Solve 3d gravity with Virasoro TQFT

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(Based on arXiv 2304.13650 by Scott Collier, Lorenz Eberhardt and MZ)
QIMG 2023, Kyoto, Japan

Princeton University

October 2023

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- A first-order formulation of 3d Einstein gravity with negative cosmological constant via $SL(2,\mathbb{R}) \times SL(2,R)$ Chern-Simons theory was proposed and studied (Witten 88;...) for decades.

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- The bulk gravitational path-integral is studied via various approaches(Maloney, Witten 07; Giombi, Maloney, Yin 08; Cotler, Jensen 18-20; Eberhardt 22;...), while these approaches usually work only for a specific class of 3d manifolds and still have some unsatisfactory subtleties.

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- Topological nature of 3d gravity suggests a holographic description. Provided that the asymptotic symmetry of AdS_3 is two copies of Virasoro symmetry with $c = \frac{3L}{2G_N}$(Brown, Henneaux 86), there is a persistent exploration of potential holographic CFT_2 dual of 3d gravity (Witten 07; Maloney, Witten 07; Benjamin, Collier, Maloney 20; Cotler, Jensen 20; Maxfield, Turiaci 20; Chandra, Collier, Hartman, Maloney 22;...)

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- Its phase space coincides with the gravity one. It's computationally approachable via the (generalized) Heegaard splitting of 3d manifolds and leads to finite results for all hyperbolic manifolds.
- Motivated from the recent developments in 2d gravity(Saad, Shenker, Stanford 19), the partition functions on connected manifolds with multiple asymptotic boundaries are computed in proposed TQFT, which provides implications on the non-Gaussian statistics of the ensemble of 2d holographic CFT.

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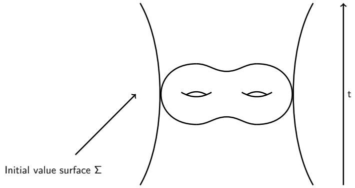
I. Canonical quantization of AdS_3 gravity

The Hilbert space

- Consider the canonical quantization of lorentzian AdS_3 gravity on $\Sigma \times \mathbb{R}$. By requiring the metric to be nondegenerate, we are equivalently quantizing the Teichmuller space $\mathcal{T}_\Sigma$ associated with Σ.



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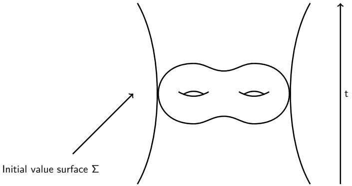

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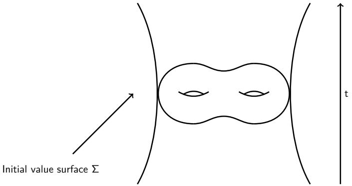

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- From AdS/CFT, we expect $Z_{\text{bulk}}(M) \subset Z_{\text{CFT}}(\partial M)$.The bulk TQFT partition function $Z_{\text{bulk}}(M)$ corresponds to vectors in $\mathcal{H}_{TQFT}(\partial M)$. On the other hand, $Z_{CFT}(\partial M)$ can be spanned by Virasoro conformal block. Therefore, we expect $\mathcal{H}_{TQFT}(\partial M)$ to be also spanned by Virasoro conformal blocks.

The Hilbert space

- An inner product structure on this Hilbert space was proposed(Verlinde 89; Eberhardt, Collier, MZ 23)

$$\langle \mathcal{F}_1 | \mathcal{F}_2 \rangle = \int_{\mathcal{T}} Z_{\text{bc-ghost}} Z_{\text{timelike-Liouviile}} \mathcal{F}_2 \bar{\mathcal{F}}_1. \quad (1)$$

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$$\langle \mathcal{F}^{\mathcal{C}}_{\Sigma}(\vec{P}_1) | \mathcal{F}^{\mathcal{C}}_{\Sigma}(\vec{P}_2) \rangle = \frac{\delta(\vec{P}_1 - \vec{P}_2)}{\rho^{\mathcal{C}}_{\Sigma}(\vec{P}_1)}, \quad (2)$$

where $\rho^{\mathcal{C}}_{\Sigma}(\vec{P})$ can be expressed in terms of $\rho_0(P)$, the universal Cardy density of states, and normalized DOZZ coefficient $C_0(P_1, P_2, P_3)$. The above integral converges when $\Delta_P > \frac{c-1}{24}$

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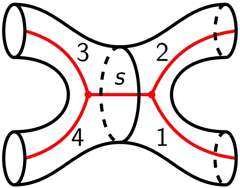
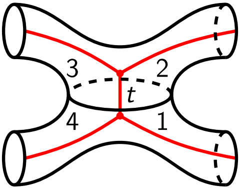


$$= \sum_s \mathbb{F}_{st} \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \tag{3}$$



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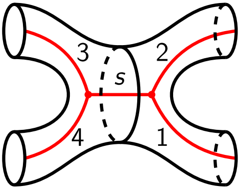

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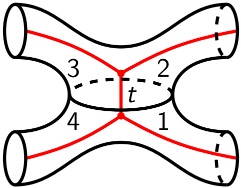

- $\mathbb{B}, \mathbb{S}, \mathbb{T}, \mathbb{F}$ satisfies the Moore-Seiberg consistency condition(Moore, Seiberg 89), and preserve the inner product $\langle \mathcal{F}_1 | \mathcal{F}_2 \rangle$.

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II. Virasoro TQFT

A TQFT related to Liouville CFT

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- Note that in this TQFT the lines are normalizable only if $\Delta_P > \frac{c-1}{24}$. Therefore, different from ordinary TQFT, the trivial line is not normalizable, i.e. the sphere partition function $Z(S^3)$ is not formally convergent.

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- The normalizable lines in this TQFT are associated to the normalizable states in Liouville CFT with Liouville momenta P. This TQFT is associated with Liouville CFT (similar to CS/WZW correspondence).

Heegaard splitting

- Any compact, oriented manifolds M in 3d admits a Heegaard splitting into two handlebodies $M = S\Sigma_g^{(1)} \cup_\gamma S\Sigma_g^{(2)}$, where $\gamma \in \mathrm{Map}(\Sigma_g)$ (This can be generalized to manifolds with boundaries).

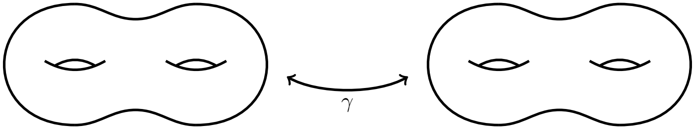

Figure: A demonstration of genus-2 Heegaard splitting.

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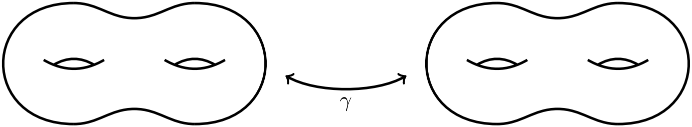

Figure: A demonstration of genus-2 Heegaard splitting.

- Two handlebodies are glued together along the boundary Riemann surface Σ_g subject to some relative mapping class group action γ. The TQFT partition function on manifold M can be written as follows

$$Z_{\mathrm{Vir}}(M) = \left\langle \mathcal{F}_g^{\mathcal{C}}(\vec{\mathbb{1}}) \,\middle|\, U(\gamma) \,\middle|\, \mathcal{F}_g^{\mathcal{C}}(\vec{\mathbb{1}}) \right\rangle . \tag{4}$$

The path integral on each handlebody prepares a state in $\mathcal{H}(\Sigma_g)$.

From TQFT to gravity partition function

- The TQFT partition function on 3d manifold M can be typically computed via Heegaard splitting, while this is still not the gravity partition function. The TQFT partition function is defined on manifold with **fixed topology**, but gravity partition function requires a sum over topologies.

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- Meanwhile, the gravity theory treat diffeomorphism group Diff(M) as gauge symmetry. From the TQFT setup, we still need to gauge the large diffeomorphism, i.e. the mapping class group.

$$Z_{\text{grav}}(M) = \sum_{\gamma \in \text{Map}(\partial M)/\text{Map}(M,\partial M)} |Z_{\text{Vir}}(M^{\gamma})|^2 \ . \tag{5}$$

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- For hyperbolic 3 manifolds, the size of $\mathrm{Map}(M)$ is finite(Rigidity theorem).

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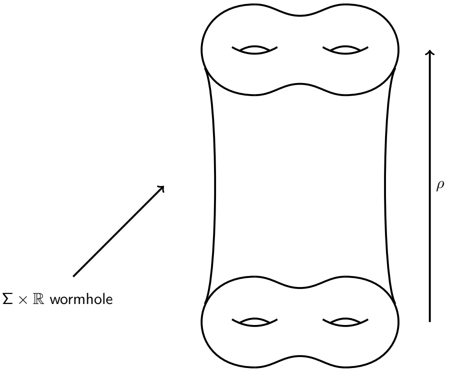
III. Explicit calculations

Euclidean wormholes

- One interesting AdS_3 gravity solutions is the Maldacena-Maoz Euclidean wormholes (Maldacena, Maoz 05), which has the following metric
$$ds^2 = d\rho^2 + \cosh^2 \rho \, ds^2_{\Sigma_{g,n}}, \tag{6}$$
where $ds^2_{\Sigma_{g,n}}$ corresponds to the hyperbolic metric on surface $\Sigma_{g,n}$.



Euclidean wormholes

- The topology of the wormhole is $\Sigma_{g,n} \times \mathbb{R}$ with two boundaries. It prepares a state in $\mathcal{H}(\Sigma_{g,n}) \otimes \mathcal{H}(\Sigma_{g,n})$. Equivalently, it corresponds to the identity $\mathbb{1} \in \text{Aut}(\mathcal{H}(\Sigma_{g,n}))$.

$$Z_{\text{Vir}}(\Sigma_{g,n} \times \mathbb{R}) = \int d^{3g-3+n}\vec{P}\ \rho^{\mathcal{C}}_{g,n}(\vec{P}) \,|\, \mathcal{F}^{\mathcal{C}}_{g,n}(\vec{P})\rangle \otimes |\, \mathcal{F}^{\mathcal{C}}_{g,n}(\vec{P})\rangle\ . \tag{7}$$

After specifying the moduli of two boundaries,

$$Z_{\text{Vir}}(\Sigma_{g,n} \times I; \mathbf{m}_1, \mathbf{m}_2) = Z_{\text{Liouville}}(\Sigma_{g,n}; \mathbf{m}_1, \mathbf{m}_2)\ . \tag{8}$$

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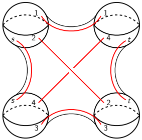
- Gravity result: $Z_{\mathrm{grav}}(\Sigma_{g,n} \times I) = \sum_{\gamma \in \mathrm{Map}(\Sigma_g)} |Z_{\mathrm{Liouville}}(\Sigma_{g,n}; \mathbf{m}_1, \gamma \cdot \mathbf{m}_2)|^2$.

- Consider one specific case of multi-boundary wormhole: four-boundary wormhole with each boundary a three-punctured sphere.

$M =$ 

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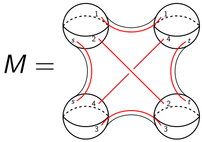
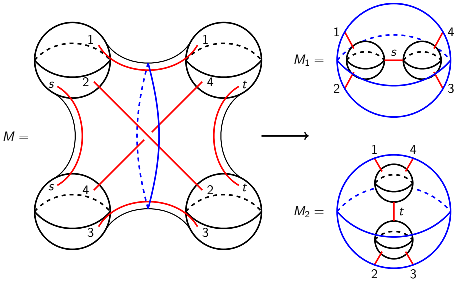


- This wormhole has the following Heegaard splitting configuration



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- Each boundary prepare a state in $\mathcal{H}(\Sigma_{0,4})$

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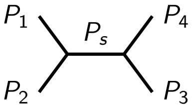

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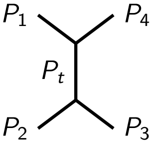

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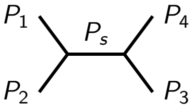

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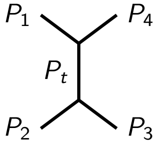

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- The gravity partition function is given by

$$Z_{\text{grav}} = |Z_{\text{Vir}}(M)|^2 = \sqrt{\overline{c_{12s}^2}\,\overline{c_{34s}^2}\,\overline{c_{14t}^2}\,\overline{c_{23t}^2}}\left|\begin{Bmatrix} P_1 & P_2 & P_s \\ P_3 & P_4 & P_t \end{Bmatrix}_b\right|^2, \tag{10}$$

which contributes to the non-gaussian corrections to the ensemble(Chandra, Collier, Hartman, Maloney 22).

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