Universal, deterministic, and exact protocol to reverse qubit-unitary and qubit-encoding isometry operations

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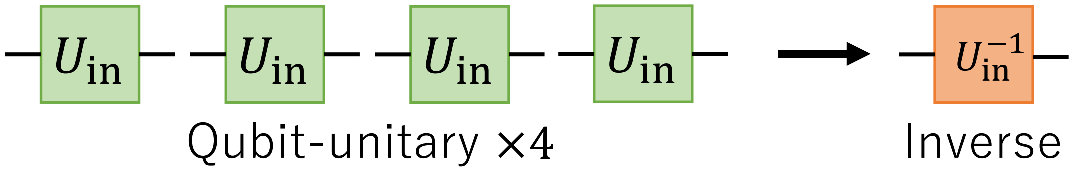
Joint work with Akihito Soeda (NII), Mio Murao (UTokyo)

arXiv:2209.02907
(Accepted in PRL)

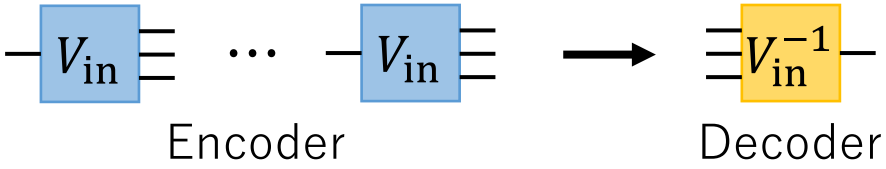
Outline

- General perspective on higher-order quantum operations

- Result 1: Deterministic exact qubit-unitary inversion



- Result 2: Isometry inversion



- Future works

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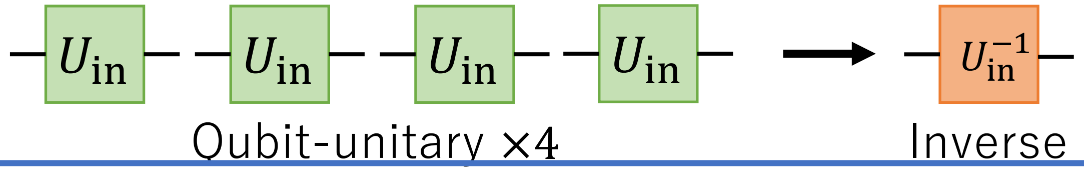
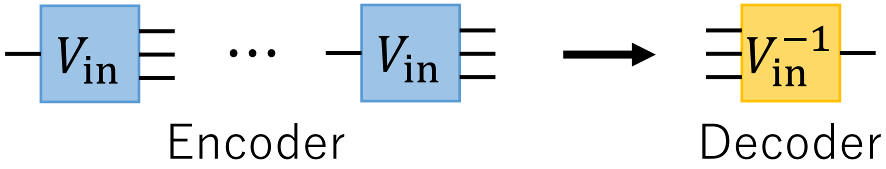


This talk

- Result 2: Isometry inversion



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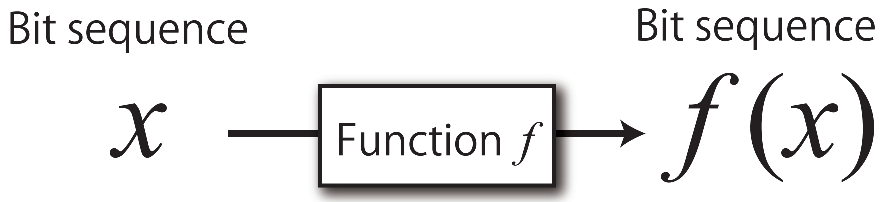
Higher-order quantum operation

- Classical information processing

\- Function



Higher-order quantum operation

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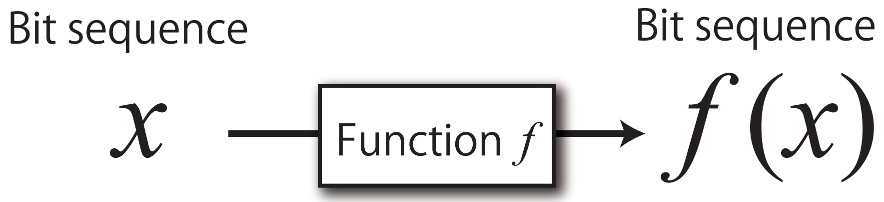
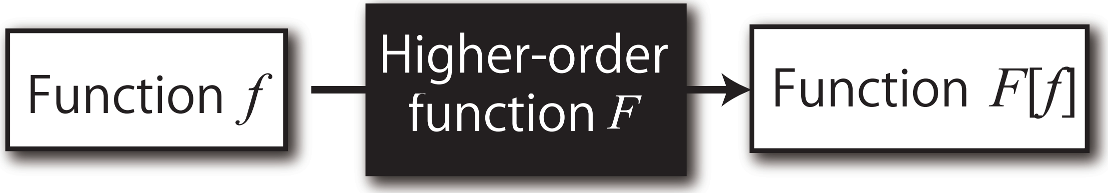


- Higher-order function



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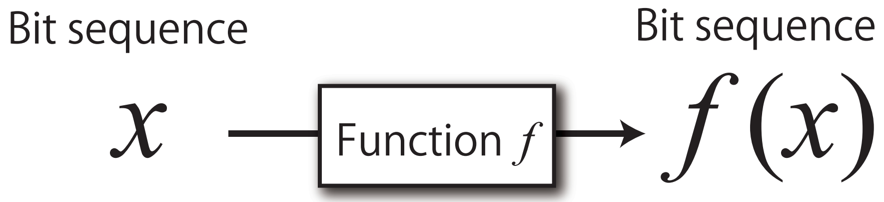

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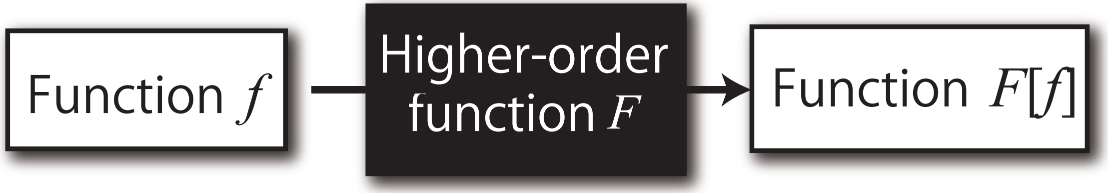


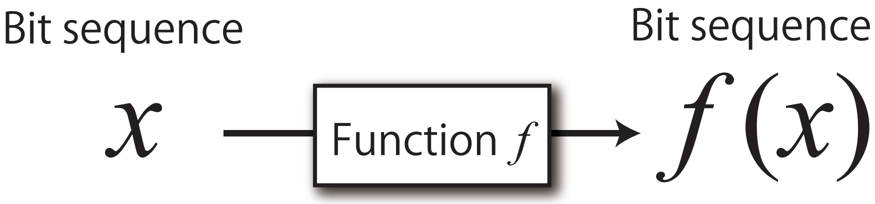
→ Functional programming

Eg. $\mathrm{Itr}(f) = f \circ f$

Higher-order quantum operation

- Classical information processing

- Function



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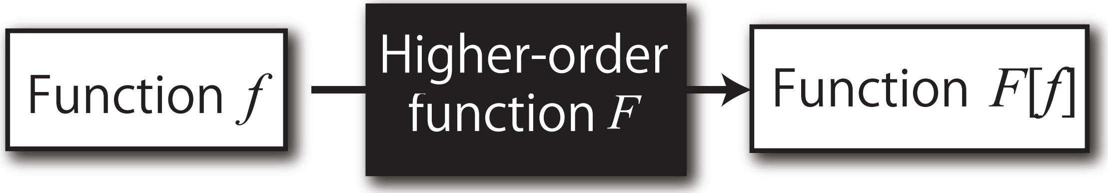
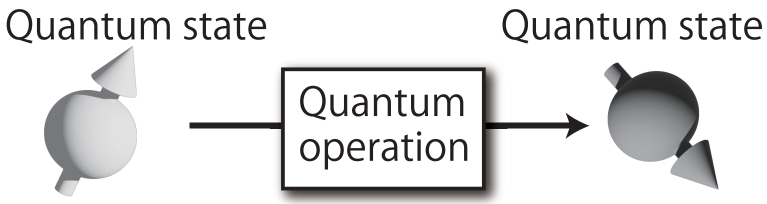
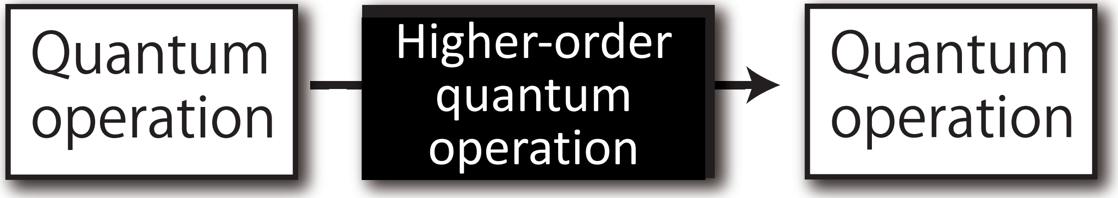


- Quantum information processing

- Quantum operation



- Higher-order quantum operation

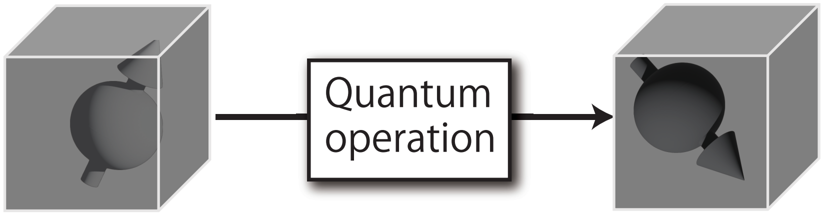


Universal transformation of quantum states

- Task

Given

ρ



$\sigma = f(\rho)$

Unknown state

W. K. Wootters and W. H. Zurek, Nature 299, 802 (1982).

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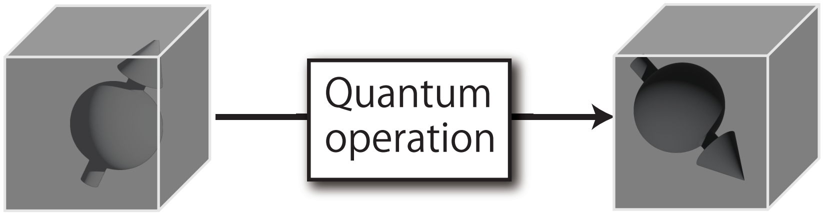

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This is NOT $|0\rangle \mapsto |\psi\rangle$

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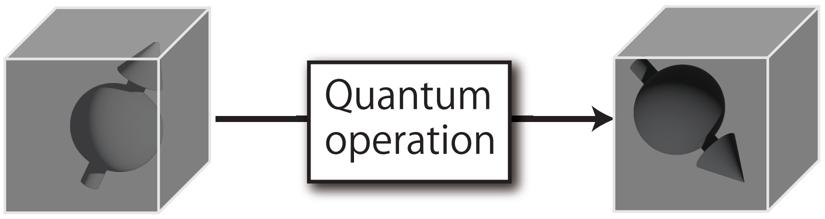

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- Eg. State cloning

$$\rho \mapsto \rho \otimes \rho$$

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Universal transformation of quantum operations

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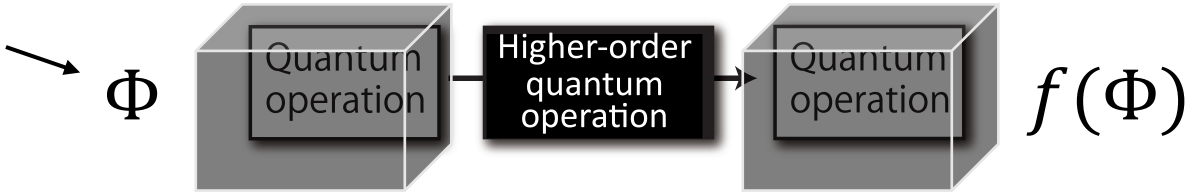
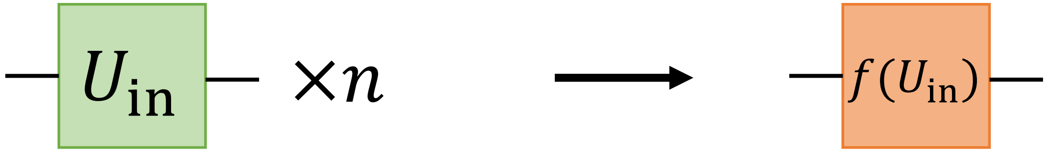


Unknown operation

- Eg. Universal transformation of unitary operation



$f(U) = U^{\otimes m}, U^{*}, U^{-1}, U^{T}, \mathrm{ctrl}-U, \ldots$

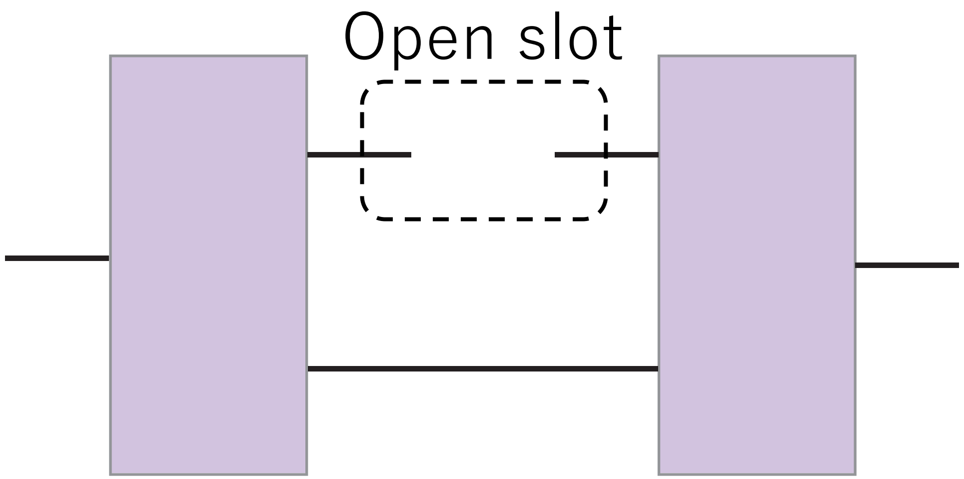
Unknown unitary

G. Chiribella et al. PRL 101, 180504 (2008). M. Quintino et al. PRL 123, 210502 (2019). D. Trillo et al. PRL 130, 110201 (2023).
J. Miyazaki et al. PRR 1, 013007 (2019). D. Ebler et al. arXiv:2206.00107. Q. Dong at al. arXiv:1911.01645. M. Araujo et al. NJP 16 093026 (2014).

Quantum combs

- How to implement transformation of quantum operations?

→ Quantum circuit with open slot(s): Quantum comb



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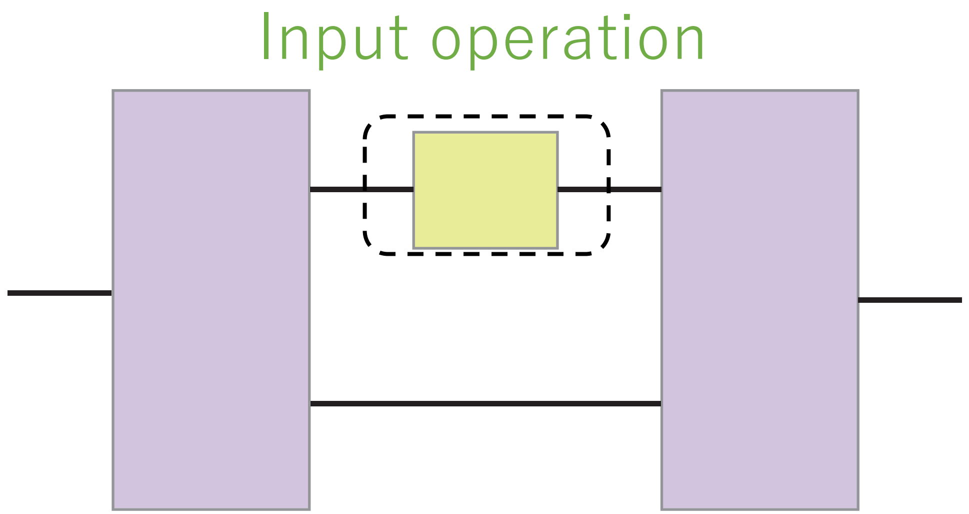

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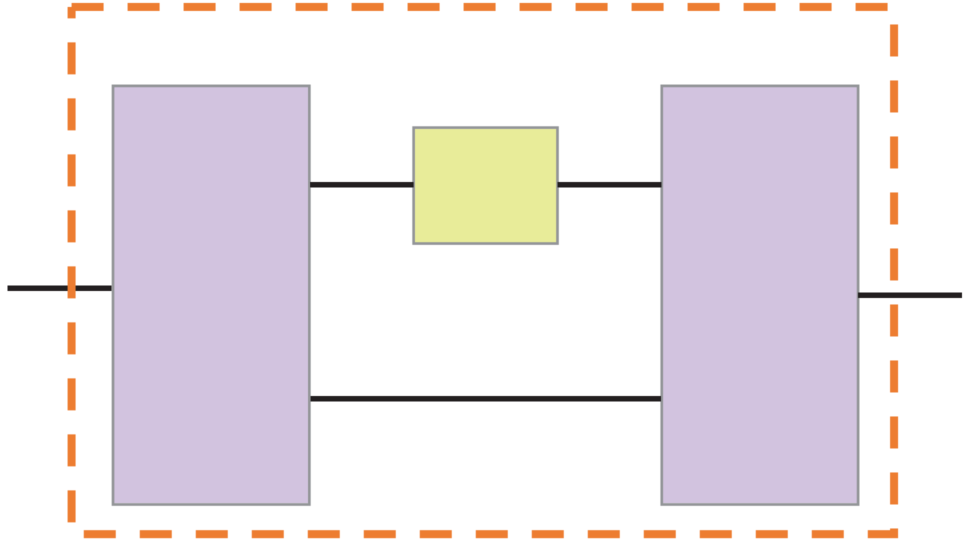

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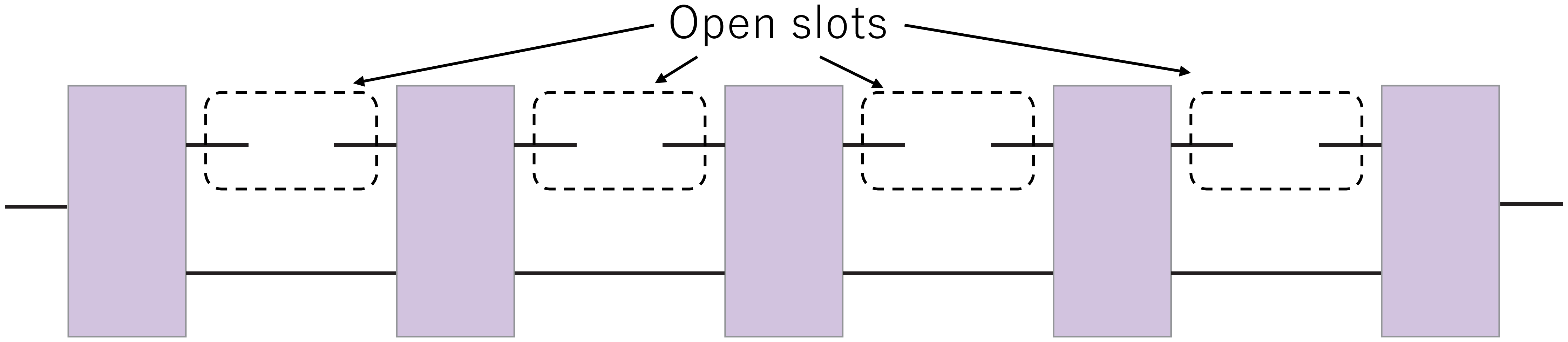

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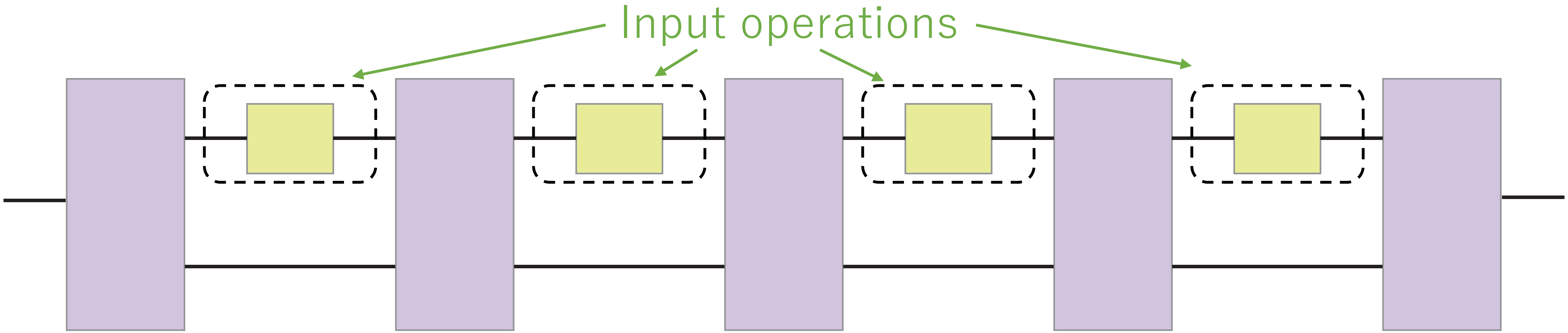

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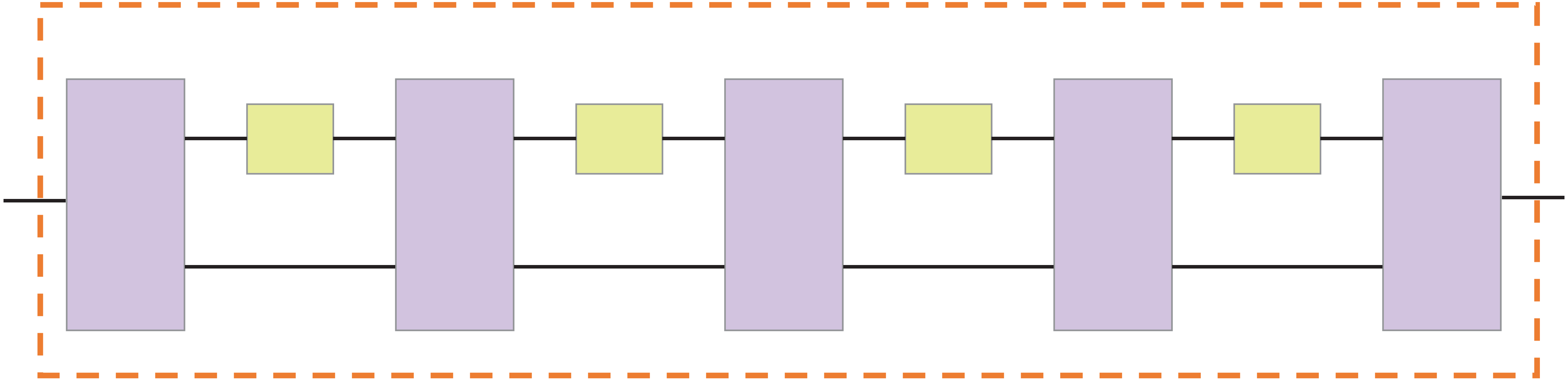

Output operation

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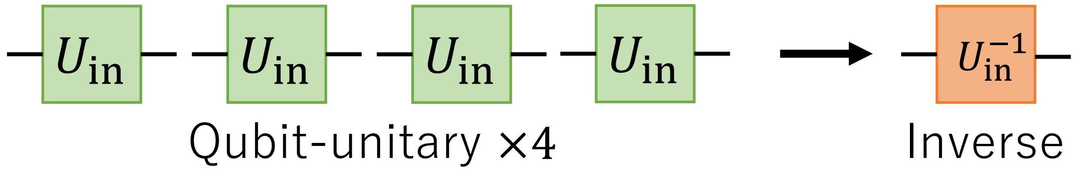
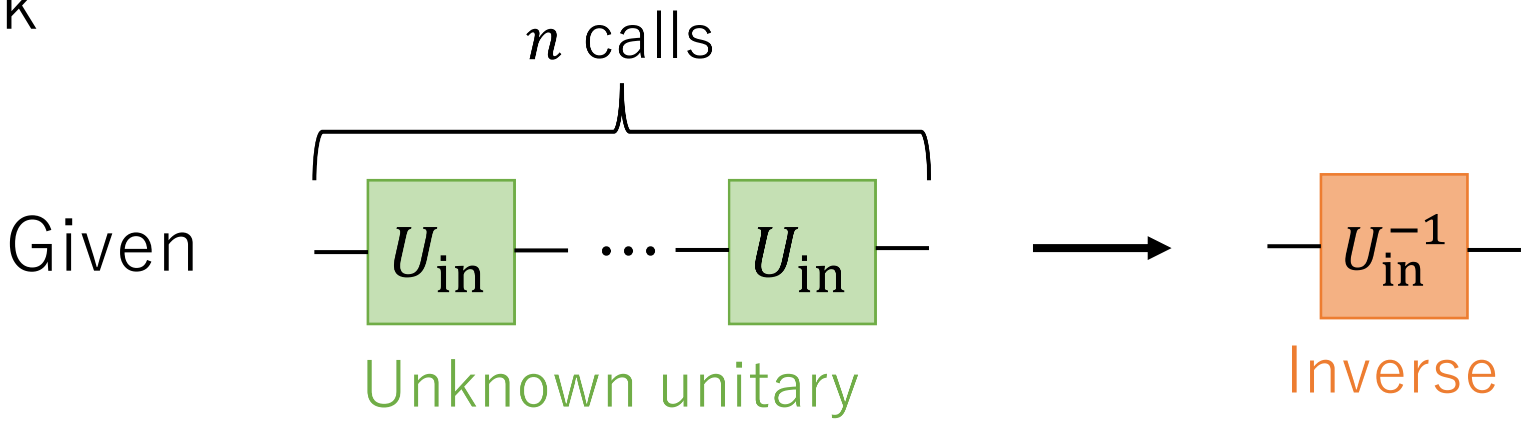


- Result 2: Isometry inversion

- Future works

Unitary inversion

- Task



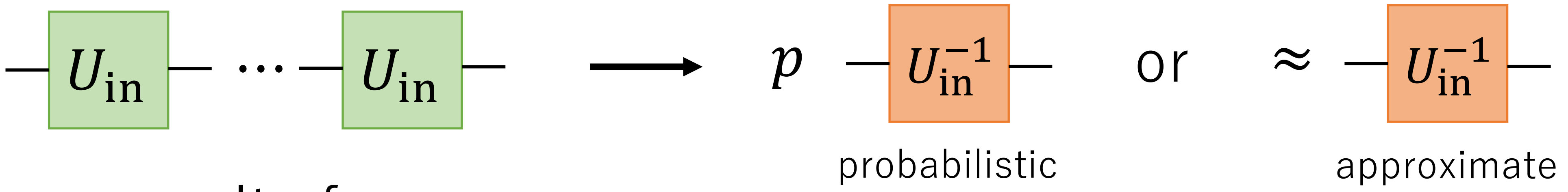
$$U_{\text{in}} = e^{-iHt} \mapsto U_{\text{in}}^{-1} = e^{iHt}$$

→ Simulation of “time inversion”

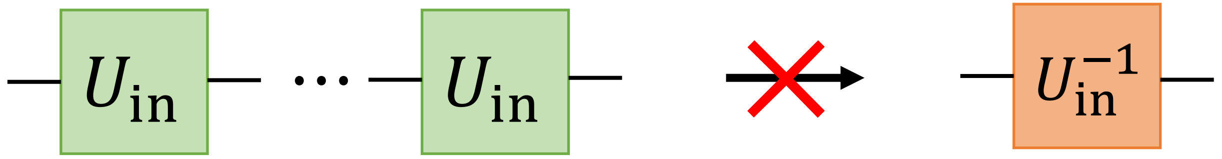
Unitary inversion

- The fundamental limitation of unitary inversion?
- Previous work:

\- Go results



\- No-go results for some cases

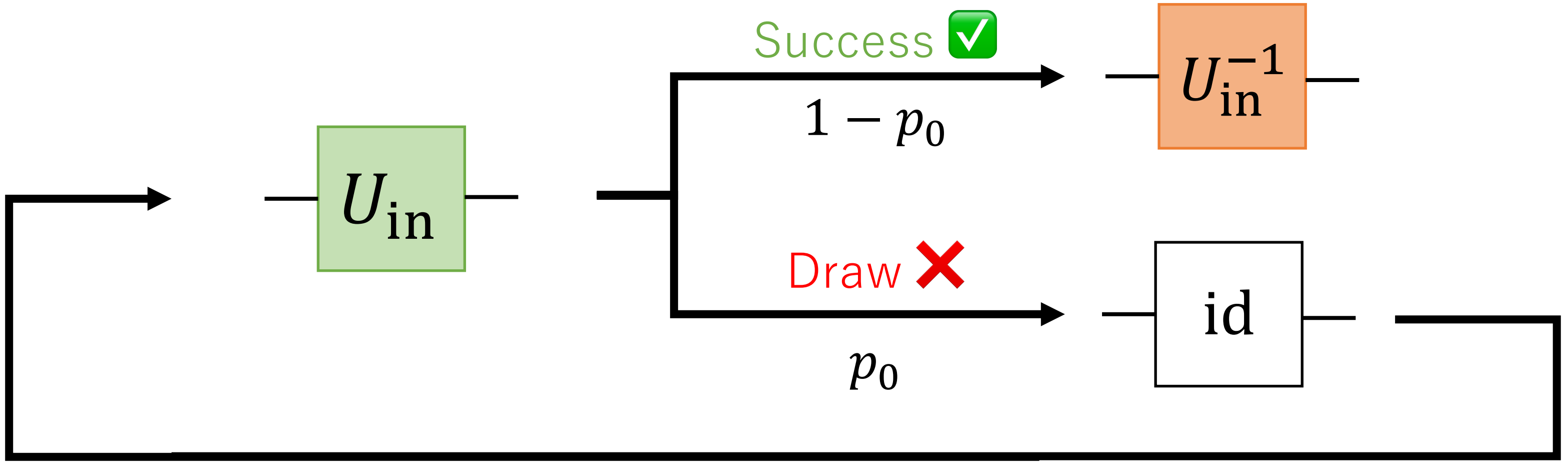


M. Sedlak et al. PRL 122, 170502 (2019), M. Navascues, PRX 8, 031008 (2018), M. Quintino et al. PRL 123, 210502 (2019), M. Quintino et al. PRA 100, 062339 (2019). M. Quintino and D. Ebler Quantum 6, 679 (2022), I. S. Sardharwalla et al. arXiv: 1602.07963, D. Ebler et al. arXiv: 2206.00107, D. Trillo et al. Quantum 4, 374 (2020), D. Trillo et al. arXiv: 2205.00131, P. Schiansky et al. arXiv: 2205.01122.

Unitary inversion

- Previous work: Go results

Best known：Success-or-draw



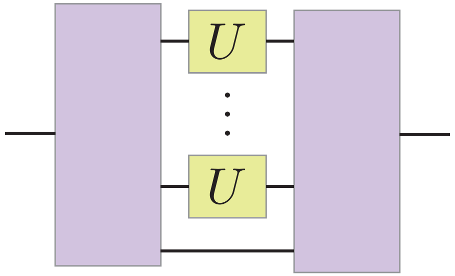
→ Success probability $p = 1 - p_0^{-O(n)} < 1$

M. Quintino et al. PRL 123, 210502 (2019), M. Quintino et al. PRA 100, 062339 (2019). M. Quintino and D. Ebler Quantum 6, 679 (2022).

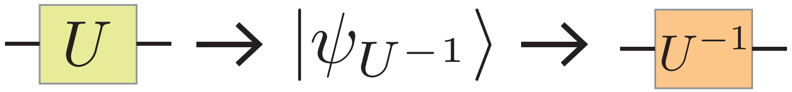
Unitary inversion

- Previous work: No-go results

Parallel



Store-and-retrieve



Upper bound
$p \leq 1 - \Omega(1/n)$

Numerics: $p_{\mathrm{opt}}(d,n), F_{\mathrm{opt}}(d,n)$ for small d, n
→ Still less than 1

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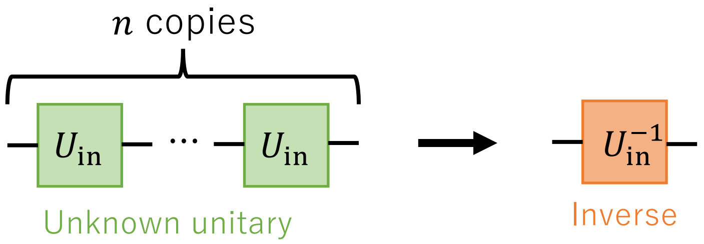
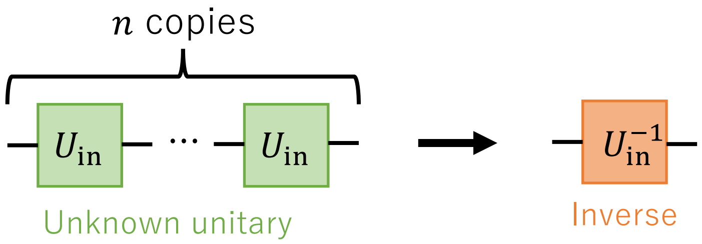


- Previous work

	Probabilistic	Deterministic
Approximate	✓	✓
Exact	✓	???

Open problem

Unitary inversion



- Previous work

	Probabilistic	Deterministic
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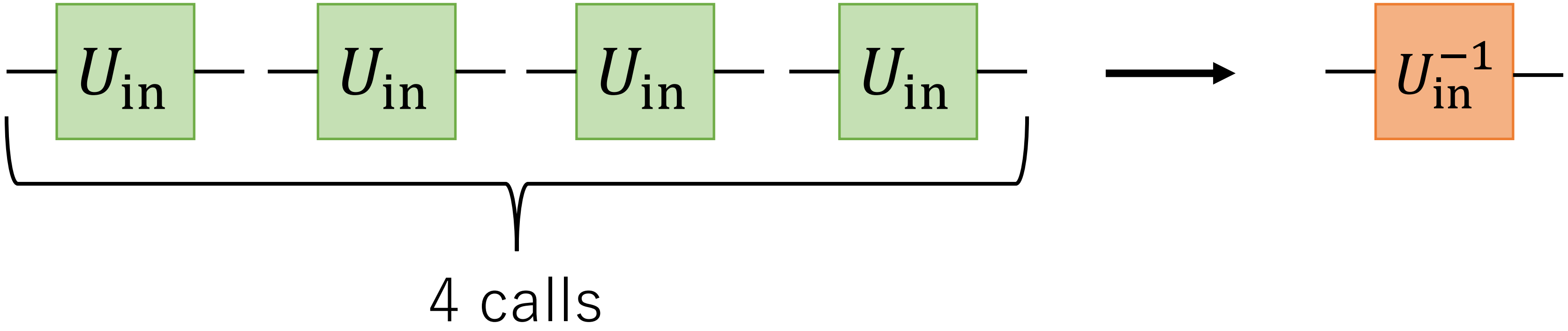
Open problem

We answer the open problem positively for $d = 2$!

Unitary inversion

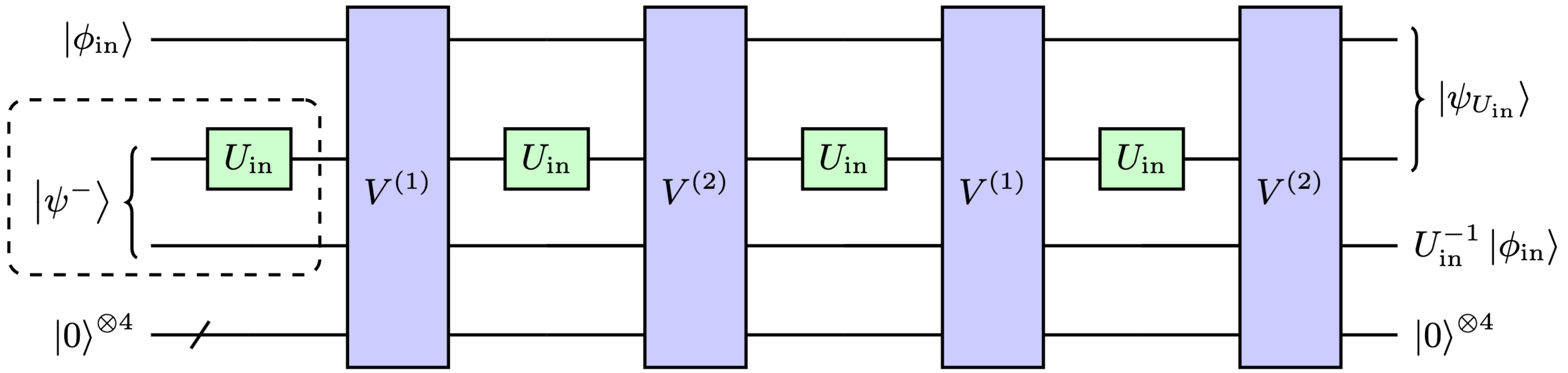
- Main result:

There exists a deterministic and exact qubit-unitary inversion protocol.



$\forall U_{\mathrm{in}} \in \mathrm{SU}(2)$

Qubit-unitary inversion protocol



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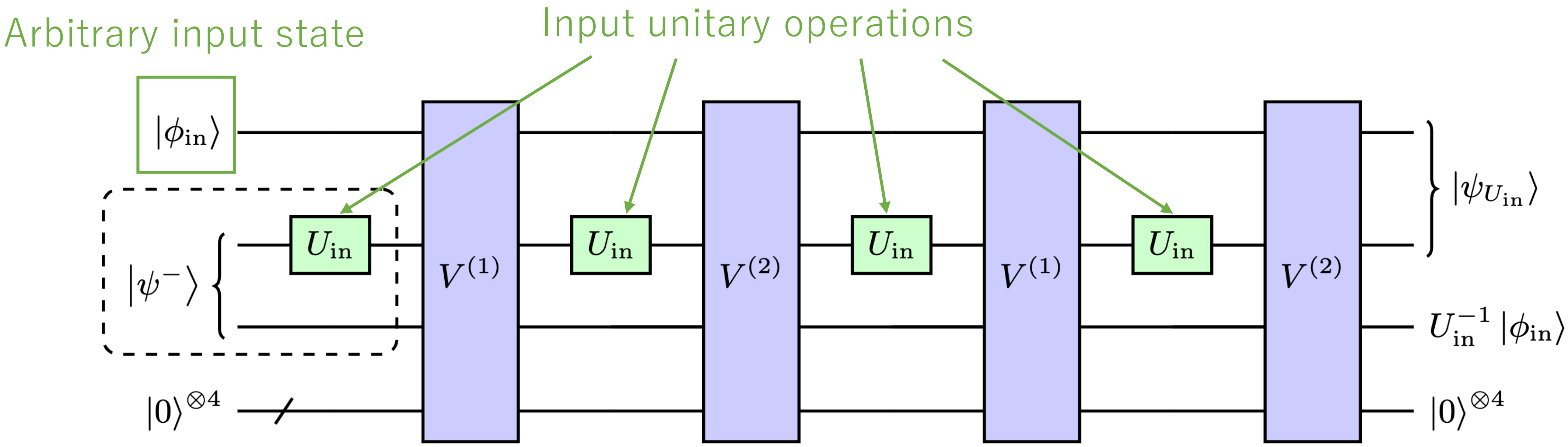

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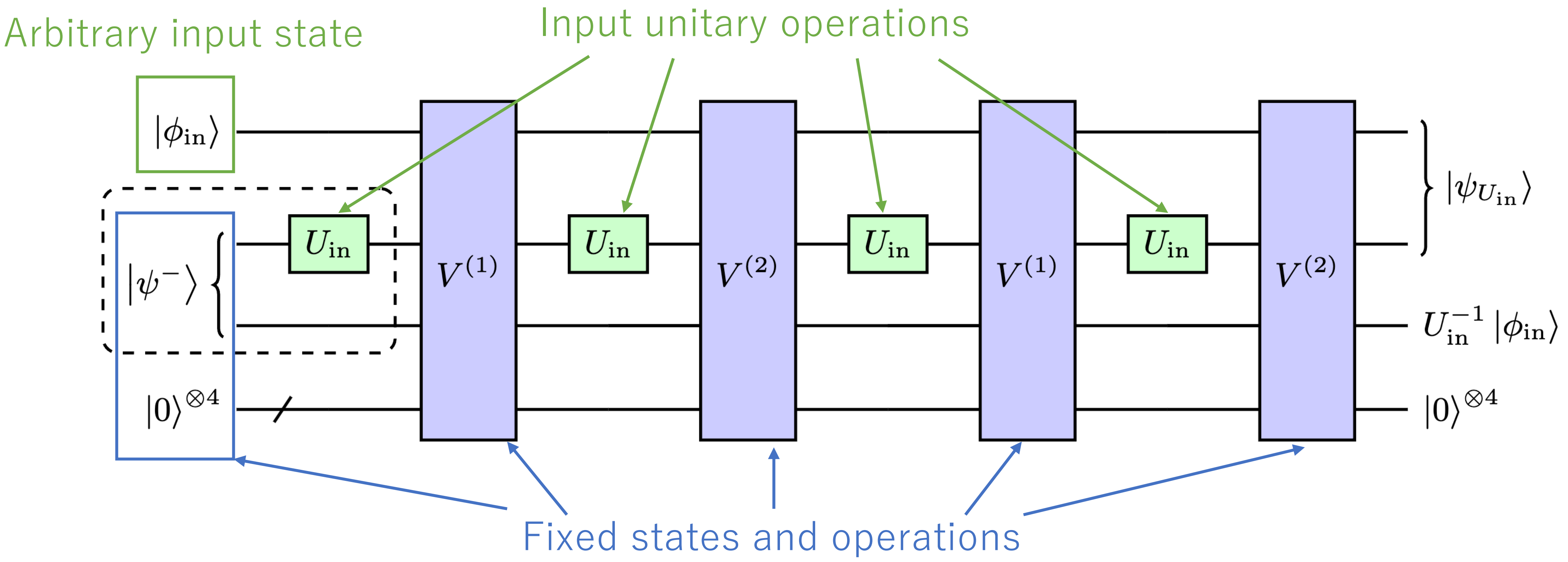

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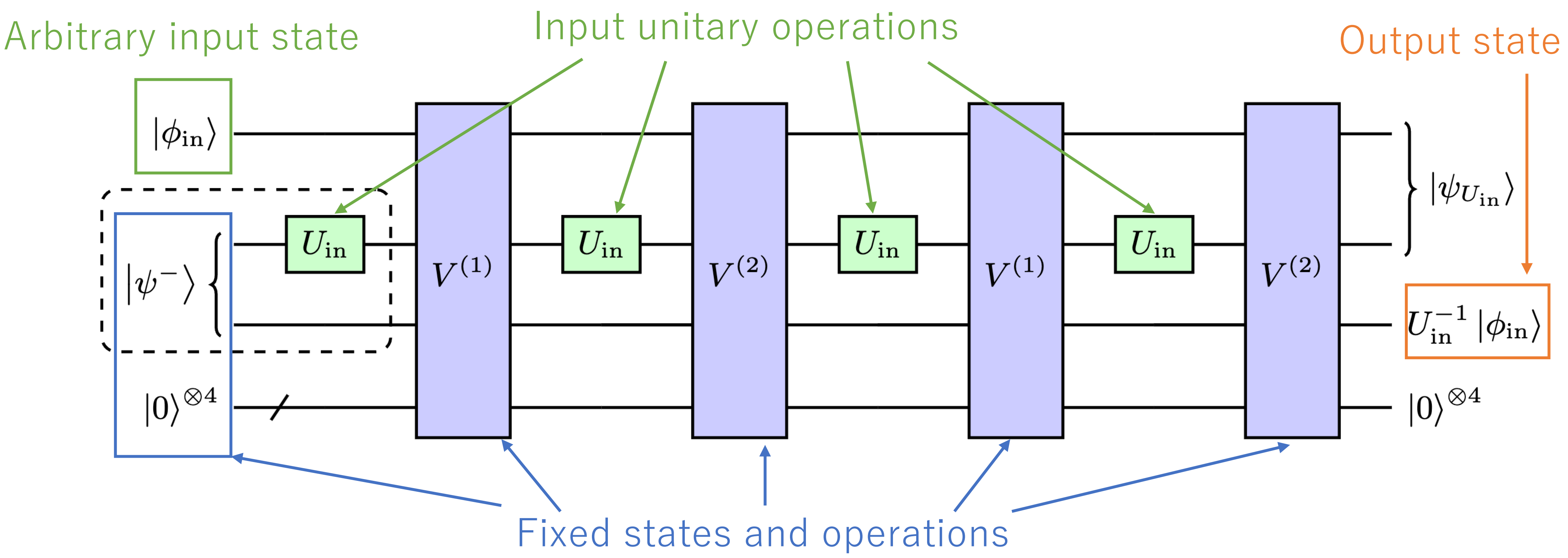

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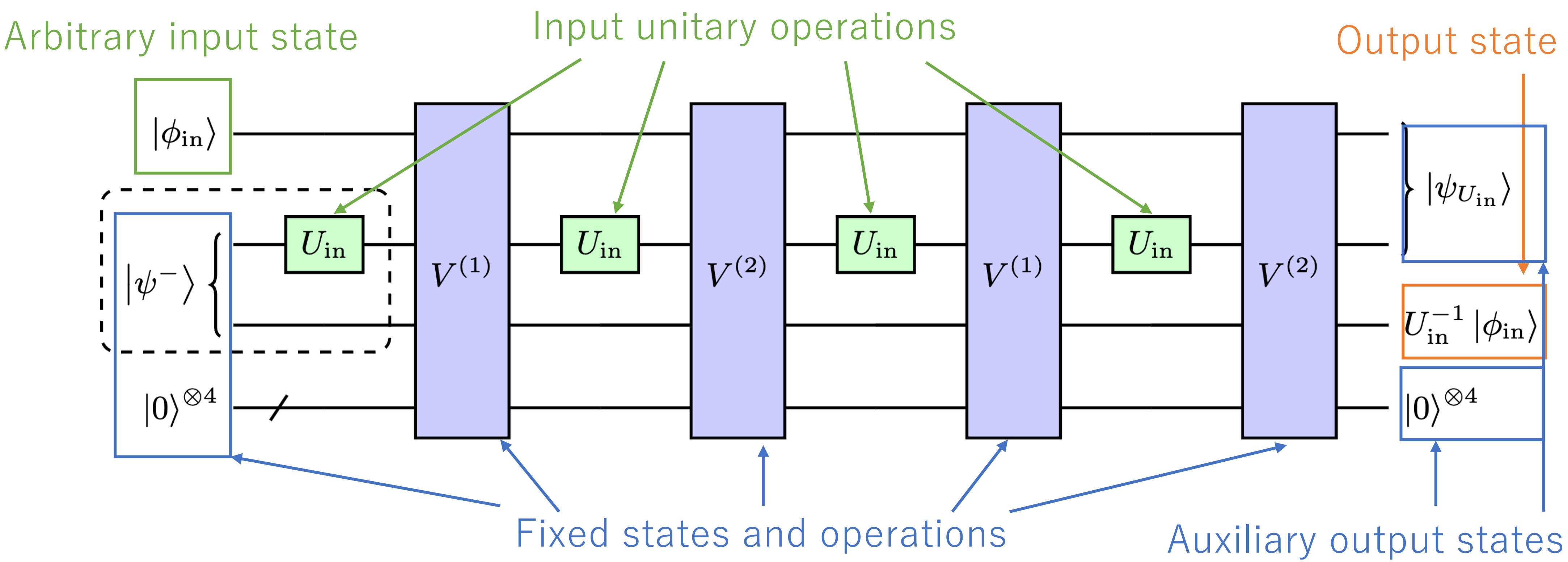

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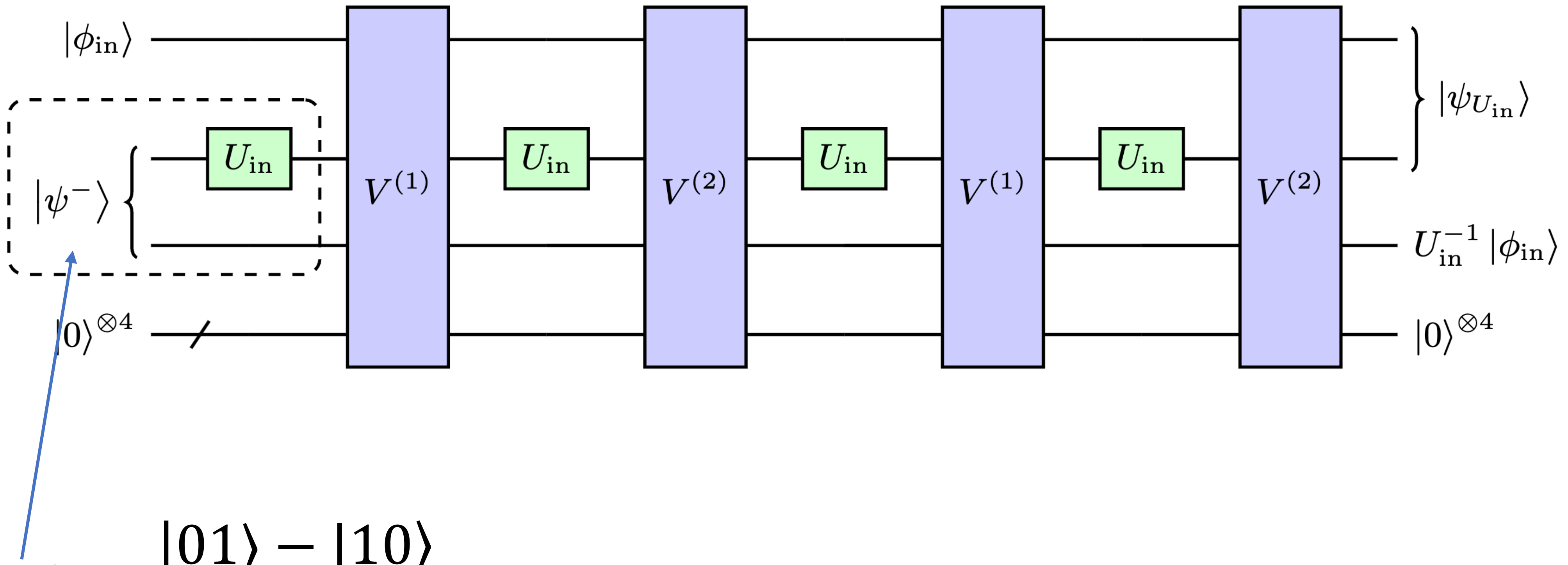
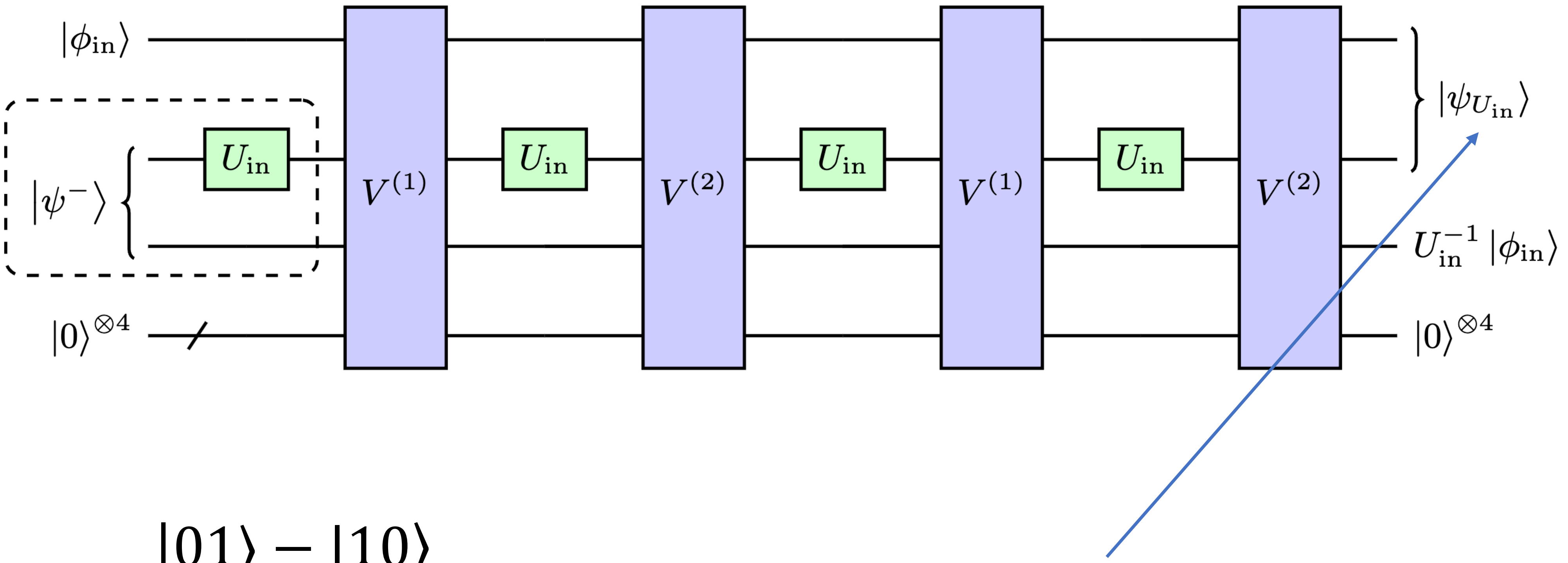


$$|\psi^-\rangle := \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Qubit-unitary inversion protocol



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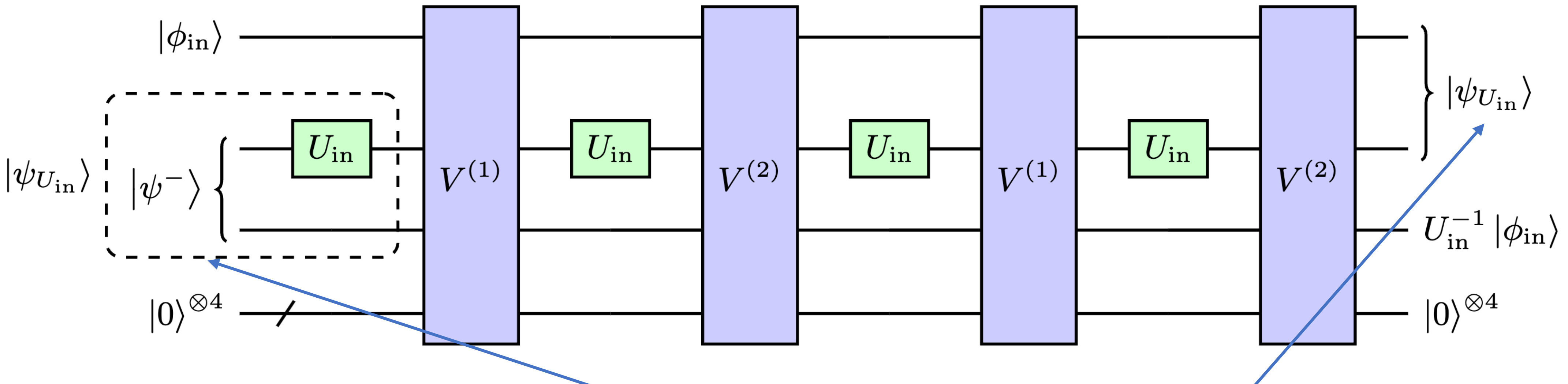

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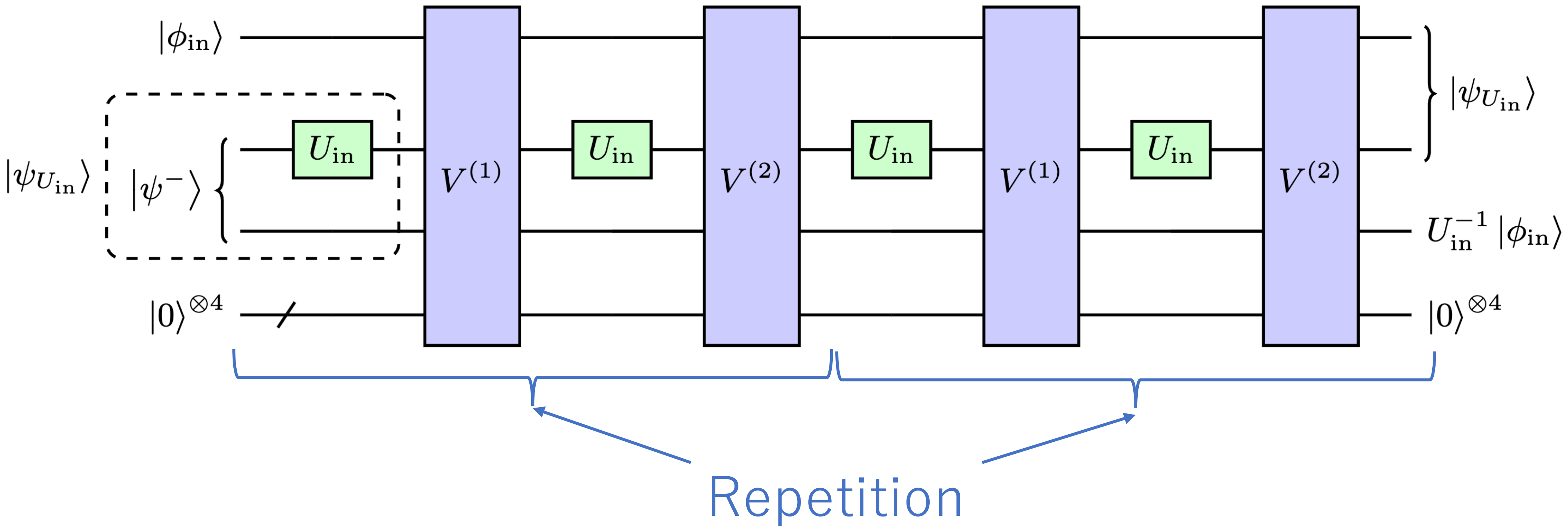

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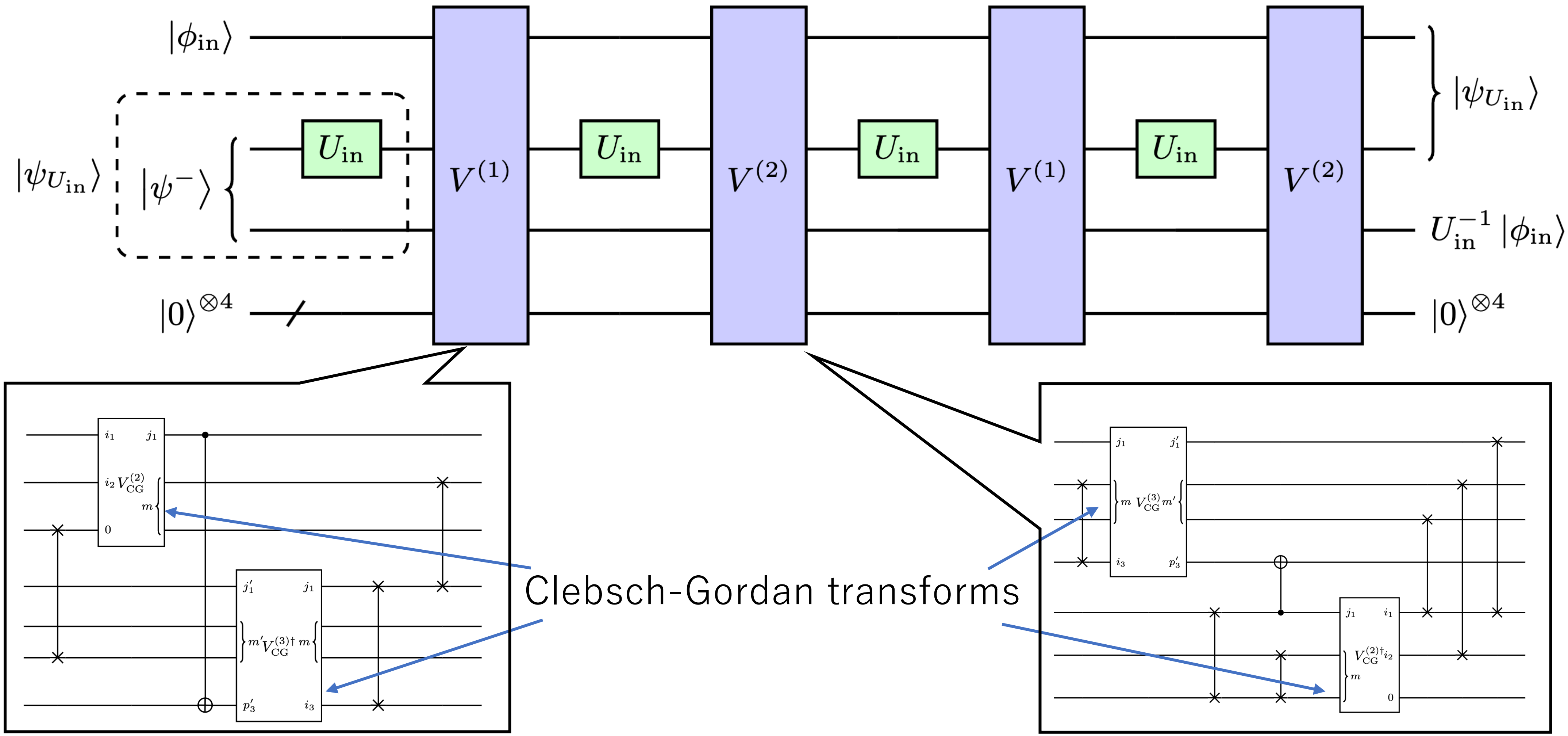
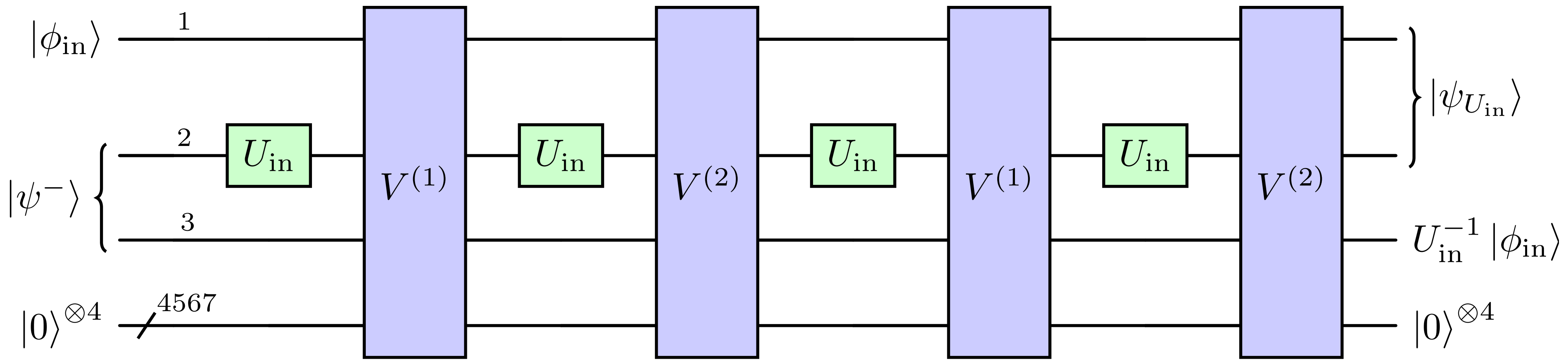


Proof sketch



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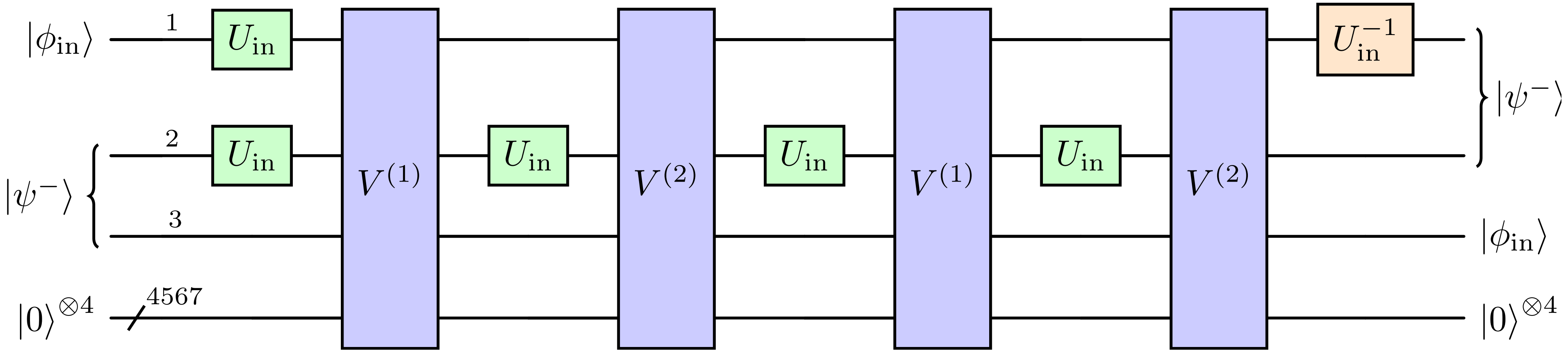

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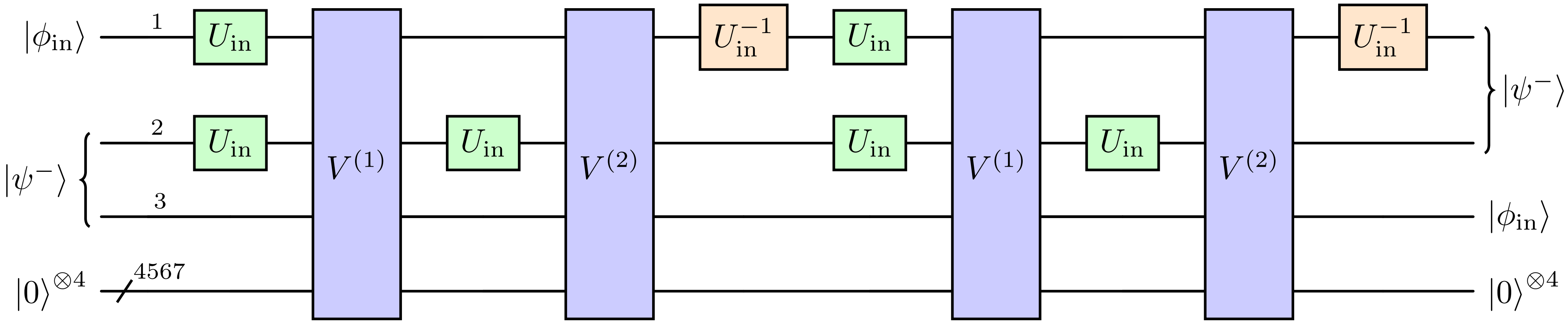

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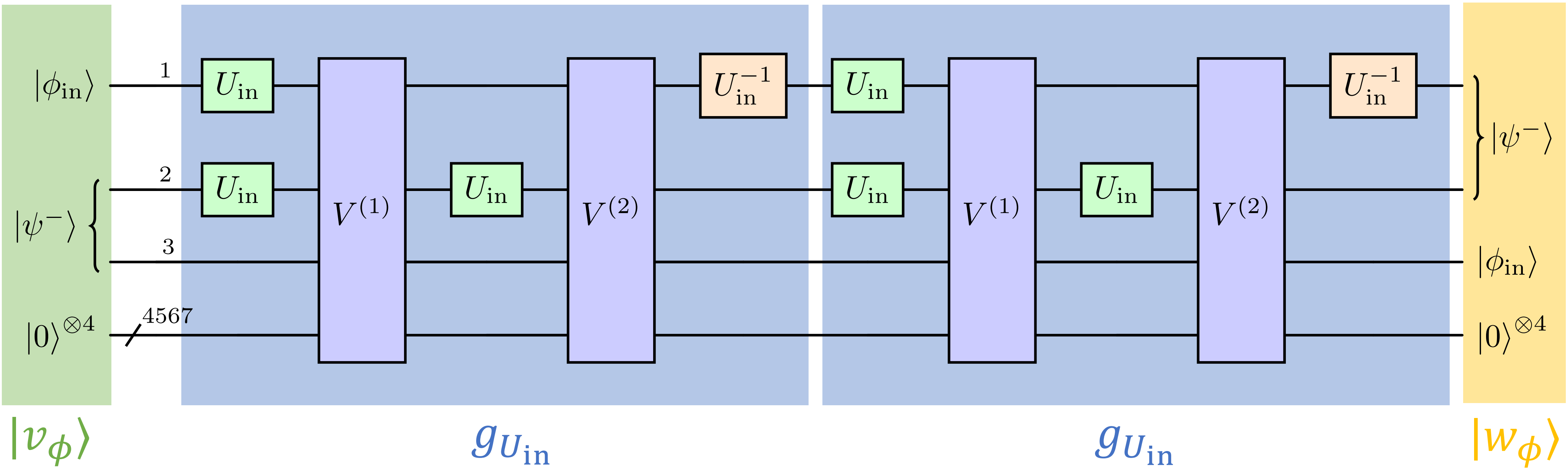

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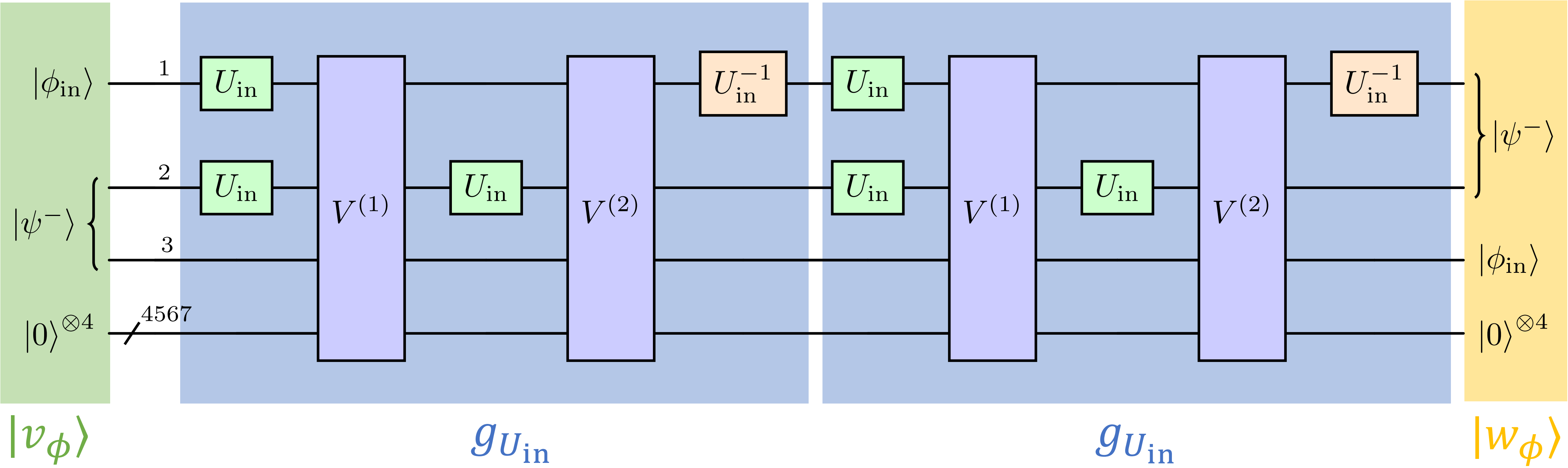

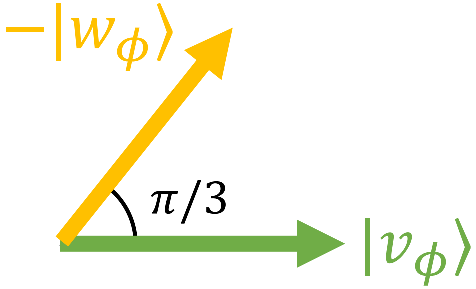

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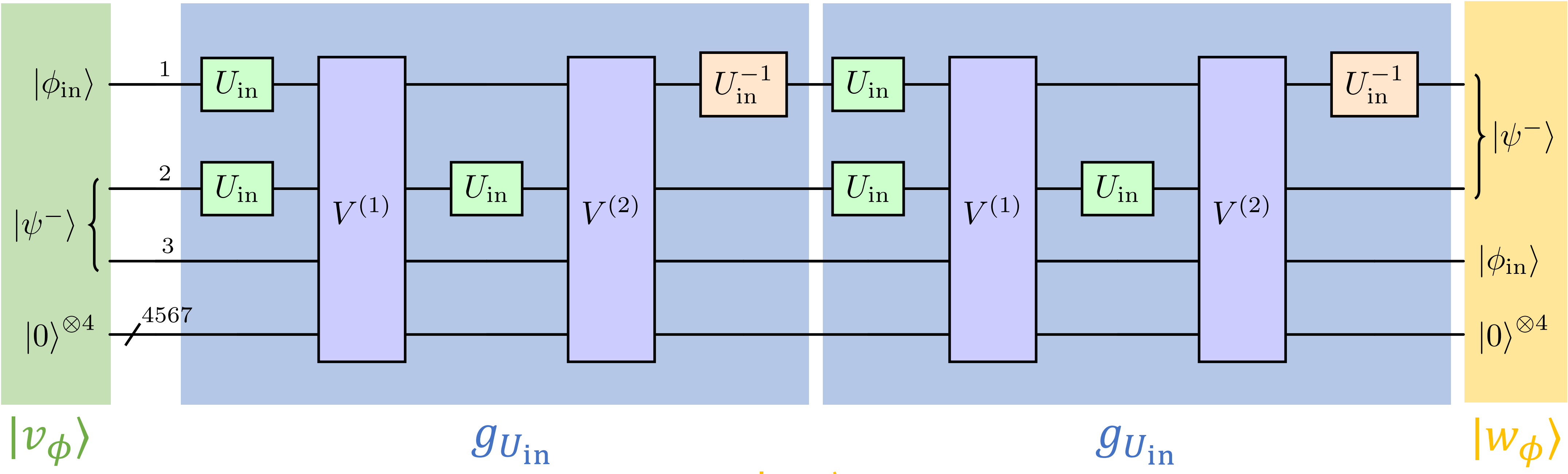

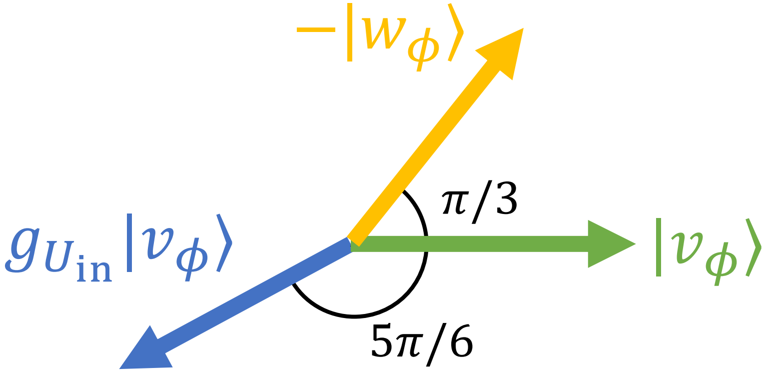

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- Catalytic use of $|\psi_{U_{\mathrm{in}}}\rangle$
- Cleanness of the protocol

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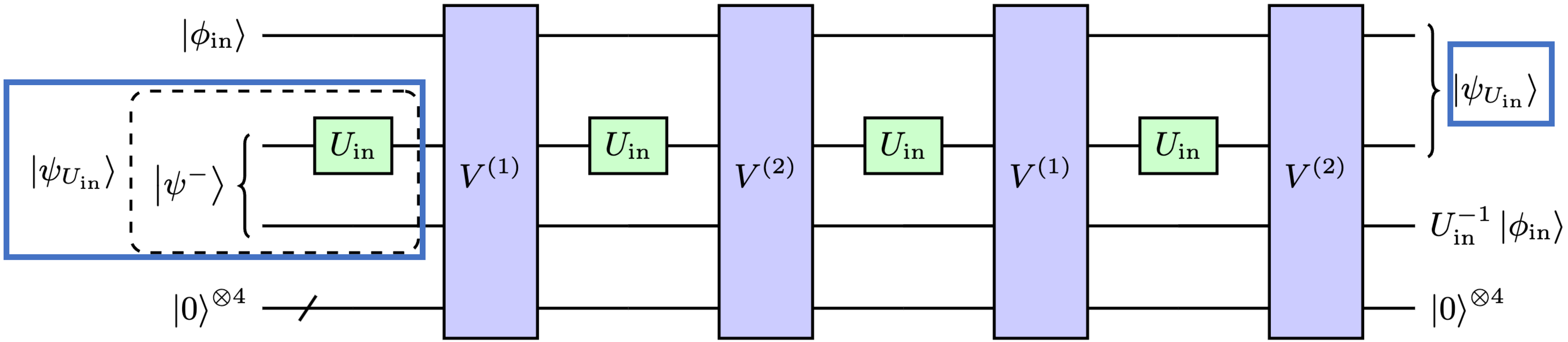

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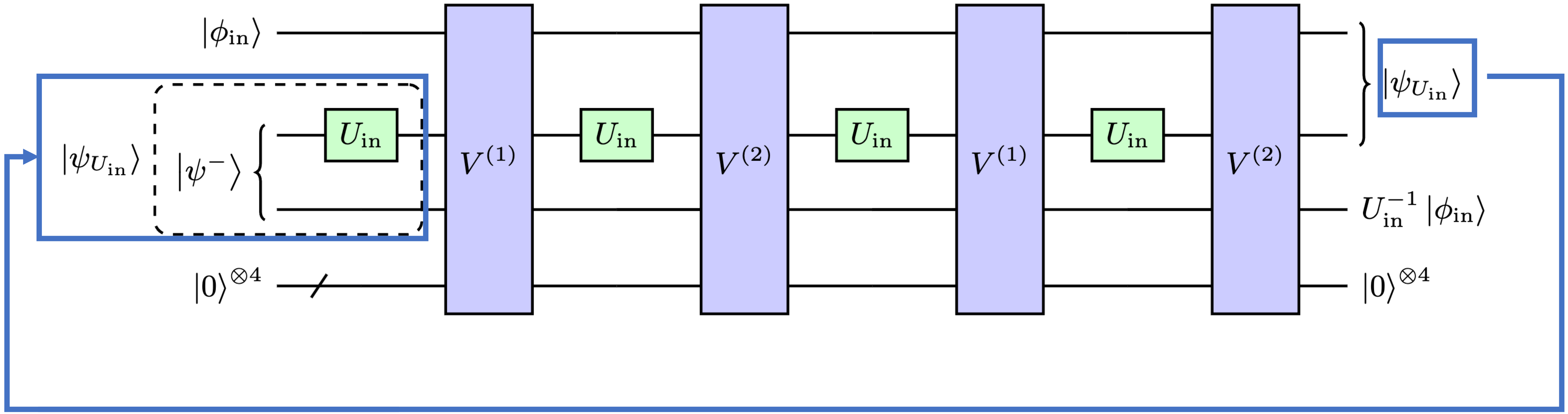

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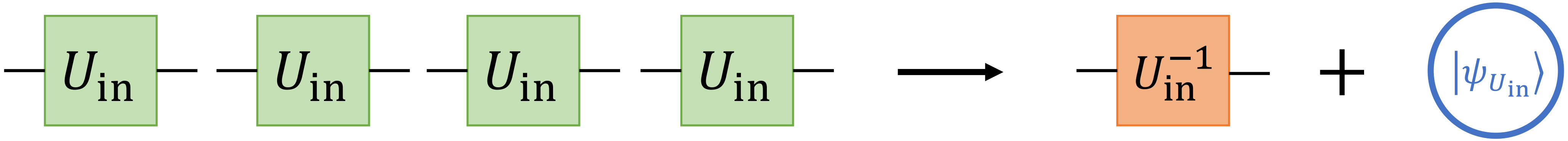

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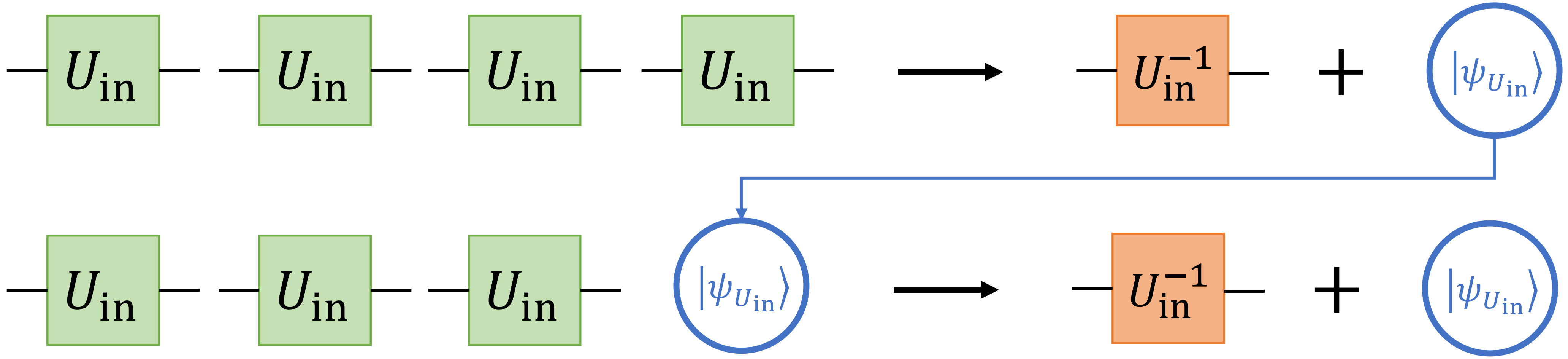

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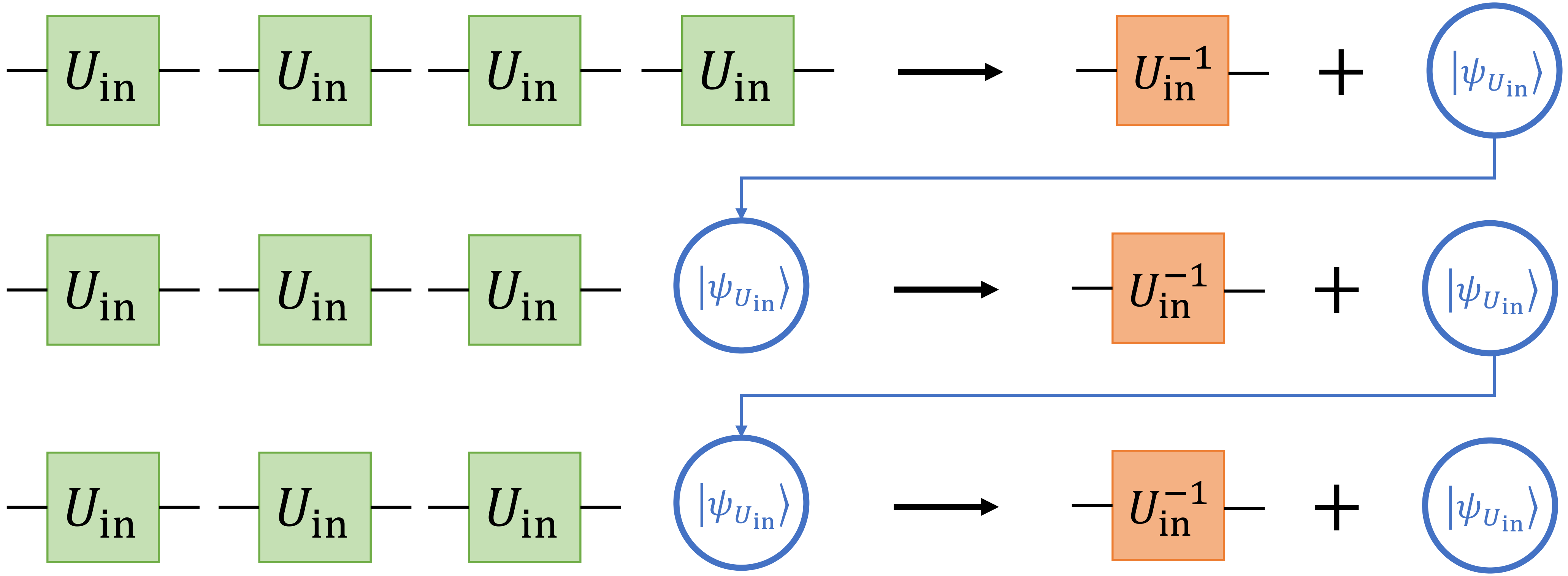

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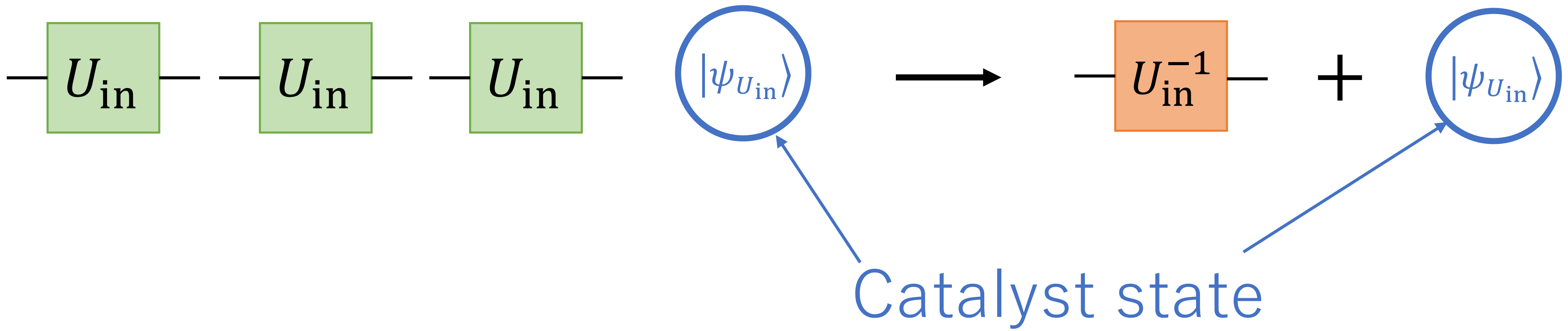

D. Jonathan, D and M. Plenio, M. B, PRL, 83, 3566 (1999).

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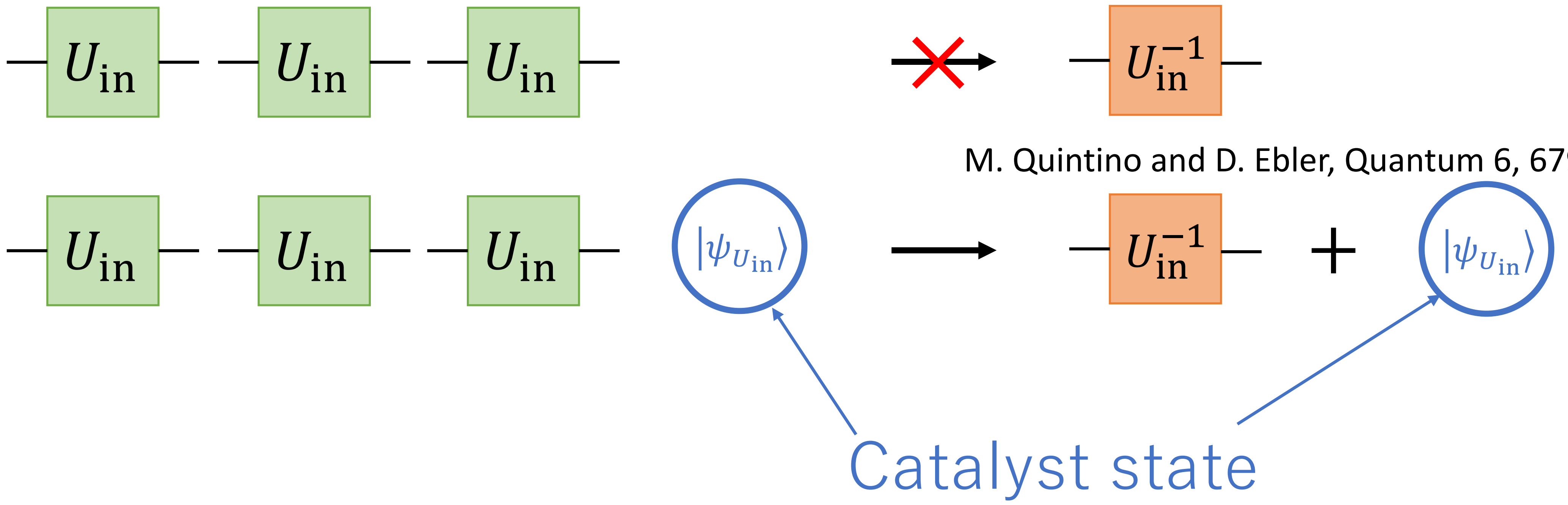

M. Quintino and D. Ebler, Quantum 6, 679 (2022)

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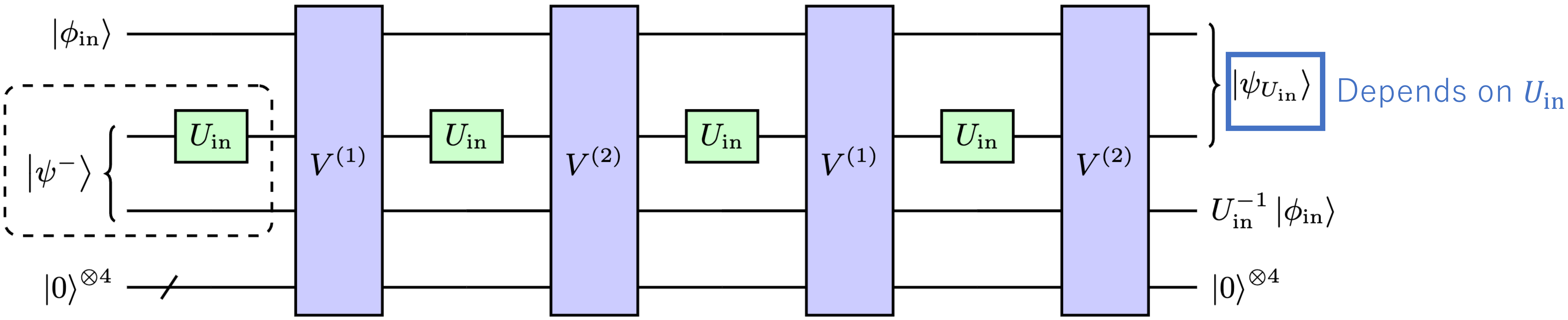
Characteristics of this protocol

- Non-clean protocol



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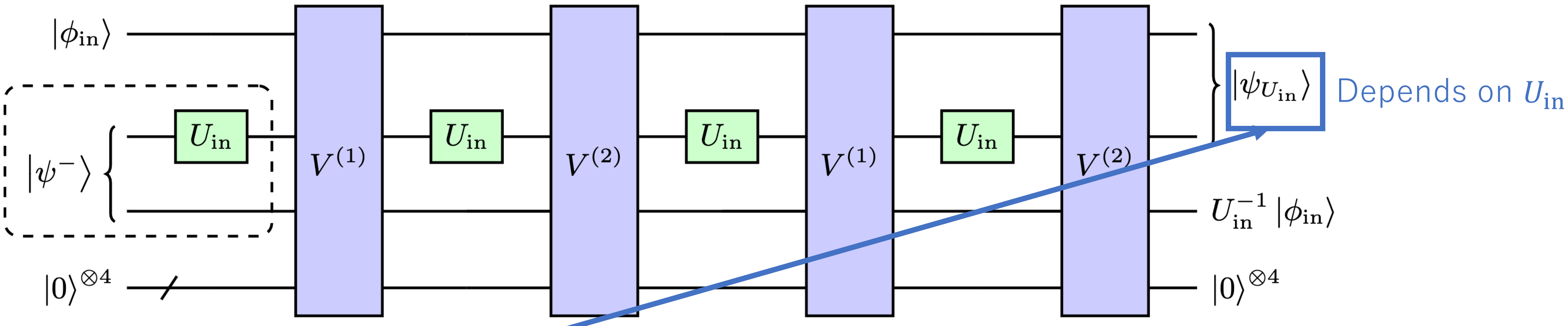

$$\rho = \int_{\mathrm{SU}(2)} dU_{\text{in}} |\psi_{U_{\text{in}}}\rangle\langle\psi_{U_{\text{in}}}| = \frac{I \otimes I}{4}$$

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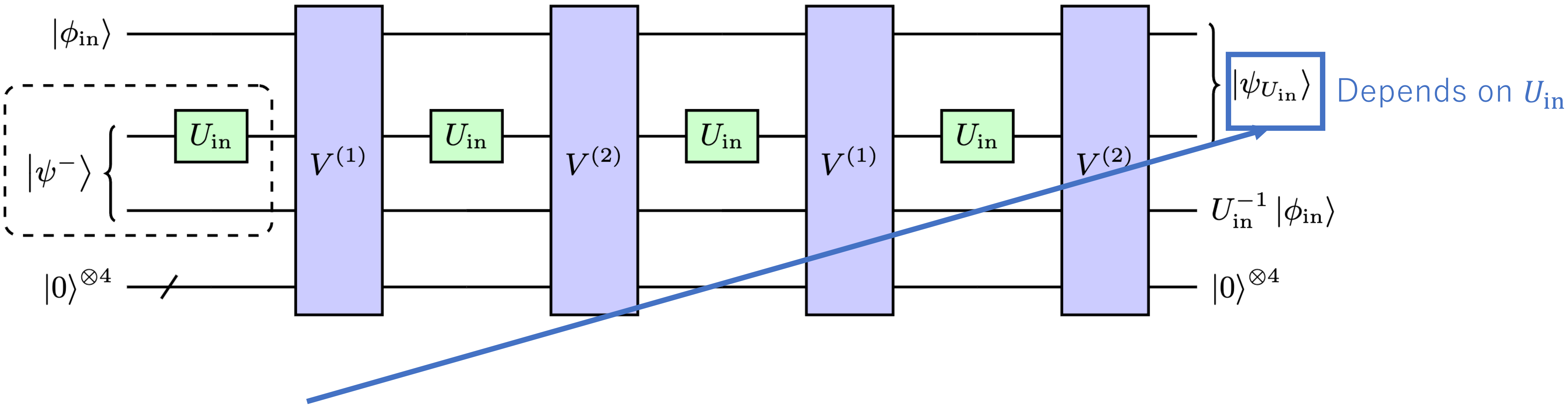

$\rho = \int_{\rm SU(2)} dU_{\rm in} |\psi_{U_{\rm in}}\rangle\langle\psi_{U_{\rm in}}| = \frac{I \otimes I}{4}$ → Initialization cost $W = k_B T H(\rho) = 2 k_B T \ln 2$

R. Landauer, IBM journal of research and development 5, 183 (1961).
F. Meier and H. Yamasaki, arXiv:2305.11212 (2023).

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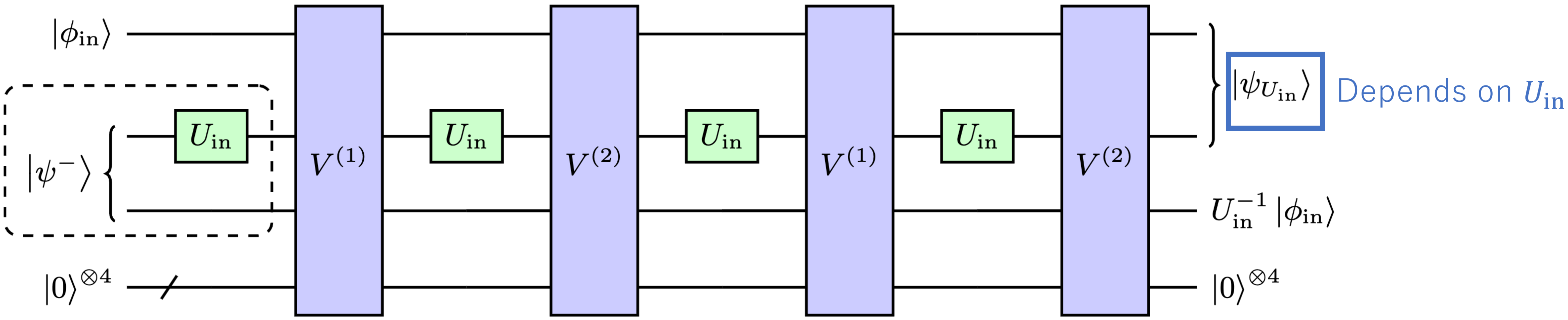

$$(I \otimes U_{\mathrm{in}})|\psi_{U_{\mathrm{in}}}\rangle = U_{\mathrm{in}}^{\otimes 2}|\psi^-\rangle = |\psi^-\rangle$$

Characteristics of this protocol

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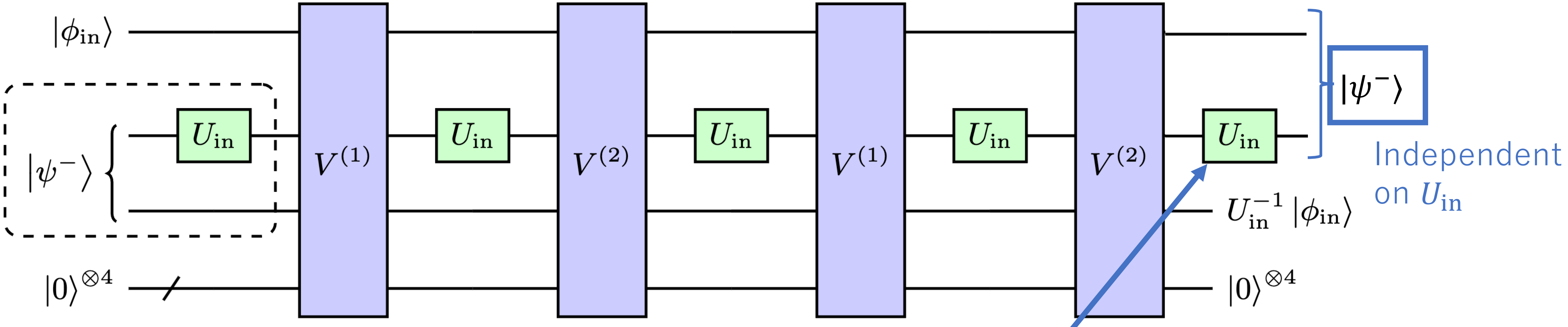

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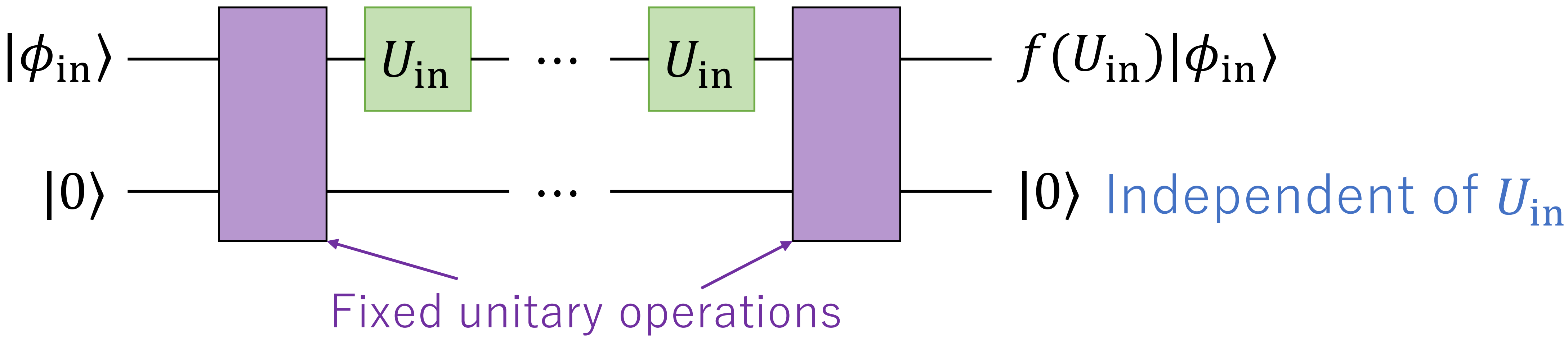
Characteristics of this protocol

- Clean protocol for $f\colon \mathrm{SU}(d) \to \mathrm{SU}(d)$



Z. Gavorová et al. arXiv:2011.10031 (2020).

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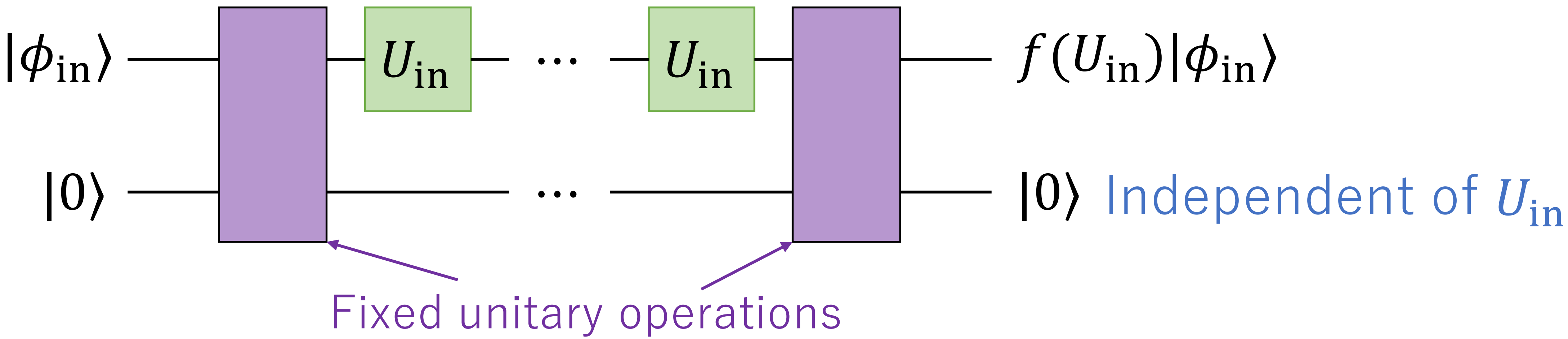

- No initialization cost

Z. Gavorová et al. arXiv:2011.10031 (2020).

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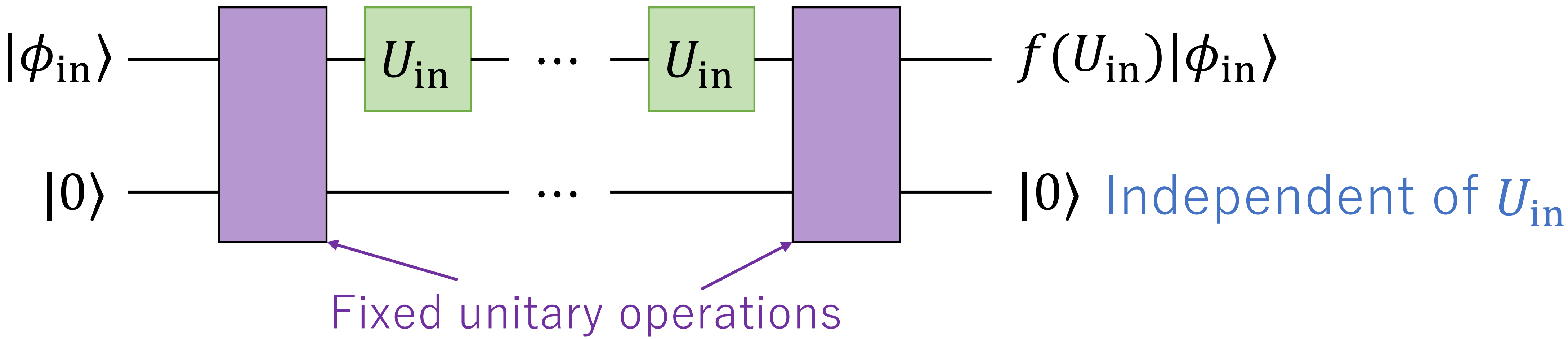

- No initialization cost

- $\mathrm{ctrl} - U_{\mathrm{in}} \to \mathrm{ctrl} - f(U_{\mathrm{in}})$

Z. Gavorová et al. arXiv:2011.10031 (2020).

Characteristics of this protocol

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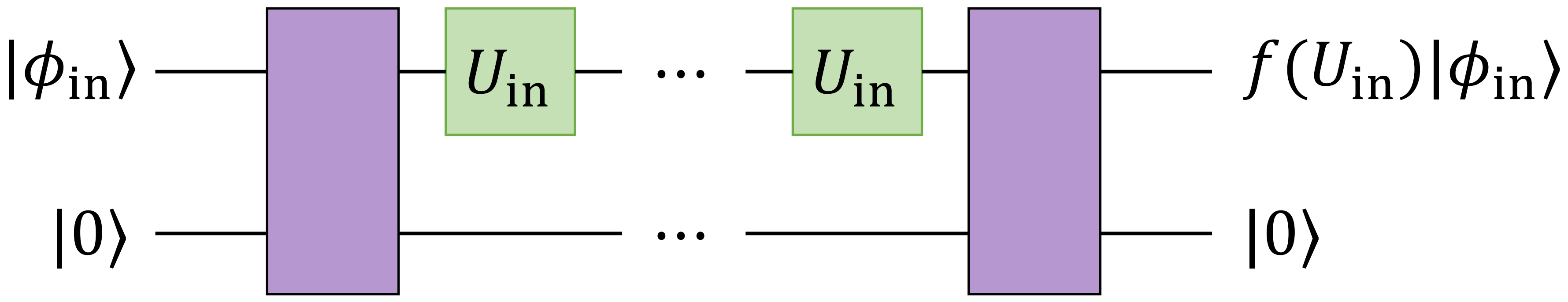

Characteristics of this protocol

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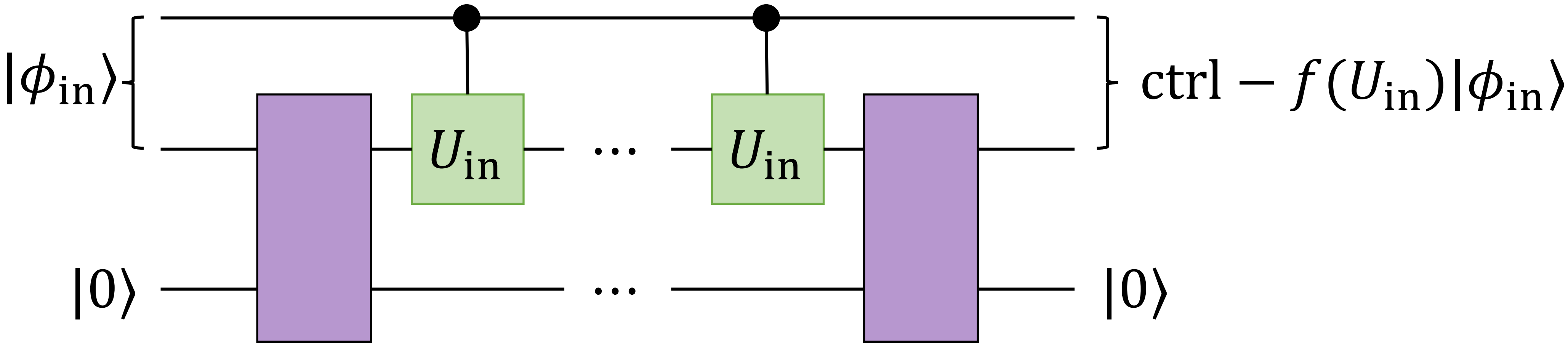

How to find this protocol?
: Numerical search + symmetry

- SDP to optimize approximation error

$$\max F_{\mathrm{ave}} := \int_{\mathrm{SU}(d)} dU_{\mathrm{in}}\, F\left[U_{\mathrm{in}}^{-1}, \mathcal{C}\left(U_{\mathrm{in}}^{\otimes n}\right)\right]$$

$$\mathrm{s.t.}\, \mathcal{C} \text{ is a quantum comb}$$

M. Quintino and D. Ebler, Quantum 6, 679 (2022)

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M. Quintino and D. Ebler, Quantum 6, 679 (2022)

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✔	✔		
$n=3$	✔			
$n=4$		???		
$\vdots$				

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$n=2$	✓	✓		
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$\vdots$				

→ Symmetry of the problem

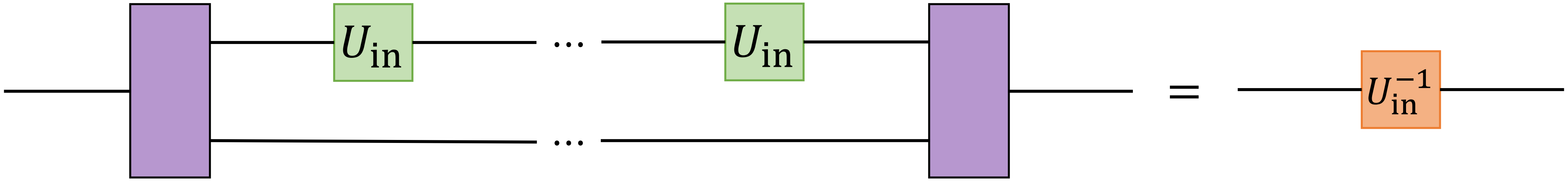
Reduction of SDP using $\mathbf{SU}(d)\times\mathbf{SU}(d)$ symmetry

M. Quintino et al. PRA 100, 062339 (2019)

- Symmetry in unitary inversion protocol

① $U_{\mathrm{in}} \mapsto V U_{\mathrm{in}} W$ for $V, W \in \mathrm{SU}(d)$

② Insert V and W to the whole circuit



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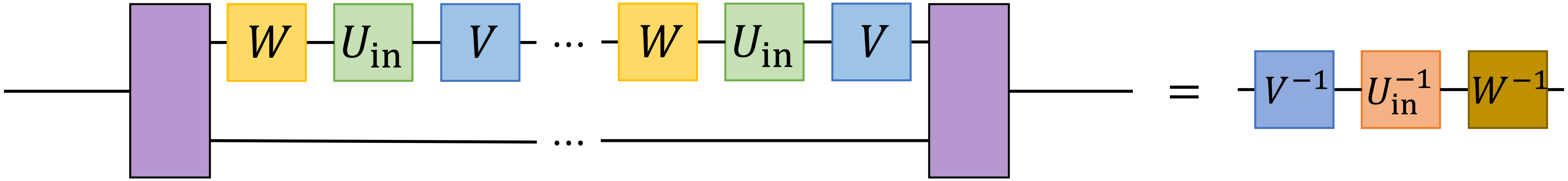

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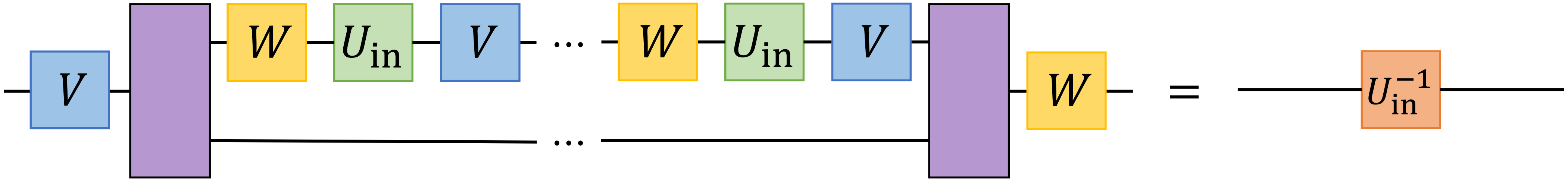

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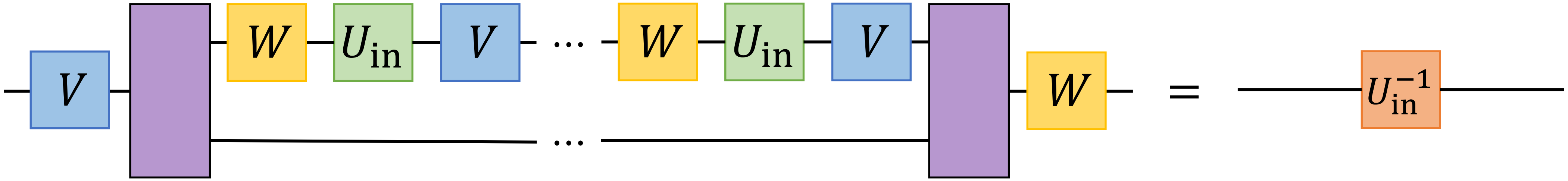


→ $[C, V^{\otimes n+1} \otimes W^{\otimes n+1}] = 0 \quad \forall V, W \in \mathrm{SU}(d)$

Numerical calculation of the SDP

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✓	✓		
$n=3$	✓			
$n=4$		???		
$\vdots$				

Numerical calculation of the SDP

	$d = 2$	$d = 3$	$d = 4$	$\cdots$
$n = 2$	✓	✓		
$n = 3$	✓			
$n = 4$		???		
$\vdots$				

→

	$d = 2$	$d = 3$	$d = 4$	$\cdots$
$n = 2$	✓	✓	✓	✓
$n = 3$	✓	✓	✓	✓
$n = 4$	✓	✓	✓	✓
$n = 5$	✓	✓	✓	✓

Numerical calculation of the SDP

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✓	✓		
$n=3$	✓			
$n=4$		???		
$\vdots$				

→

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✓	✓	✓	✓
$n=3$	✓	✓	✓	✓
$n=4$	✓	✓	✓	✓
$n=5$	✓	✓	✓	✓

Deterministic exact unitary inversion

Numerical calculation of the SDP

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✓	✓		
$n=3$	✓			
$n=4$		???		
$\vdots$				

→

	$d=2$	$d=3$	$d=4$	$\cdots$
$n=2$	✓	✓	✓	✓
$n=3$	✓	✓	✓	✓
$n=4$	✓	✓	✓	✓
$n=5$	✓	✓	✓	✓

Deterministic exact unitary inversion

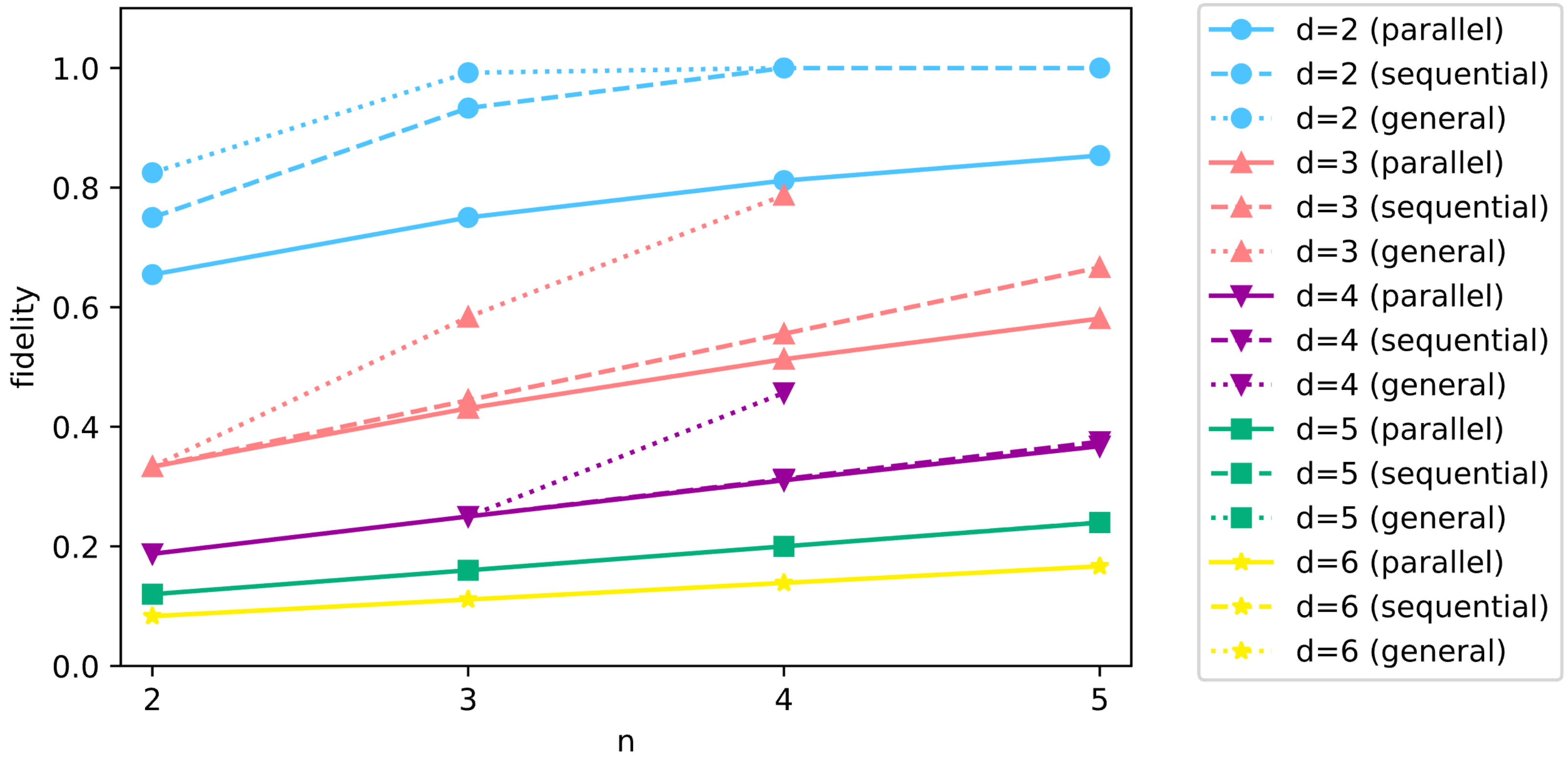
- Matrix representation of quantum comb → Quantum circuit

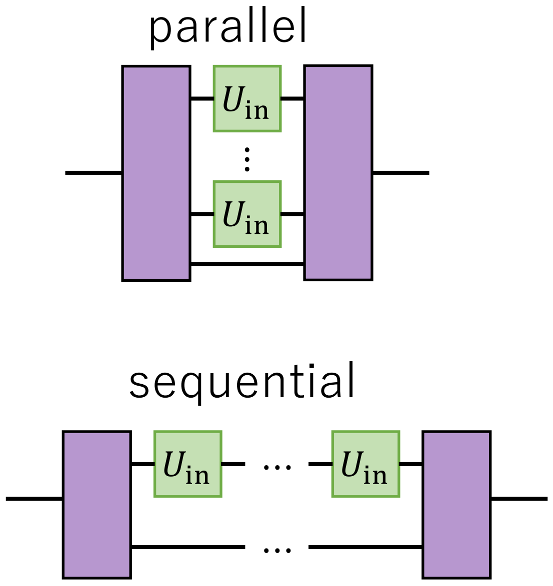
A. Bisio et al. PRA 83, 022325 (2011)

- Note: Reduction of SDP using unitary group symmetry

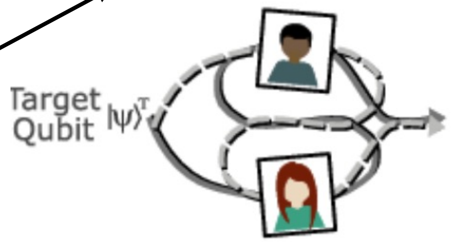
D. Grinko and M. Ozols, arXiv:2207.05713

Numerical calculation of the SDP





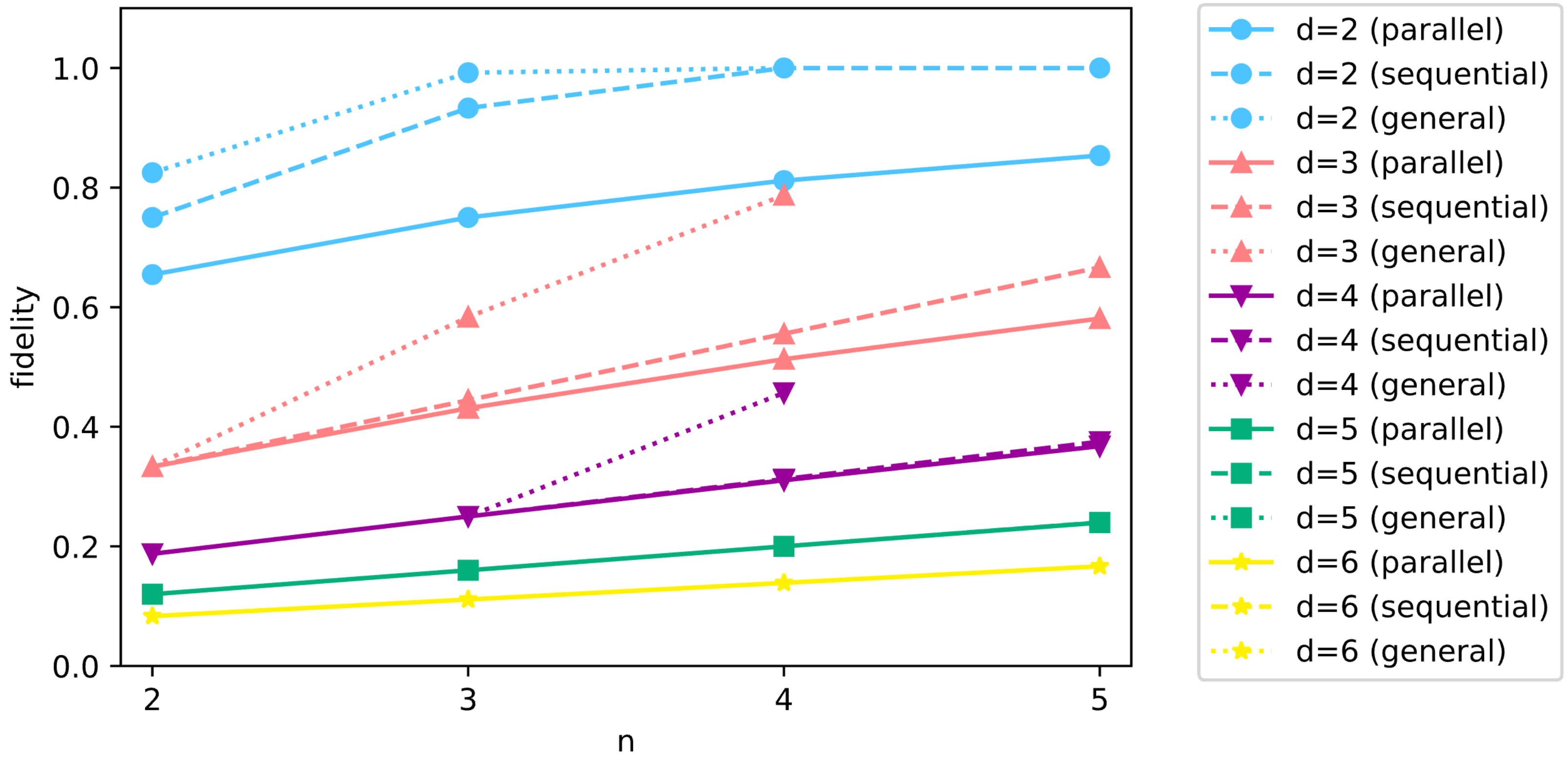
general ∋ indefinite causal order



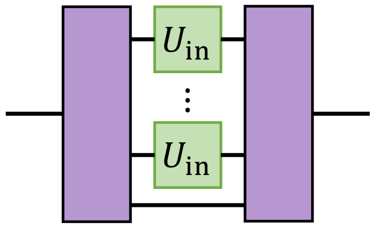
beyond circuit model
but preserves CPTP maps

Sci. Adv., 3: e1602589 (2017)

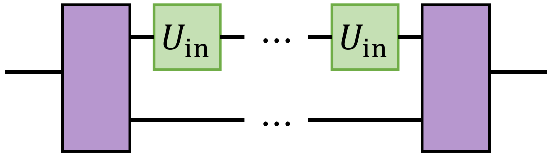
Numerical calculation of the SDP



parallel



sequential



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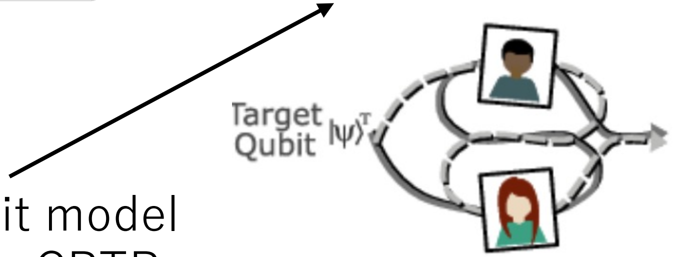


parallel ≤ sequential ≤ general, but
$n \leq d-1$ → parallel = sequential = general

beyond circuit model
but preserves CPTP maps

Sci. Adv., 3: e1602589 (2017)

Outline

- General perspective on higher-order quantum operations

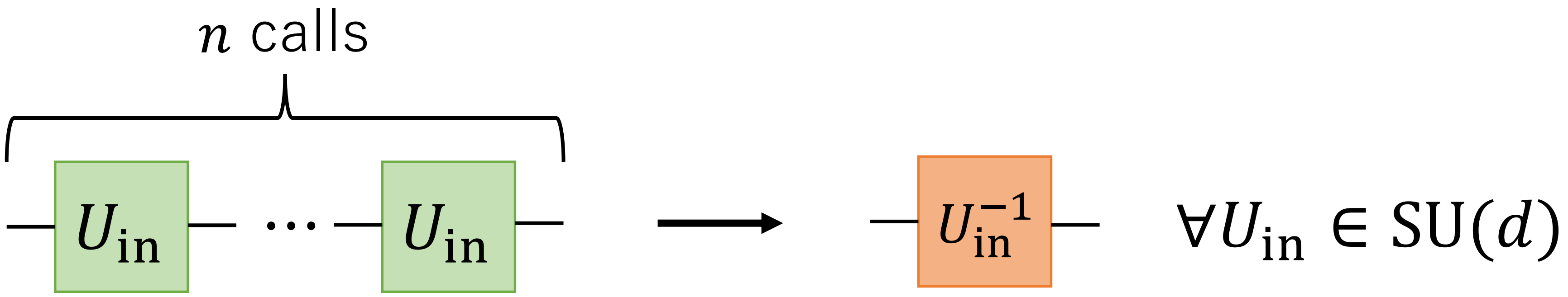
- Result 1: Deterministic exact qubit-unitary inversion

- Result 2: Isometry inversion

- Future works

Future works

- Deterministic exact unitary inversion for $d > 2$



- Is it possible for arbitrary d?
- If so, minimum number of n?

Future works

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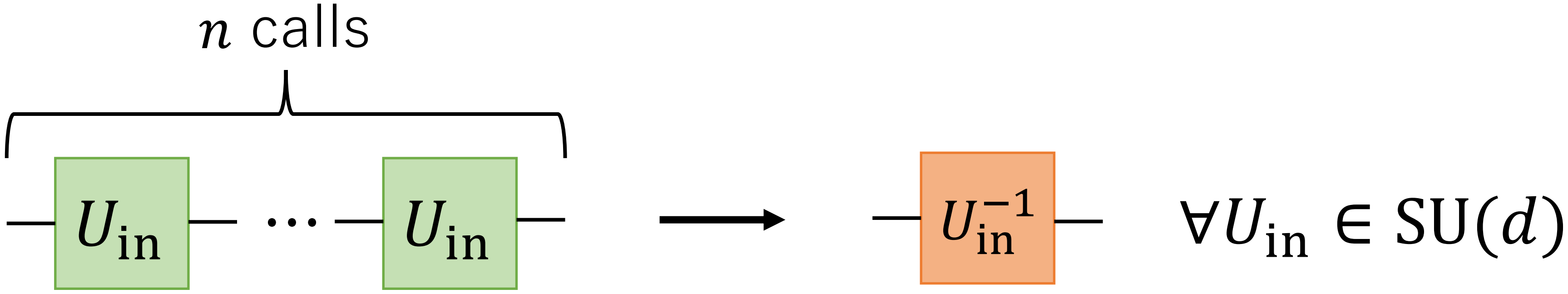

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Conjecture $n = d^2$?

Future works

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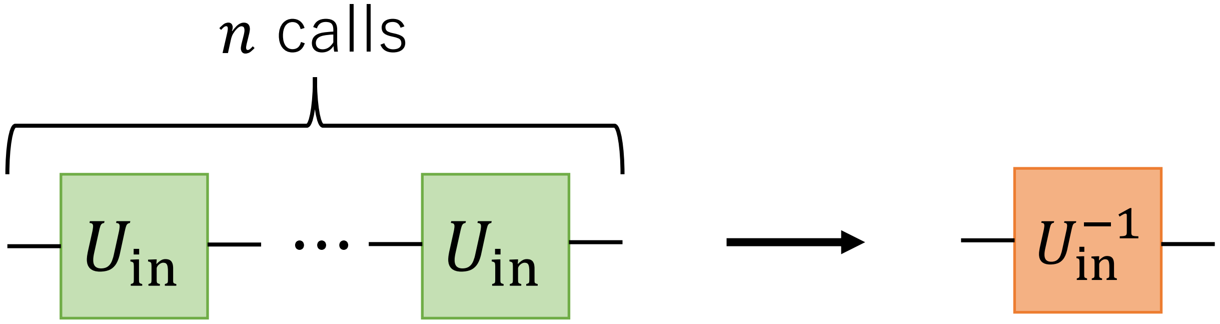


$\forall U_{\text{in}} \in \mathrm{SU}(d)$

- Is it possible for arbitrary d?

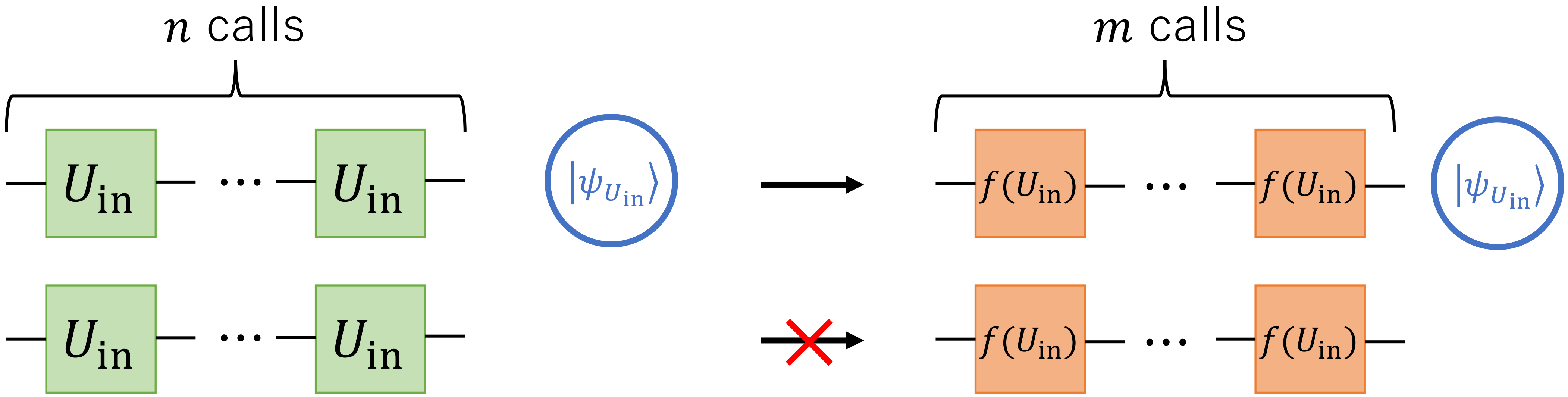
- If so, minimum number of n?

Conjecture $n = d^2$? →

- Further simplification of SDP
- Systematic understanding

Future works

- Catalytic higher-order quantum operations

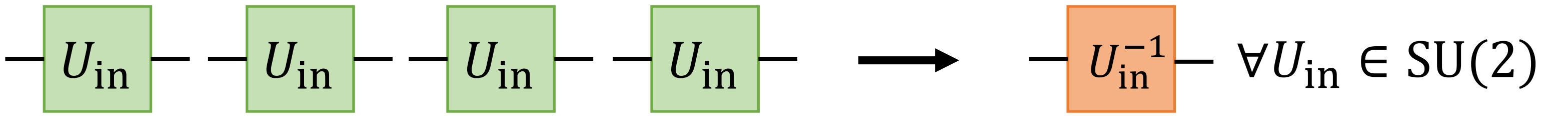


- How catalyst helps in other tasks?
- Relationship to asymptotic setting?

T. Kondra et al. PRL 127, 150503 (2021).
N. Shiraishi and T. Sagawa, PRL 126, 150502 (2021).
H. Wilming, PRL 127, 260402 (2021).

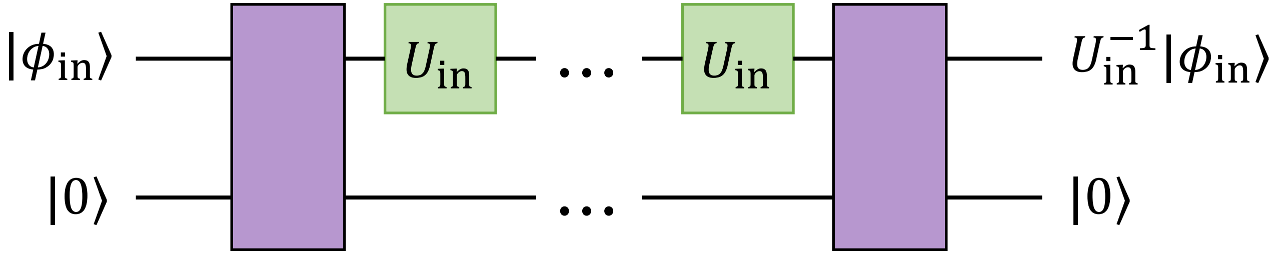
Summary

- Deterministic exact qubit-unitary inversion



- Catalyst

- Clean-version protocol



- Extension to isometry inversion (in preparation)

arXiv:2209.02907
(Accepted in PRL)

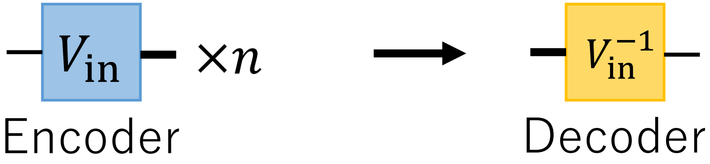
Supplementary materials

Outline

- General perspective on higher-order quantum operations

- Result 1: Deterministic exact qubit-unitary inversion

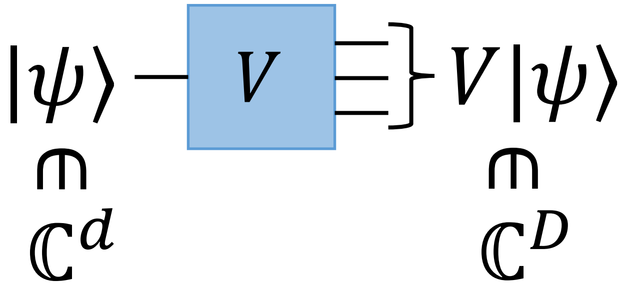
- Result 2: Isometry inversion



- Future works

Transformations of isometry operations

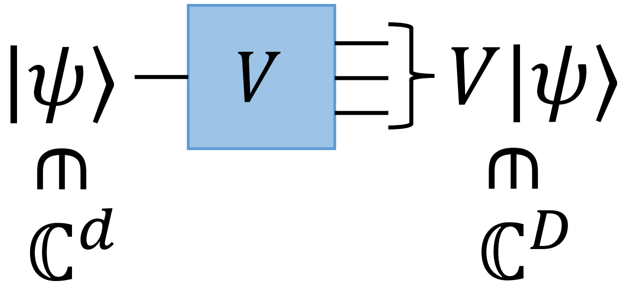
- Isometry operations



$D \geq d$

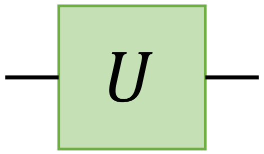
Transformations of isometry operations

- Isometry operations

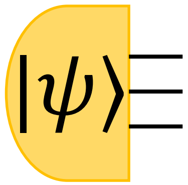


$D \geq d$

- Isometry ⊃ [Unitary ∪ Pure state]



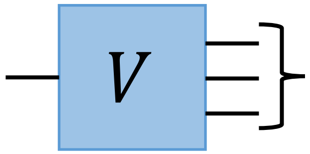
$D = d$



$d = 1$

Transformations of isometry operations

- Isometry operations

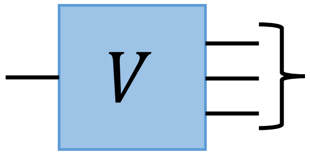
Eg. $\alpha|0\rangle + \beta|1\rangle$ 

$\alpha|000\rangle + \beta|111\rangle$

Encoder

Transformations of isometry operations

- Isometry operations

Eg. $\alpha|0\rangle + \beta|1\rangle$ 

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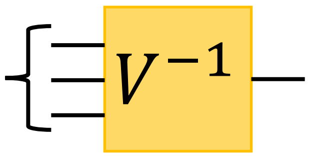
$\alpha|000\rangle + \beta|111\rangle$ 

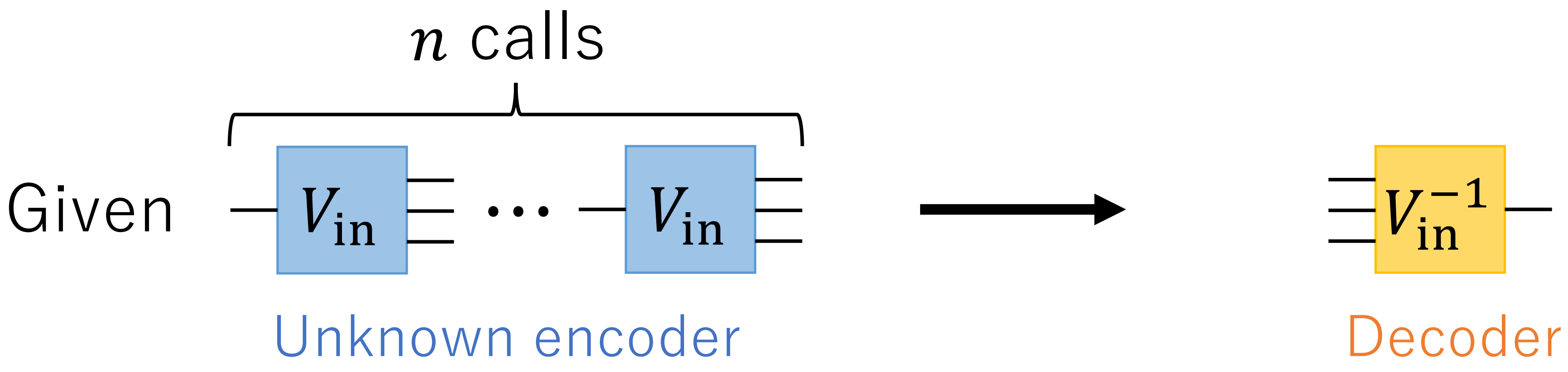
$\alpha|0\rangle + \beta|1\rangle$

Decoder

Isometry inversion

SY, A. Soeda and M. Murao, Quantum 7, 957 (2023)

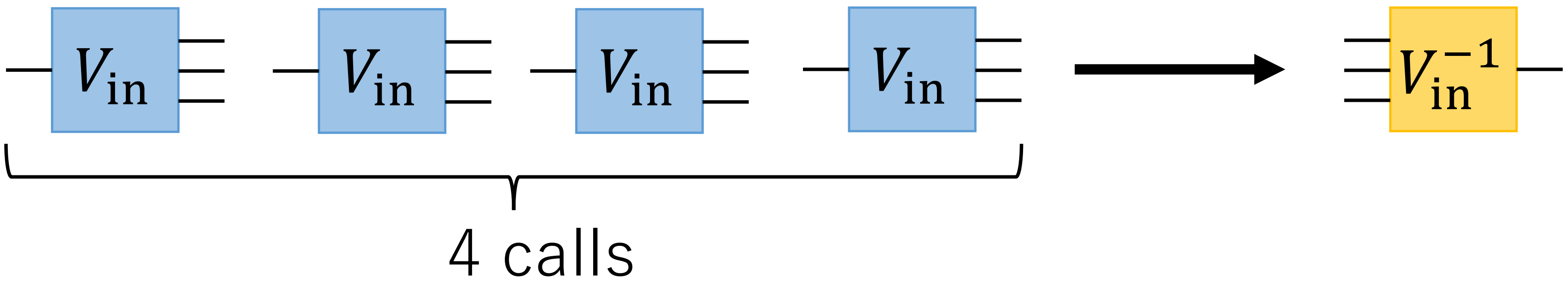
- Isometry inversion:



Isometry inversion

- Result:

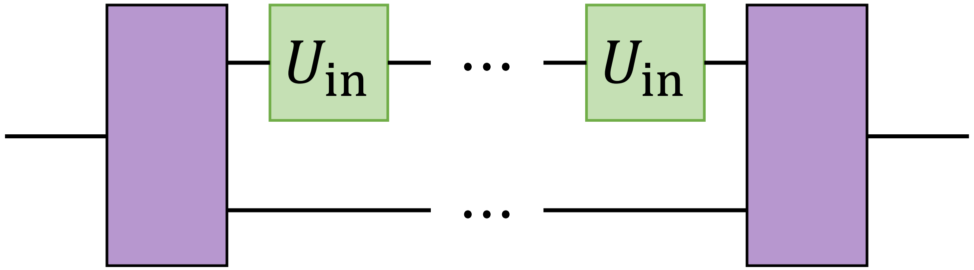
There exists a deterministic exact protocol to reverse any qubit-encoding $(d = 2)$ isometry operations.



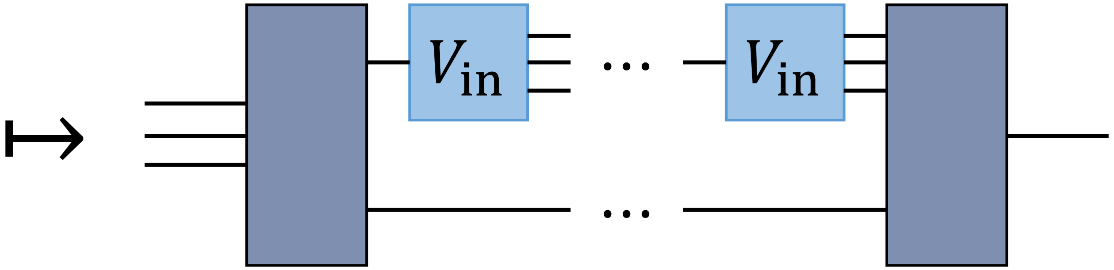
SY, A. Soeda and M. Murao, In preparation

Proof sketch

- Key idea



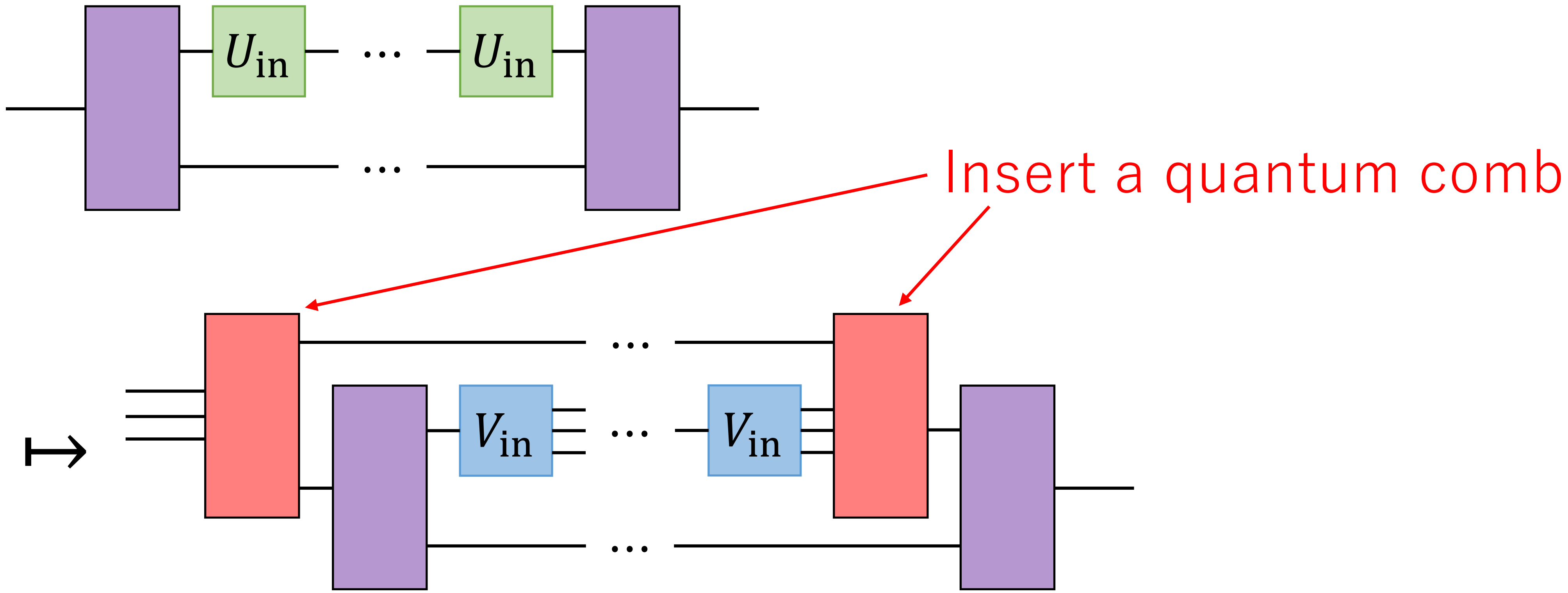
Unitary inversion



Isometry inversion

Proof sketch

- Key idea



Proof sketch

- Key idea