$

$$D_i\Psi = \partial_i\Psi \mp iA_i\Psi + \frac{i}{2}\omega_i\sigma^3\Psi$$

$$e^{A}{}_{\mu} = \begin{pmatrix} c_{eff} & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & +\sin\phi & \cos\phi \end{pmatrix}, \qquad A_i = \partial_i\lambda \qquad \omega_i = \partial_i\phi$$

A subtlety

- Remember that the gauge field scaled inversely with the lattice spacing;

$$A \sim \frac{\epsilon}{a}\tilde{A} \qquad \tilde{A} \sim O(1)$$

- But this mixes up the naive derivative expansion;

$$0 = e^{\mu}{}_{A}\gamma^{A} D_{\mu}\Psi \pm i a\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} C^{\sigma\mu\nu} D_{\mu} D_{\nu}\Psi + a^{2}\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} D^{\sigma\mu\nu\rho} D_{\mu} D_{\nu} D_{\rho}\Psi + O(a^{3})$$

Spin connection correction

$$a e^{\mu}{}_{A}\gamma^{A} D_{\mu}\Psi \sim a\partial\Psi + \epsilon\tilde{A}\Psi + \epsilon a\partial\Psi + \epsilon a\Psi$$

Frame correction

Terms of same order

$$i a^{2}\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} D_{\mu}\left(C^{\sigma\mu\nu} D_{\nu}\Psi\right) \sim a^{2}\partial^{2}\Psi + \epsilon a\tilde{A}\partial\Psi + \epsilon a(\partial\tilde{A})\Psi + \epsilon^{2}\tilde{A}^{2}\Psi + \epsilon a^{2}\partial^{2}\Psi + \epsilon a^{2}\partial\Psi + \epsilon a^{2}\Psi$$

A subtlety

Consistent structure

- We find that a consistent structure requires consistent truncation in both ϵ and a expansions;

$$0 = e^{\mu}{}_{A}\gamma^{A} D_{\mu}\Psi \pm ia\,\eta_{AB}\gamma^{A} e^{B}{}_{\sigma} C^{\sigma\mu\nu} D_{\mu}D_{\nu}\Psi + a^{2}\,\eta_{AB}\gamma^{A} e^{B}{}_{\sigma} D^{\sigma\mu\nu\rho} D_{\mu}D_{\nu}D_{\rho}\Psi + O(a^{3})$$

Covariant derivatives included	Gauge field contributions	Metric contributions
Dirac term only	$\frac{\epsilon}{a}$	Trivial flat metric
Dirac + two derivative	$\frac{\epsilon}{a},\ \epsilon,\ \frac{\epsilon^2}{a}$	ϵ
Dirac, two and three derivatives	$\frac{\epsilon}{a},\ \epsilon,\ \epsilon a,\ \frac{\epsilon^2}{a},\ \epsilon^2$	$\epsilon,\ \epsilon a,\ \epsilon^2$

- Hence first metric corrections require including two derivative term…

Effective theory

Some details…

- We expand in orders of ϵ and a;

$$t_n(\vec{x}) = 1 + \epsilon\delta_1 t_n(\vec{x}) + \epsilon^2\delta_2 t_n(\vec{x}) + \ldots$$

$$\delta_k t_n(\vec{x}) = \delta_{k,0} t_n(\vec{x}) + a\delta_{k,1} t_n(\vec{x}) + a^2\delta_{k,2} t_n(\vec{x}) + \ldots$$

- Write Schrödinger system;

$$\frac{i\hbar}{T}\partial_t A_{\vec{x}_A} - \sum_n t_n(\vec{x}_A + \frac{a\vec{\ell}_n}{2}) B_{\vec{x}_A + a\vec{\ell}_n} = 0$$

$$\frac{i\hbar}{T}\partial_t B_{\vec{x}_B} - \sum_n t_n(\vec{x}_B - \frac{a\vec{\ell}_n}{2}) A_{\vec{x}_B - a\vec{\ell}_n} = 0$$

- Write wavefunctions including frame and gauge;

$$A_{\vec{x}_A}(t) = \psi_{1,2}(t, \vec{x}_A) f(\vec{x}) e^{\frac{i}{2}(-\frac{\pi}{2} \pm \phi(\vec{x}_A))} e^{\pm \frac{i\Phi(\vec{x}_A)}{a}}$$

$$B_{\vec{x}_B}(t) = \psi_{2,1}(t, \vec{x}_B) f(\vec{x}) e^{-\frac{i}{2}(-\frac{\pi}{2} \pm \phi(\vec{x}_B))} e^{\pm \frac{i\Phi(\vec{x}_B)}{a}}$$

Effective theory

Some details…

- What is $f(\vec{x})$?

- There is a natural choice;

$$\sqrt{|g_{ij}|}\bar{\Psi}\gamma^t\bar{\Psi} = |A|^2 + |B|^2 \longrightarrow f = (\det g_{ij})^{1/4}$$

- Then the number density of Dirac field is the same as for the microscopic electron density

- However we leave it for the equations to determine for now.

Effective theory

Some details...

$$A_{\vec{x}_A}(t) = \psi_{1,2}(t, \vec{x}_A) f(\vec{x}) e^{\frac{i}{2}(-\frac{\pi}{2} \pm \phi(\vec{x}_A))} e^{\pm \frac{i\Phi(\vec{x}_A)}{a}}$$

- Expand as;

$$\Phi(\vec{x}) = -\frac{4\pi}{3\sqrt{3}} x + \sum_{n=1}^{\infty} \epsilon^n \delta_n \Phi(\vec{x}) \ , \quad \phi(\vec{x}) = \sum_{n=1}^{\infty} \epsilon^n \delta_n \phi(\vec{x}) \ , \quad f(\vec{x}) = 1 + \sum_{n=1}^{\infty} \epsilon^n \delta_n f(\vec{x})$$

$$\delta_n \Phi = \sum_{m=0}^{\infty} a^m \delta_{n,m} \Phi$$ and similarly for $\delta_n \phi, \delta_n f$

- For $\epsilon = 0$ then; $\frac{1}{a}\Phi(\vec{x}) = \vec{K} \cdot \vec{x}$

- Order of limits — we require;

$$e^{\frac{i\Phi(\vec{x})}{a}} \simeq e^{i\vec{K}\cdot\vec{x}} \left(1 + \frac{i\epsilon}{a} \delta_1 \Phi + \frac{i\epsilon^2}{a} \delta_2 \Phi - \frac{\epsilon^2}{2a^2} (\delta_1 \Phi)^2 + O(\epsilon^3) \right)$$

- May think of expanding in a and $\lambda = \dfrac{\epsilon}{a}$

$$e^{\frac{i\Phi(\vec{x})}{a}} \simeq e^{i\vec{K}\cdot\vec{x}} \left(1 + i\lambda \left(\delta_{1,0}\Phi + a\delta_{1,1}\Phi + a^2 \delta_{1,2}\Phi + O(a^3) \right) - \lambda^2 \left(\frac{1}{2} (\delta_{1,0}\Phi)^2 + a\delta_{1,0}\Phi \delta_{1,1}\Phi - 2ia\delta_{2,0}\Phi + O(a^2) \right) + O(\lambda^3) \right)$$

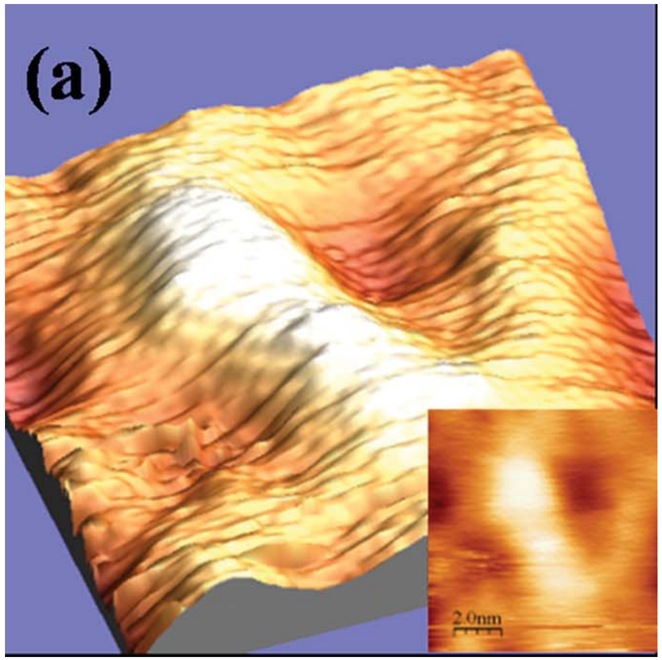
Effective theory

An aside on ripples

[Meyer et al ; Fasolino, Los, Katsnelson]

- For ripples; height $\sim 0.5nm$, wavelength $\sim 5nm$ as compared to $a \sim 0.25nm$.
- In our units, corresponds to; $a \sim 0.05$, $\epsilon \sim 0.01$ so $\lambda \sim 0.2$

STM image: Zan et al ‘12



Effective theory

Some details...

$$\lambda = \frac{\epsilon}{a}$$

- Schrodinger system; $$0 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a^p \lambda^q \mathcal{O}_{p,q}(\vec{x}) \begin{pmatrix} \psi_1(\vec{x}) \\ \psi_2(\vec{x}) \end{pmatrix}$$

$$A_\mu = \left(0, A^i\right) \ , \quad A_i = \sum_{n=1}^{\infty} \epsilon^n \delta_n A_i \ , \quad \delta_n A_i = \frac{1}{a} \sum_{m=0}^{\infty} a^m \delta_{n,m} A_i$$

$$e^\mu{}_A = \begin{pmatrix} \frac{1}{c_{eff}} & 0 \\ 0 & e^i{}_I \end{pmatrix} \ , \quad e^i{}_I = \delta^i_I + \sum_{n=1}^{\infty} \epsilon^n \delta_n e^i{}_I \ , \quad \delta_n e^i{}_I = \sum_{m=0}^{\infty} a^m \delta_{n,m} e^i{}_I$$

- Consider a term; $a^M Q^{\mu_1 \ldots \mu_M} D_{\mu_1} \cdots D_{\mu_M} \Psi$
- Contributes to equations; $\mathcal{O}_{p,q} \quad \text{for} \quad p \geq M$

Effective theory

Some details…

- Unlike usual effective theory

$$a^M Q^{\mu_1 \cdots \mu_M} D_{\mu_1} \cdots D_{\mu_M} \Psi = a^M T_{M,0} + \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} a^p \lambda^q T_{p,q}$$

	a^0	a^1	$\dots$	a^{M-1}	a^M	a^{M+1}	a^{M+2}	$\dots$
ϵ^0	0	0	$\dots$	0	$T_{M,0}$	0	0	$\dots$
ϵ^1	0	0	$\dots$	$T_{M,1}$	$T_{M+1,1}$	$T_{M+2,1}$	$T_{M+3,1}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
ϵ^{M-1}	0	$T_{M,M-1}$	$\dots$	$T_{2M-2,M-1}$	$T_{2M-1,M-1}$	$T_{2M,M-1}$	$T_{2M+1,M-1}$	$\dots$
ϵ^M	$T_{M,M}$	$T_{M+1,M}$	$\dots$	$T_{2M-1,M}$	$T_{2M,M}$	$T_{2M+1,M}$	$T_{2M+2,M}$	$\dots$
ϵ^{M+1}	$T_{M+1,M+1}$	$T_{M+2,M+1}$	$\dots$	$T_{2M,M+1}$	$T_{2M+1,M+1}$	$T_{2M+2,M+1}$	$T_{2M+3,M+1}$	$\dots$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		

Effective theory

Some details…

- Leading order $\mathcal{O}_{p,q} \; for \; p \leq 1$

- Recover flat Dirac equation with large gauge field

Effective theory

Some details…

$$\sqrt{|g|}\, C^{ijk}(\vec{x}) = -\frac{1}{3}\epsilon_{kl}\sum_n \ell_n^i \ell_n^j \ell_n^l$$

$$\sqrt{|g|}\, D^{ijkl}(\vec{x}) = \frac{1}{9}\sum_n \ell_n^i \ell_n^j \ell_n^k \ell_n^l$$

- Subleading order $\mathcal{O}_{p,q}\ for\ p \leq 2$
- Curved space Dirac equation and higher covariant derivative term

$$0 = a e^{\mu}{}_{A}\gamma^{A} D_{\mu}\Psi \pm i a^2\, \eta_{AB}\gamma^{A} e^{B}{}_{\sigma} C^{\sigma\mu\nu} D_{\mu} D_{\nu}\Psi + O(\epsilon^3, \epsilon^2 a, \epsilon a^2, a^3)$$

$$e^{i}{}_{I} = \delta^{i}_{I} + \epsilon\left(\delta_{1,0} e^{i}{}_{I} + O(a)\right) + O(\epsilon^2)\,, \quad f = 1 + \epsilon\left(\delta_{1,0} f + O(a)\right) + O(\epsilon^2)$$

$$A_i = \frac{\epsilon}{a}\left(\delta_{1,0} A_i + a\delta_{1,1} A_i + O(a^2)\right) + \frac{\epsilon^2}{a}\left(\delta_{2,0} A_i + O(a)\right) + O(\frac{\epsilon^3}{a})\,.$$

- **Key point**; effect of curved frame *at same order* as higher derivative
- Interesting point; consistency require the usual torsion free connection and requires the canonical scaling of $f(\vec{x})$

$$\sqrt{|g_{ij}|}\,\bar{\Psi}\gamma^{t}\bar{\Psi} = |A|^2 + |B|^2 \qquad f = (\det g_{ij})^{1/4}$$

Effective theory

Some details…

- Subsubleading order $\mathcal{O}_{p,q}\ for\ p \leq 3$
- Curved space Dirac equation and higher covariant derivative term

$$0 = a e^{\mu}{}_{A}\gamma^{A} D_{\mu}\Psi \pm i a^{2}\,\eta_{AB}\gamma^{A} e^{B}_{\ \sigma} D_{\mu}\left(C^{\sigma\mu\nu} D_{\nu}\Psi\right) + a^{3}\,\eta_{AB}\gamma^{A} e^{B}_{\ \sigma} D^{\sigma\mu\nu\rho} D_{\mu} D_{\nu} D_{\rho}\Psi + O(\epsilon^{4}, \epsilon^{3} a, \epsilon^{2} a^{2}, \epsilon a^{3}, a^{4})$$

$$e^{i}{}_{I} = \delta^{i}_{I} + \epsilon\left(\delta_{1,0} e^{i}{}_{I} + a\delta_{1,1} e^{i}{}_{I} + O(a^{2})\right) + \epsilon^{2}\left(\delta_{2,0} e^{i}{}_{I} + O(a)\right) + O(\epsilon^{3})$$

$$A_{i} = \frac{\epsilon}{a}\left(\delta_{1,0} A_{i} + a\delta_{1,1} A_{i} + a^{2}\delta_{1,2} A_{i} + O(a^{3})\right) + \frac{\epsilon^{2}}{a}\left(\delta_{2,0} A_{i} + a\delta_{2,1} A_{i} + O(a^{2})\right) + \frac{\epsilon^{3}}{a}\left(\delta_{3,0} A_{i} + O(a)\right) + O(\frac{\epsilon^{4}}{a})$$

- Now includes quadratic corrections in curvature; again connection is torsion free.
- This order is required to see corrections to the dispersion relation.

Effective theory

Result

- To this order we find and ‘*electrometric*’;

$$f = (\det g_{ij})^{1/4} + O(\epsilon^3, \epsilon^2 a, \epsilon a^2)$$

$$\Delta^2 = (\sum_n t_n^2)^2 - 2(\sum_m t_m^4)$$

$$g_{ij} = \frac{3}{\Delta^2} \sum_n (\delta_{ij} - \frac{4}{3} \ell_n^i \ell_n^j) t_n^2 + O(\epsilon^3, \epsilon^2 a, \epsilon a^2)$$

- And the gauge field (up to gauge transformation);

$$A_i = \frac{1}{a\Delta^2} \epsilon_{ij} \sum_m \Big[\ell_m^j \Delta t_m \left(2 + \sum_n (3\delta_{mn} \Delta t_n) + \sum_{n,p} \left(\left(\frac{1}{3} + 2\delta_{mn} - 3\delta_{np} \right) \Delta t_n \Delta t_p \right) \right) + a^2 \left(\frac{1}{4} \ell_m^j \ell_m^k \ell_m^l - \frac{3}{8} K^{jkl} + \frac{1}{6} \delta^{jk} \ell_m^l \right) \partial_k \partial_l \Delta t_m \Big] + O(\frac{\epsilon^4}{a}, \epsilon^3, \epsilon^2 a, \epsilon a^2)$$

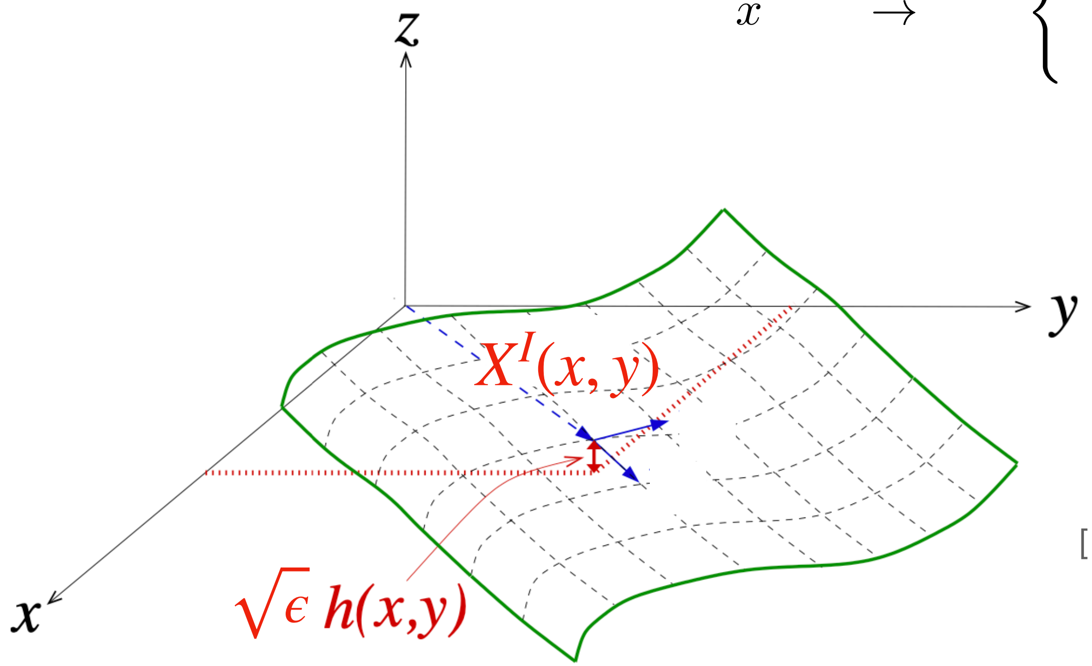
$$\Delta t_n = t_n - 1$$

- Note — if tune so gauge field vanishes then electrometric is exact

Elastic strain

- We may embed lattice as;

$$\begin{array}{ccc} \mathbb{R}^2_{lat} & \to & \mathbb{R}^3_{lab} \\ \vec{x} & \to & \begin{cases} X^I(\vec{x}) = \delta^I_i \left(x^i + \epsilon v^i(\vec{x})\right) \\ Z(\vec{x}) = \sqrt{\epsilon} h(\vec{x}) \end{cases} \end{array}$$



[from review Bowick, Travesset]

- This gives an induced metric and strain tensor;

$$g^{(ind)}_{ij} = \delta_{ij} + \epsilon \left(\delta_{ik} \frac{\partial v^k}{\partial x^j} + \delta_{jk} \frac{\partial v^k}{\partial x^i} + \frac{\partial h}{\partial x^i} \frac{\partial h}{\partial x^j} \right) + \epsilon^2 \delta_{kl} \frac{\partial v^k}{\partial x^i} \frac{\partial v^l}{\partial x^j}$$

$$\sigma_{ij} = \frac{1}{2} \left(g^{(ind)}_{ij} - \delta_{ij} \right)$$

Elastic strain

- We then choose a simple bond model;

$$\frac{T_{n,A}}{T} = F\left(\frac{L_{n,A}}{a} - 1\right)$$

$$F(0) = 1\ , \quad F'(0) = -\beta\ , \quad F''(0) = (\tau - 1)\beta$$

- This assumes only dependence of bond length

- And then we can express the hopping functions in terms of strain;

$$t_n(\vec{x}) = 1 - \beta\left(\ell_n^i \ell_n^j \sigma_{ij}(\vec{x}) + \frac{a^2}{24}\ell_n^i \ell_n^j (\ell_n^k \partial_k)^2 \sigma_{ij}(\vec{x})\right) + \frac{\beta\tau}{2}\left(\ell_n^i \ell_n^j \sigma_{ij}(\vec{x})\right)^2 + O(\epsilon a^3, \epsilon^2 a^2, \epsilon^3)$$

Elastic strain

- Now we can write the effective theory in terms of strain.
- To leading order;

$$K^{ijk} = \frac{4}{3}\sum_n \ell_n^i \ell_n^j \ell_n^k$$

$$A_i(\vec{x}) = -\frac{\beta}{2a}\epsilon_{ij}\left(K^{jkl}\sigma_{kl}(\vec{x}) + O(\epsilon^2, \epsilon a^2)\right) \ , \quad g_{ij}(\vec{x}) = \delta_{ij} + 2\beta\sigma_{ij}(\vec{x}) + O(\epsilon^2, \epsilon a)$$

- Now for an in-plane diffeomorphism, $\sigma_{ij} = \epsilon\partial_{(i}v_{j)}$
- Then the electrometric is also flat — so no `analog gravity'
- Note — it is not the same as the induced metric though!

$$g_{ij}^{ind}(\vec{x}) = \delta_{ij} + 2\sigma_{ij}(\vec{x}) + \ldots$$

Elastic strain

- Going beyond leading order;

$$\sigma_i = K^{ijk}\sigma_{jk}$$

$$A_i(\vec{x}) = -\frac{\beta\epsilon_{ij}}{2a}\Big(K^{jkl}\left(\sigma_{kl}(\vec{x}) + \frac{(\beta-\tau)}{2}\sigma_{km}(\vec{x})\sigma_{ml}(\vec{x}) - \frac{(3\beta+\tau)}{8}\sigma_k(\vec{x})\sigma_l(\vec{x})\right)$$
$$+\frac{a^2}{12}\left(9\partial_j\partial_k\sigma_k(\vec{x}) - 3\partial_k\partial_k\sigma_j(\vec{x}) - 7K^{klm}\partial_k\partial_l\sigma_{jm}(\vec{x})\right) + O(\epsilon a^3, \epsilon^2 a^2, \epsilon^3)\Big)$$

$$g_{ij}(\vec{x}) = \delta_{ij} + 2\beta\sigma_{ij}(\vec{x}) + 4\beta^2\sigma_{ik}(\vec{x})\sigma_{kj}(\vec{x}) + \frac{\beta(\beta+\tau)}{4}\left(\delta_{ij}\left(\sigma_{kk}(\vec{x})\right)^2 - 4\sigma_{ij}(\vec{x})\sigma_{kk}(\vec{x}) - \sigma_i(\vec{x})\sigma_j(\vec{x})\right) + O(\epsilon a^2, \epsilon^2 a, \epsilon^3)$$

- Very interestingly we find pure *in-plane* strain results in a *curved* electrometric at quadratic order!

- So we can have analog gravity effects

General graphene theory

- Consider the lattice model to leading order. We have;

$$0 = a e^{\mu}{}_{A} \gamma^{A} D_{\mu} \Psi + O(\epsilon^2, \epsilon a) \ , \quad A_i(\vec{x}) = -\frac{\beta}{2a} \epsilon_{ij} K^{ijk} \sigma_{jk}(\vec{x}) + O(\frac{\epsilon^2}{a}, \epsilon)$$

- Simply controlled by symmetries.
- Hence for 'real graphene' and in-plane deformation conjecture the same expressions, but now the constants c_{eff}, a, β should be determined from 'experiment'

General graphene theory

- Natural to then expect that beyond leading order the effective theory of real graphene takes the form controlled by symmetries;

$$0 = a e^{\mu}{}_{A} \gamma^{A} D_{\mu} \Psi \pm c_2 i a^2 \, \eta_{AB} \gamma^{A} e^{B}{}_{\sigma} D_{\mu} \left(C^{\sigma\mu\nu} D_{\nu} \Psi \right) + c_3 a^3 \, \eta_{AB} \gamma^{A} e^{B}{}_{\sigma} D^{\sigma\mu\nu\rho} D_{\mu} D_{\nu} D_{\rho} \Psi + O(\epsilon^4, \epsilon^3 a, \epsilon^2 a^2, \epsilon a^3, a^4)$$

$$\begin{aligned} A_i(\vec{x}) = -\frac{\beta}{2a} \epsilon_{ij} \Big(& K^{jkl} \left(\sigma_{kl}(\vec{x}) + \xi_1 \sigma_{km}(\vec{x}) \sigma_{ml}(\vec{x}) + \xi_2 \sigma_k(\vec{x}) \sigma_l(\vec{x}) \right) \\ & + a^2 \left(\alpha_1 \partial_j \partial_k \sigma_k(\vec{x}) + \alpha_2 \partial_k \partial_k \sigma_j(\vec{x}) + \alpha_3 K^{klm} \partial_k \partial_l \sigma_{jm}(\vec{x}) \right) + O(\epsilon^3, \epsilon^2 a, \epsilon a^2) \Big) \end{aligned}$$

$$g_{ij}(\vec{x}) = \delta_{ij} + \beta \Big(\chi_1 \sigma_{ij}(\vec{x}) + \chi_2 \sigma_{ik}(\vec{x}) \sigma_{kj}(\vec{x}) + \chi_3 \delta_{ij} \left(\sigma_{kk}(\vec{x}) \right)^2 + \chi_4 \sigma_{ij}(\vec{x}) \sigma_{kk}(\vec{x}) + \chi_5 \sigma_i(\vec{x}) \sigma_j(\vec{x}) \Big) + O(\epsilon^3, \epsilon^2 a, \epsilon a^2)$$

- We know the various constants in the lattice model. For real graphene we should derive them from ‘experiment’

- Subtlety; presumably require different constants for conduction and valence bands

Summary

- The effective theory for the graphene Dirac cones has a subtle structure

- This is due to the local gauge symmetry and large magnetic field

- There is a curved space theory describing the deformed tight-binding model BUT it must include higher derivatives (which explicitly break Lorentz invariance)