Equivalence between fermion-to-qubit mappings in two spatial dimensions

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Motivations

Why do we need fermion-to-qubit mappings?

- Duality between fermionic and bosonic systems
- Quantum simulation of fermionic Hamiltonians
- Exactly solvable models (transversal-field Ising model, Kitaev's honeycomb model)
- Quantum error corrections (Majorana error-correcting codes [1], related to CSS codes)

Jordan-Wigner transformation

$$c_i^\dagger \to \Big(\prod_{j<i} Z_j\Big) \otimes \sigma_i^+, \qquad c_i \to \Big(\prod_{j<i} Z_j\Big) \otimes \sigma_i^- \tag{1}$$

fermionic modes : number of qubits = 1 : 1.

$$c_i^\dagger c_k = \sigma_i^+ \Big(\prod_{i<j<k} {\color{red} Z_j}\Big) \sigma_k^- \tag{2}$$

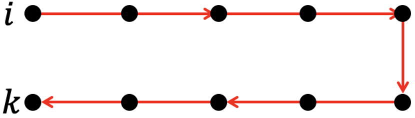


Figure: In the 2d square lattice, a choice of ordering is required. The vertical hopping operator of fermions becomes non-local after Jordan-Wigner transformation.

Mechanism of local fermion-to-qubit mappings

Solution: introduce entanglement to restore the locality in 2d.

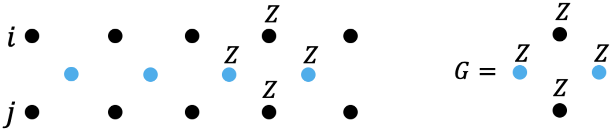


Figure: Double the number of qubits and impose gauge constraints to enable local readout of non-local fermion parity (Z string operator).

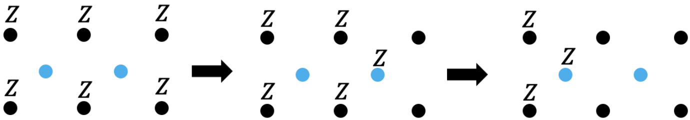


Figure: Procedure of local readout of fermion parity

Many Fermion-to-qubit mappings

TABLE I. Comparison between fermion-to-qubit mappings on the 2d square lattice.

	Qubit-fermion ratio r	Fermion parity weight	Hopping weight	Stabilizer weight
Verstraete-Cirac mapping [3][a]	2	1	3–4	6
BKSF encoding [11][b]	2	4	2–6	6
Kitaev's honeycomb model [4]	2	2	2–5	6
Exact bosonization [2]	2	4	2–6	6
MLSC [7]	2	3	3–4	4–10
Compact fermion-to-qubit mapping [9]	1.5	1	3	8
Supercompact fermion-to-qubit mapping	1.25	1–2	2–6	12

Different mappings have different overheads and logical operators. Are they related to each other?

Main results

- Equivalence (local unitary+ancilla) between all 2d local fermion-to-qubit mappings[1] (Bravyi-Kitaev superfast simulation, Verstraete-Cirac mapping, 2d exact bosonization, Majorana loop stabilizer codes, and compact fermion-to-qubit mapping).
- New fermion encoding with 1.25 qubits per fermionic modes
- General construction of 2d local fermion-to-qubit mapping

[1]Haah proves that $\mathbb{Z}_2$ Pauli stabilizer codes must be copies of toric codes. Hence all fermion-to-qubit mappings in 2d are based on the emergent fermions in toric codes.

Fermion-to-qubit mappings as stabilizer codes

- Physical Hilbert space: qubit array
- Logical Hilbert space: fermionic modes
- Stabilizer constraints: moving a fermion along a closed loop $\propto$ identity
- Logical operators: Pauli strings that satisfy even fermionic algebra

Each fermion-to-qubit mapping is a dictionary that maps any product of an even number of fermionic operators to Pauli string operators.

Review of 2d exact bosonization

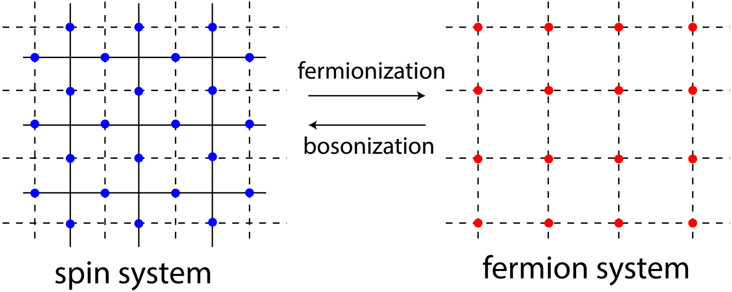
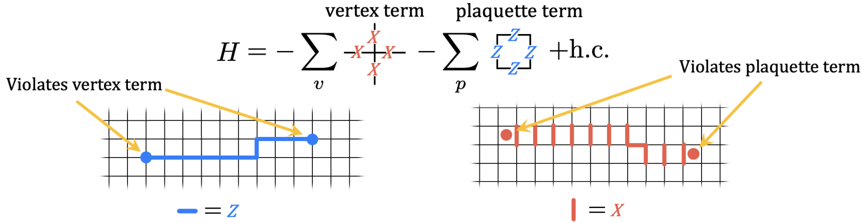
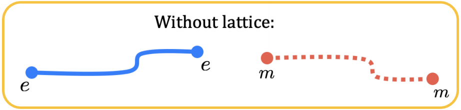


Figure: Bosonization and fermionization, each blue vertex consists a qubit, and each red vertex consists two Majorana modes γ, γ'.

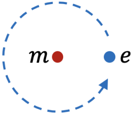
Emergent fermions on $\mathbb{Z}_2$ toric code



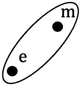
e : electric charge excitation



m : magnetic flux excitation



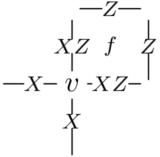
generators -1 sign (Aharonov–Bohm) $\Rightarrow$ $\epsilon =$



is a fermion.

Review of 2d exact bosonization

- Logical space: 1 (vacuum), ϵ (fermion)
- Stabilizer (gauge constraint) on vertex v:

$G_v =$  $= 1.$ (3)

- Enlarged Hilbert space: 2 qubits (edges) per fermionic mode (face).
- Codespace: fermionic subspace of a toric code.

Logical operators of 2d exact bosonization

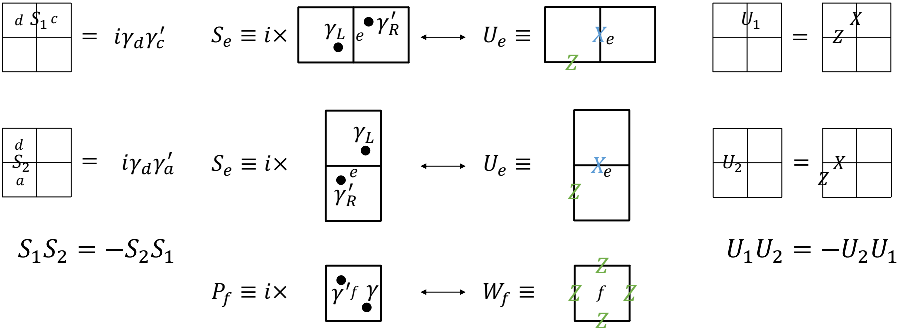


Figure: Dictionary of even Majorana and qubit interactions

Equivalence between 2d fermion-to-qubit mappings

Equivalence ≈ $O(1)$-depth quantum circuit + ancilla

Case 1. The same Hilbert space. Two mappings have different gauge constraints G_1 and G_2. If there exists a finite-depth local unitary circuit U such that

$$UG_1U^\dagger = G_2, \tag{4}$$

then they are equivalent.

- Two mappings are equivalent ⇔ one can convert the codespaces from one to the other by a finite-depth Clifford circuit.

To define the equivalence between mappings in different Hilbert spaces, Circuit + ancilla are needed (formally: generalized local unitary by Chen, Gu, and Wen, 2010) [2, 3])

Generalized local unitary (gLU)

Case 2. Different Hilbert space. Some qubits can be disentangled (decoupled) by a Clifford circuit.

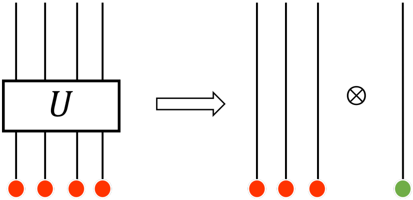


Figure: Disentanglement by gLU

We argue 2d local fermion-to-qubit mappings can be obtained by this way.

gLU disentanglement

Consider a 3-qubit repetition code with check operators $G = \{Z_1 Z_2, Z_2 Z_3\}$ and codewords

$$|0\rangle_L = |000\rangle\,, \quad |1\rangle_L = |111\rangle\,. \tag{5}$$

We may disentangle the third qubit by applying $\mathrm{CNOT}_{2\to 3}$ such that

$$(\mathrm{CNOT}_{2\to 3})\; G\; (\mathrm{CNOT}_{2\to 3})^\dagger = \{Z_1 Z_2, Z_3\}. \tag{6}$$

Then, we have disentangled the third qubit.

gLU disentanglement

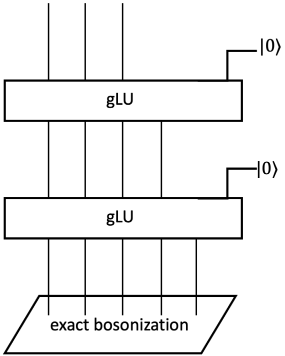


Figure: Disentanglement procedure, we disentangle $O(N)$ qubits

We can keep disentangling qubits, and obtain mappings requires different overheads.

General construction and equivalence relation

After applying follow Clifford circuit, we will obtain a fermion-to-qubit mapping with $r = 1.5$

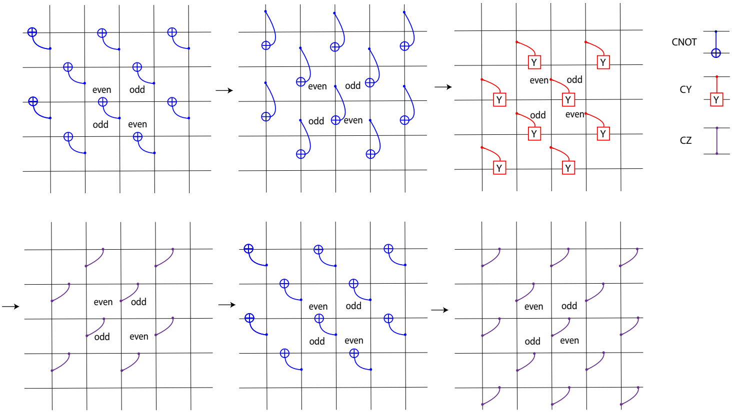
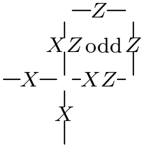
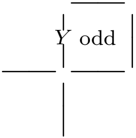


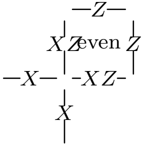
Figure: Clifford circuit to construct mapping with $r = 1.5$

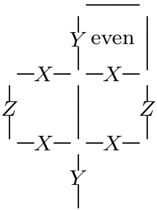
General construction and equivalence relation

$G_{odd} =$ 

$\longrightarrow$ $(-1) \times$ 

, (7)

$G_{even} =$ 

$\longrightarrow$ $(-1) \times$ 

. (8)

It is compact fermion-to-qubit mapping [4].

Relation to Jordan Wigner transformation

If we remove all the stabilizers, then the mapping reduce to Jordan-Wigner transformation.

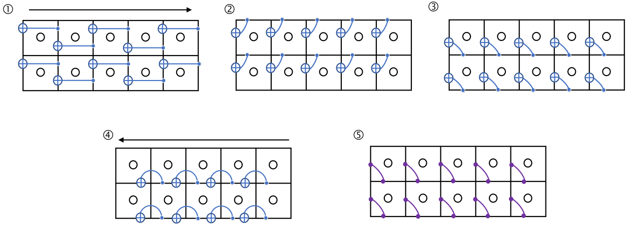


Figure: Linear-depth circuit to convert exact bosonization to 1d Jordan-Wigner

Outlooks

- Stabilizer weight is high, can we reduce it by the idea of Floquet codes?
- Due to the non-trivial QCA in three dimension, there may exist different families of fermion-to-qubit mappings in 3d.
- Interplay between symmetries of fermions and their bosonic counterparts.

References I

S. Bravyi, B. M. Terhal, and B. Leemhuis, “Majorana fermion codes,” *New Journal of Physics*, vol. 12, p. 083039, Aug 2010.

X. Chen, Z.-C. Gu, and X.-G. Wen, “Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order,” *Phys. Rev. B*, vol. 82, p. 155138, Oct 2010.

B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, “Quantum information meets quantum matter – from quantum entanglement to topological phase in many-body systems,” 2018.

C. Derby, J. Klassen, J. Bausch, and T. Cubitt, “Compact fermion to qubit mappings,” *Phys. Rev. B*, vol. 104, p. 035118, Jul 2021.