Complexity for Interacting quantum field theory

Arpan Bhattacharyya

**Yukawa Institute for Theoretical Physics,
Kyoto**

**based on JHEP 1810 (2018) [arXiv: 1808.03105[hep-th]]
with Arvind Shekar, Aninda Sinha**

Outline

– **A brief Introduction**:

– **Circuit Complexity for interacting QFT**

Setup

Assumptions

Results (Relation with RG flows)

– **Future directions**

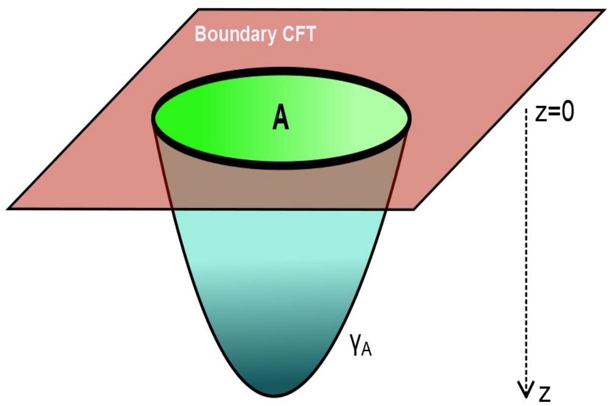
Introduction

In recent time tools from QI has played important role
to advance our understand about the mechanism of AdS/CFT

For eg: Entanglement entropy

Ryu-Takayanagi prescription:

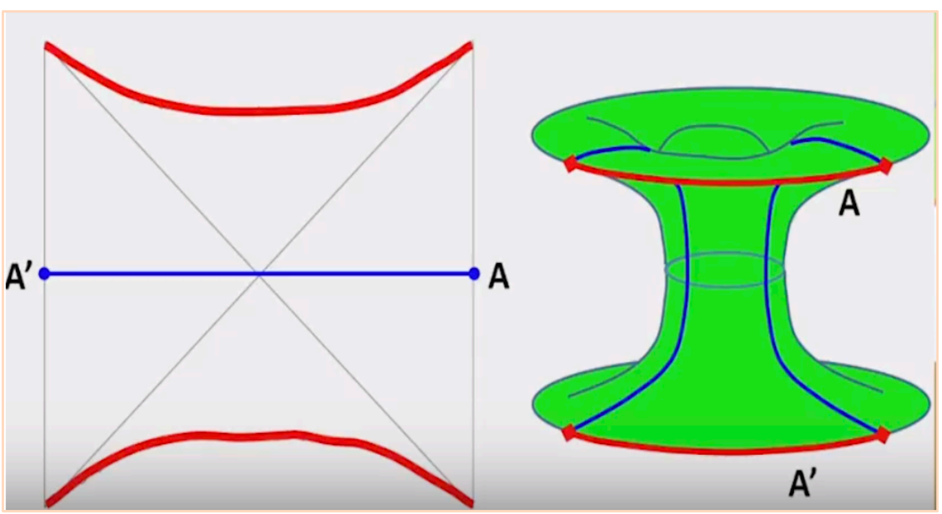
(Ryu -Takayanagi, Phys.Rev.Lett.96:181602,2006)

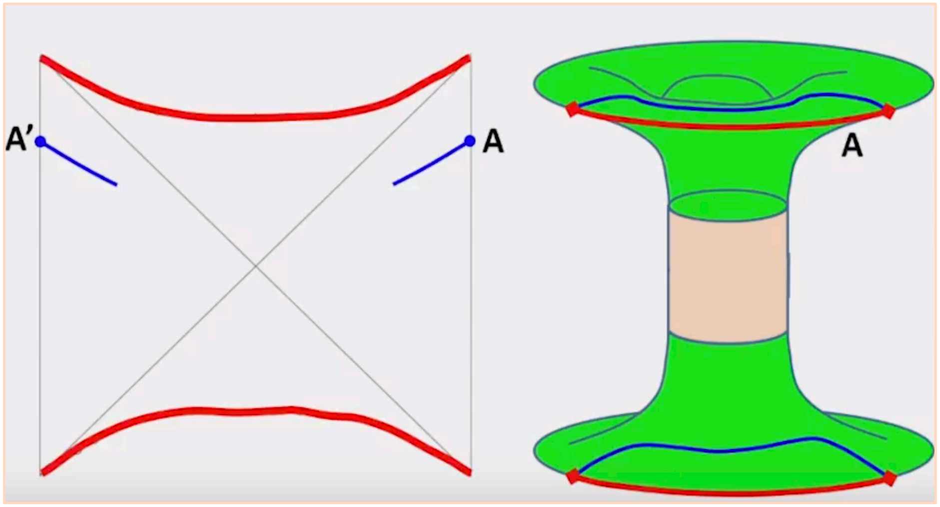


This duality becomes more stimulating in the context of Black hole

Eternal Black Hole ⟷ Thermofield Double

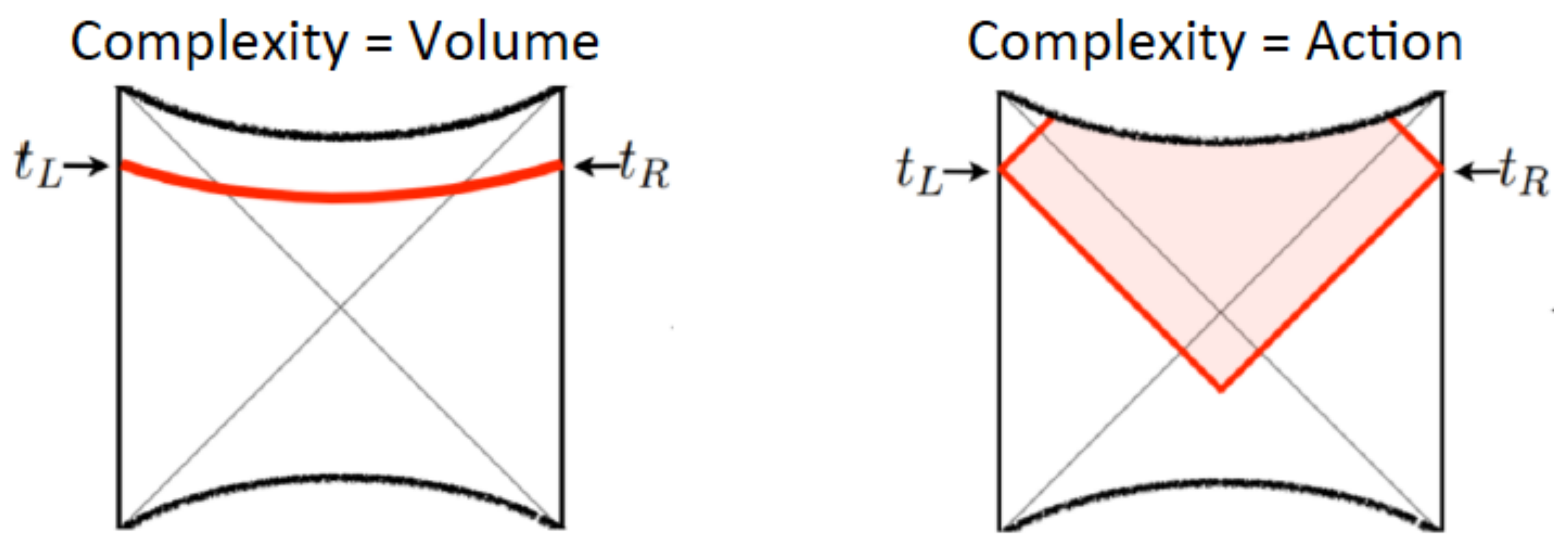
AdS/CFT





EE is not a good probe for physics behind horizon

Two interesting objects probing the interior of black hole



(Brown, Roberts, Swingle, Susskind & Zhao)

(Carmi, Chapman, Lehner,Myers, Marrochio, Poisson, Sorkin,Sugishita et al…)

$$\mathcal{C}_V(\Sigma) = \max\left[\frac{\mathcal{V}(\mathcal{B})}{G_N\, l}\right] \qquad\qquad \mathcal{C}_A(\Sigma) = \frac{I_{WDW}}{\pi\hbar}$$

(picture courtesy Jefferson-Myers, 1707.08570 [hep-th])

Grows with time and keep growing even after the thermalization time

"Complexity" is dual to these two objects ?
Can we compute it field theory ?

Computational Complexity

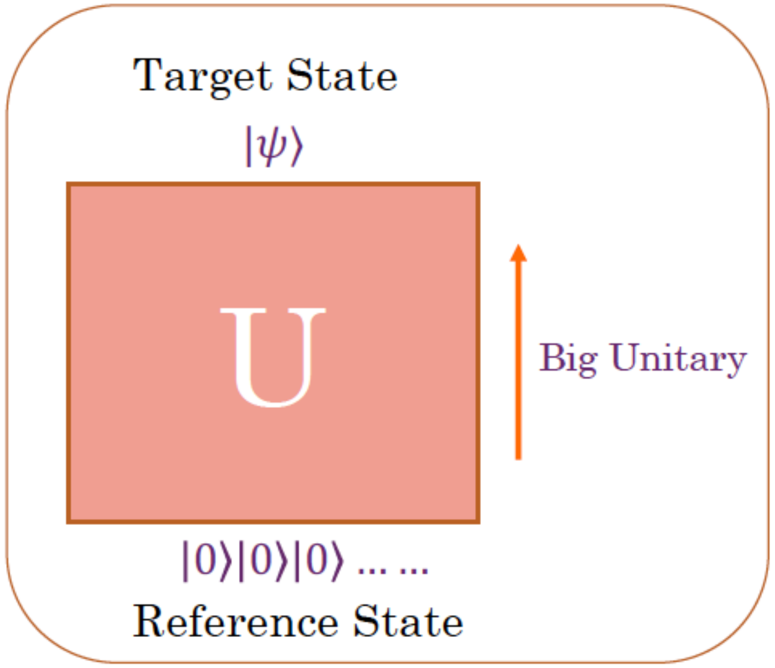
Generically: How difficult is to implement a task ?

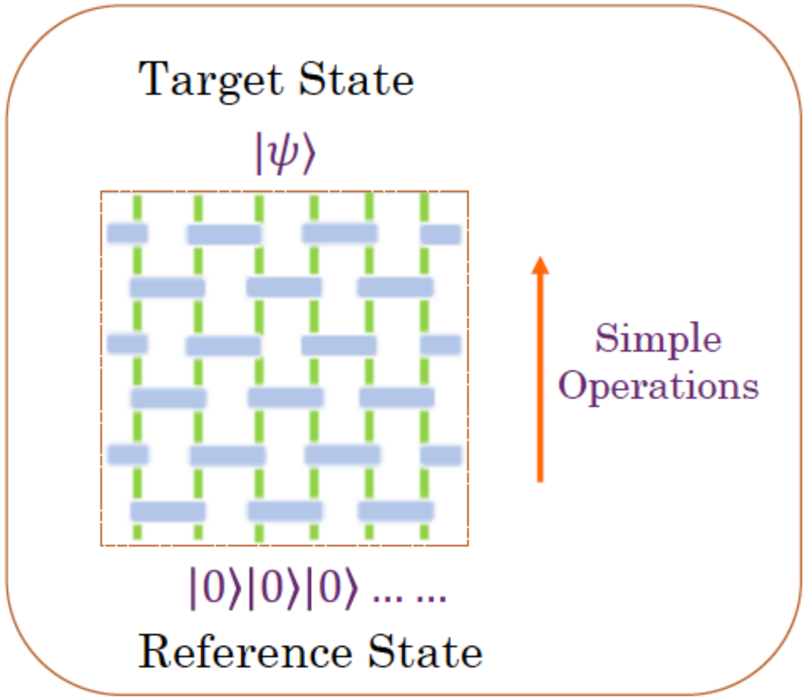
Important applications in QI and Quantum Many body physics

(Vidal '03, '04, F. Verstraete and I.Cirac '06,09
N. Schuch, I. Cirac, and F. Verstraete '08,
D. Aharonov, I. Arad, Z. Landau, and U.
Vazirani '11)

Here we will use the notion of "Circuit complexity"

how difficult is to prepare a particular state ?





"minimize the number of operations"

will depend on the choice of the reference state

Free QFT computation: Jefferson Myers '17 using Nielsen approach

(for other approaches refer to Chapman, Heller, Marrochio, Pastawski (arXiv:1707.08582)[Phys. Rev. Lett. 120,121602(2018)], Caputa, Kundu, Miyaji, Takyanagi,Watanabe arXiv: 1706.07056 [JHEP11(2017)097])

But to make contact with holography we need to understand this interacting QFT.

Jordan-Lee-Preskill (2012): Non-perturbative computation of n-particle scattering for ϕ^4 theory by a quantum computer provides an exponential advantage over perturbative method which uses Feynman Diagrams.

Then question naturally arises how a quantum computer would compute other interesting quantities that are calculated by conventional means

Motivated by all these we ask what other important aspects of QFTs can be capture of "Complexity"

RG flow is one important aspects: what we can we say about it in terms complexity ?

Circuit complexity for Interacting QFT

$\lambda\phi^4$ theory: $\mathcal{H} = \frac{1}{2}\int d^{d-1}x\Big[\pi(x)^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2 + \frac{\hat{\lambda}}{12}\phi(x)^4\Big]$

AB, A.Shekar, A. Sinha, JHEP 1810 (2018) 140, arXiv: 1808.03105[hep-th]

$$X(\vec{n}) = \delta^{d/2}\phi(\vec{n}),\; P(\vec{n}) = \pi(\vec{n})/\delta^{d/2},\; M = \frac{1}{\delta}, \omega = m,\; \Omega = \frac{1}{\delta},\; \lambda = \frac{\hat{\lambda}}{24}\delta^{-d}.$$

Discretize: $$\mathcal{H} = \sum_{a=0}^{N-1}\frac{1}{2}\Big[p_a^2 + \omega^2 x_a^2 + \Omega^2(x_a - x_{a+1})^2 + 2\lambda\, x_a^4\Big]$$

$$x_a = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\exp\Big(\frac{2\pi\, i\, k}{N}\,a\Big)\tilde{x}_k,$$

Normal Mode: $$p_a = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\exp\Big(-\frac{2\pi\, i\, k}{N}\,a\Big)\tilde{p}_k$$

Hamiltonian gets diagonalized

$$\mathcal{H} = \frac{1}{2}\sum_{k=0}^{N-1}\Big[|\tilde{p}_k|^2 + \tilde{\omega}_k^2|\tilde{x}_k|^2\Big] + \frac{\lambda}{N}\sum_{\alpha=N-k'-k_1-k_2 \bmod N, k', k_1, k_2=0}^{N-1}\tilde{x}_\alpha\tilde{x}_{k'}\tilde{x}_{k_1}\tilde{x}_{k_2}.$$

$$\sum_{k=1}^{d-1}\tilde{\omega}_{i_k}^2 = m^2 + \frac{4}{\delta^2}\sum_{k=1}^{d-1}\sin^2\Big(\frac{\pi\, i_k}{N}\Big),$$

Sum of Anhamronic oscillator

We solve the ground state perturbatively, linear order in $\lambda \ll 1$

and compute circuit complexity (minimal circuit depth) for it.

Circuit complexity using Nielsen approach:

(Nielsen quant-ph/0502070,
Nielsen, Dowling, Gu, Doherty,quant-ph/0603161
M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004)

$$U(s) = \overleftarrow{\mathcal{P}} \exp(i \int_0^s ds\, Y^I(s)\, O_I(s)),$$

Path ordering

Generators of the elementary gates

control functions

Boundary conditions: $\psi^T(\tilde{x}_0, \cdots \tilde{x}_{N-1}) = U(s=1)\psi^R(\tilde{x}_0, \cdots \tilde{x}_{N-1})$

Ground state of $\lambda\phi^4$

Unentangled State

Optimal Circuit: We need to find optimal $Y^I(s)$ achieved by minimizing some kind of action "***Cost function***" $\mathcal{F}(U, \dot{U})$ for these $Y^I(s)$

(Nielsen quant-ph/0502070,
Nielsen, Dowling, Gu, Doherty,
quant-ph/0603161
M.~A. Nielsen and M.~R. Dowling,
quant-ph/0701004,
Jefferson-Myers, 1707.08570 [hep-th],
Hackl-Myers, 1803.10638 [hep-th],
Guo-Hernandez-Myers-Ruan,
1807.07677[hep-th])

We choose: $$\mathcal{F}(U, \dot{U}) = \sum_I p_I |Y^I(s)|$$

p_I penalty factor: we fix it such that we recover free theory result for $\lambda = 0$

(Jefferson-Myers, 1707.08570 [hep-th])

Complexity: $$\mathcal{C}_{\kappa=1}(U) = \int_0^1 \mathcal{F}(U, \dot{U}) ds$$

To elaborate: Let us first focus on N=2 oscillator case in d=1+1

AB, A.Shekar, A. Sinha, JHEP 1810 (2018) 140, arXiv: 1808.03105[hep-th]

Target state: $$\psi^T(\tilde{x}_0, \tilde{x}_1) = \mathcal{N} \exp\left[-\frac{1}{2}(v_a . A(s=1)_{a\,b} . v_b)\right]$$

$$\vec{v} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_0\tilde{x}_1, \tilde{x}_0^2, \tilde{x}_1^2\} \rightarrow A(s=1) = \left(\begin{array}{cc:ccc} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & \tilde{b}\, a_5 & 0 & 0 \\ 0 & 0 & 0 & a_3 & \frac{1}{2}(1-\tilde{b})a_5 \\ 0 & 0 & 0 & \frac{1}{2}(1-\tilde{b})a_5 & a_4 \end{array}\right)$$

$$\begin{aligned} a_0 &= \frac{3\lambda}{8}\Big(\frac{3}{4\tilde{\omega}_0^3} + \frac{3}{4\tilde{\omega}_1^3} + \frac{\tilde{\omega}_0\tilde{\omega}_1 + \tilde{\omega}_0^2 + \tilde{\omega}_1^2}{\tilde{\omega}_0^2\tilde{\omega}_1^2(\tilde{\omega}_0 + \tilde{\omega}_1)}\Big), \\ a_1 &= \tilde{\omega}_0 + \frac{1}{\tilde{\omega}_0}\Big(3a_3 + \frac{a_5}{2}\Big), \quad a_2 = \tilde{\omega}_1 + \frac{1}{\tilde{\omega}_1}\Big(3a_4 + \frac{a_5}{2}\Big), \\ a_3 &= \frac{\lambda}{4\tilde{\omega}_0}, \quad a_4 = \frac{\lambda}{4\tilde{\omega}_1}, \quad a_5 = \frac{3\lambda}{(\tilde{\omega}_1 + \tilde{\omega}_0)} \end{aligned}$$

Gaussian

Non-Gaussian

$$det(A(s=1)) > 0, 0 < \tilde{b} < 1 + \frac{1}{6}\sqrt{\frac{(\tilde{\omega}_0 + \tilde{\omega}_1)^2}{\tilde{\omega}_0\tilde{\omega}_1}}(\tilde{\omega}_1 > \tilde{\omega}_0)$$

Reference state: $$\psi^R(\tilde{x}_0, \tilde{x}_1) = \mathcal{N} \exp\left[-\frac{1}{2} v_a . A(s=0)_{a\,b} . v_b)\right]$$

$$A(s=0) = \begin{pmatrix} \tilde{\omega}_{ref} & 0 & 0 & 0 & 0 \\ 0 & \tilde{\omega}_{ref} & 0 & 0 & 0 \\ 0 & 0 & 3\lambda\,\tilde{\omega}_{ref} & 0 & 0 \\ 0 & 0 & 0 & \frac{\lambda\,\tilde{\omega}_{ref}}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda\,\tilde{\omega}_{ref}}{2} \end{pmatrix}.$$

Can be easily checked that in original basis it does not have any entanglement

Now we will have:

$$A(s=1) = U(s=1).A(s=0).U(s=1)^T$$

$$U(s) = \overleftarrow{\mathcal{P}}\exp(i\int_0^s ds\, Y^I(s)\, O_I(s)),$$

We take them to be GL(R) generators

For this case of two oscillator: N=2, d=2 $U(s)$ ***is a GL(5,R) unitary***

Given the block structures of A(s) we parametrize U(s) it in the following way,

$$U(s) = \begin{pmatrix} (A_1)_{3\times 3} & 0 \\ 0 & (A_3)_{2\times 2} \end{pmatrix}, A_1 = \begin{pmatrix} \exp(y_1(s)-\rho_1(s)) & 0 & 0 \\ 0 & \exp(y_1(s)+\rho_1(s)) & 0 \\ 0 & 0 & \exp(y_2(s)) \end{pmatrix},$$

$$A_2 = e^{y_3(s)}\begin{pmatrix} (\cos(\tau_3(s))\cosh(\rho_3(s)) - \sin(\theta_3(s))\sinh(\rho_3(s))) & (\cos(\theta_3(s))\sinh(\rho_3(s)) - \cosh(\rho_3(s))\sin(\tau_3(s))) \\ (\cosh(\rho_3(s))\sin(\tau_3(s)) + \cos(\theta_3(s))\sinh(\rho_3(s))) & (\cos(\tau_3(s))\cosh(\rho_3(s)) + \sin(\theta_3(s))\sinh(\rho_3(s))) \end{pmatrix}$$

this is nothing but $R^3 \times SL(2,R)$

Metric:

penalty factor proportional to λ

$$\begin{aligned} ds^2 &= G_{IJ}dY^I dY^J, \\ &= 2\Big(dy_1^2 + dy_2^2 + d\rho_1^2 + p[dy_3^2 + d\rho_3^2 \\ &+ \cosh(2\rho_3)\sinh^2(\rho_3)d\theta_3^2 + \cosh(2\rho_3)\cosh^2(\rho_3)d\tau_3^2 - \sinh^2(2\rho_3)d\theta_3 d\tau_3]\Big) \end{aligned}$$

Geodesic: Given this we know find the geodesics with the following boundary conditions

Initial condition: s=0: $U(s=0)=I \Rightarrow \{\rho_1(0)=\rho_3(0)=y_1(0)=y_2(0)=y_3(0)=0\}$

Final condition : s=1

$$A(s=1)=U(s=1).A(s=0).U(s=1)^T$$

The simplest solution is a “straight line geodesic”

$$\begin{aligned}
&y_1(s)=y_1(1)s,\ \rho_1(s)=\rho(1)\,s,\\
&y_3(s)=y_3(1)s,\ \rho_3(s)=\rho(1)s,\\
&\tau_3(s)=0,\theta_3(s)=\theta_0,\\
&y_2(s)=y_2(1)s
\end{aligned}$$

Jefferson-Myers, 1707.08570 [hep-th]

AB, A.Shekar, A. Sinha, arXiv: 1808.03105[hep-th]

Then, $\mathcal{C}_{\kappa=1}(U)=\int_0^1 \mathcal{F}(U,\dot{U})ds$, $\mathcal{F}(U,\dot{U})=\sum_I p_I|Y^I(s)|$

gets minimized on this geodesic and the minimum value corresponds to the circuit complexity

Next we analyze the results for arbitrary N and arbitrary “d”

Results

AB, A. Shekar, A. Sinha,
JHEP 1810 (2018) 140,
arXiv: 1808.03105[hep-th]

$$\mathcal{C}_{\kappa=1} = \frac{1}{2}\sum_{k=1}^{d-1}\left[\sum_{i_k=0}^{N-1}\left|\log\frac{\tilde{\omega}_{i_k}}{\tilde{\omega}_{ref}}\right| + \frac{3\lambda\,\delta}{2\,N}\sum_{i_k=0}^{N-1}\frac{1}{\tilde{\omega}^3_{i_k}} + \mathcal{O}(\lambda^2)\right.$$

Continuous Limit: $N \to \infty, \delta \to 0, N\,\delta \to \text{finite}$

Also we will rewrite everything in terms of Renormalized quantities

At 1-loop Renormalization order:

d=2: $$(m\,\delta)^2 = (m_R\,\delta)^2 - \frac{\lambda_R\,\delta^2}{2}\left[C_0 - 2\,C_1\log(m_R\,\delta) - C_2(m_R\,\delta)^2 + \frac{1}{32\,\pi}(m_R\,\delta)^2\log((m_R\,\delta)^2) + \mathcal{O}((m_R\,\delta)^4)\right]$$

$$C_0 = 0.28, C_1 = 0.08, C_2 = 0.02.$$

$d \geq 3$ $$(m\,\delta)^2 = (m_R\,\delta)^2 - \frac{\lambda_R\,\delta^{4-d}}{2}\left[C_0 - C_2(m_R\,\delta)^2 + \frac{1}{16\pi^2}(m_R\,\delta)^2\log((m_R\,\delta)^2)|_{d=4} + \mathcal{O}((m_R\,\delta)^4)\right]$$

C_i	d=3	d=3.99	d=4	d=5
C_0	0.21	0.15	0.15	0.11
C_2	0.06	0.03	0.03	0.015

extra log term
for d=4

also: $\lambda = \lambda_R$

Finally we will also expand in terms $m_R\,\delta$ and keep only leading terms

Finally we get,

d=2: $$\mathcal{C}^{(1)}_{\kappa=1} \approx \frac{V}{2\delta}\left[\log\left(\frac{1}{\tilde{\omega}_{ref}\delta}\right) + 2a_1 - \lambda_R \delta^2 \frac{C_0 - 2C_1\log(m_R\delta)}{2m_R\delta} c_1\right] + \cdots$$

$d \geq 3$ $$\mathcal{C}^{(1)}_{\kappa=1} \approx \frac{V}{2\delta^{d-1}}\left[\log\left(\frac{1}{\tilde{\omega}_{ref}\delta}\right) + 2a_{d-1} - \lambda_R \delta^{4-d} C_0 (c_2 + b_2 \log(m_R\delta)) + \frac{b_2}{2})\right]$$
$$+ \frac{\lambda_R}{16}\delta^{6-2d} V^{\frac{d-2}{d-1}} \left(f_1\{(m_R\,\delta)^{d-4}|_{d\neq 4} + \log(m_R\,\delta)|_{d=4}\} + f_0\right) + \cdots$$

vanishes for $d \geq 4$

fractional volume dependence

For large V
this is the leading
contribution at linear of order of λ_R
Perturbation theory
breaks down for $d > 4$

We can understand this break down of perturbation theory intititutively invoking RG picture, as for $d > 4$ Gaussian fixed points are stable compared to the Wilson-Fisher fixed point.

A Flow equation for Complexity

Now armed with all these we derive a flow equation for complexity:

Define: $$\widetilde{\Delta\mathcal{C}} \equiv (\mathcal{C}_{\kappa=1} - \mathcal{C}_{\kappa=1}|_{\lambda_R=0})\frac{\delta^{d-1}}{V}$$ change of complexity per unit degree of freedom

Scale Transformations: $$\lambda_R \to b^{d-4}\lambda'_R, \quad \delta \to b\,\delta,$$
$$b = 1 + db, \lambda_R = \lambda_R + d\lambda_R$$

Now in large volume "V" keeping only leading order term in small δ

$$\frac{d\widetilde{\Delta\mathcal{C}}}{db} = 2(4-d)\widetilde{\Delta\mathcal{C}} + \mathcal{O}(\lambda_R^2)$$

Similar to the flow equation for coupling: $$\frac{d\lambda_R}{db} = (4-d)\lambda_R + O(\lambda_R^2)$$

For $d < 4:$ *Wilson Fisher fixed point is favored in term of complexity*

$d > 4$ *Gaussian fixed point is favored in term of complexity*

This matches nicely with the intuitive idea of RG flow!!!

Outlook

- Our methods works for O(N) scalar model and we can classify the fixed points in terms of complexity
- Extend it for other non-trivial interacting theories for eg:Fermions, Gauge theories?
- How to understand fractional subleading volume dependence from holography ?
- Comparison with other methods ?
  (for eg: Fubini-Study method by Chapman, Heller, Marrochio, Pastawski (arXiv:1707.08582), Path Integral method: AB, Caputa, Kundu, Miyaji, Takyanagi arXiv: 1804.01999)
- Hamiltonian Complexity
  (To appear with A. Sinha and P.Nandi)
- Applications to quantum quench

Many more…..

Thank you