OPE for conformal defects and Holography

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Introduction

Introduction

Defects = Non-local objects in QFTs

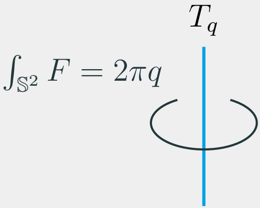
- Defined by
  - boundary conditions around them
  - coupled to low-dimentional system
- Many examples:

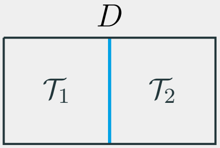
1-dim : Line operators (Wilson-'t Hooft loops)

2-dim : Surface operators

Codim-1 : Domain walls and boundaries

Codim-2 : Entangling surface for entanglement entropy



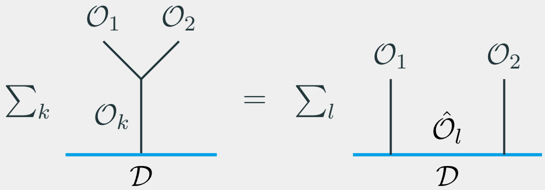


Why Defects?

We can probe the part of theory which is inaccesible without defects

- allow us to characterize the phase of theory
    - wilson loop in gauge theory
    - higher-form symmetry

In fixed point (**CFT**), constrain bulk CFT data in defect-CFT by conformal **bootstrap** [Liendo-Rastelli-van Rees 12]



Conformal defects

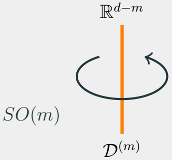
Especially, we consider particular class of defects:

Conformal defects (codimension-m)

defects preserving $\underbrace{SO(d-m+1,1)}_{\text{conformal sym. on defect}} \times \underbrace{SO(m)}_{\text{rotational sym. around defect}}$

- conformal defects allow defect local operators $\hat{\mathcal{O}}(x)$
- additional dynamical information appears

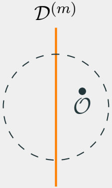
- coupling to defects, defect local operators,...



OPE for conformal defects

There are two types of OPE in Defect-CFT

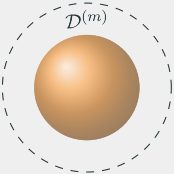
- Bulk-to-defect OPE : [Cardy 84, McAvity-Osborn 95]



$= \sum_n b^{(m)}_{\mathcal{O}\,\hat{\mathcal{O}}_n}$

$+$ (descendants)

- Defect OPE : [Berenstein-Corrado-Fischler-Maldacena 98, Gadde 16]



$= \sum_n c^{(m)}_{\mathcal{O}_n}$

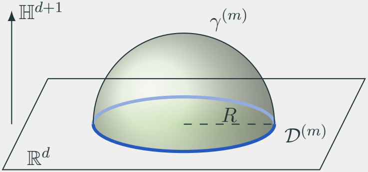
$+$ (descendants)

Several questions about DCFT

1. To what extent are we able to determine the structure of the **defect OPE** by conformal symmetry?
    - Decomposition by the irreducible representations

$$\mathcal{D}^{(m)} = \sum_{n\,\in\,\text{primaries}} \mathcal{B}^{(m)}[\mathcal{O}_n]$$

2. Can we probe the bulk AdS information by conformal defects on the boundary?



3. Is there any extention of CFT?
    - **Spinning defects**

Overview of our results

1. Give the integral representation of the **defect OPE blocks**

$$\mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \, \langle \mathcal{O}_n(x) \rangle_{\mathcal{D}^{(m)}} \tilde{\mathcal{O}}_n(x)$$

$\tilde{\mathcal{O}}$: shadow operator with $\tilde{\Delta} = d - \Delta$ for $\mathcal{O}$ with Δ

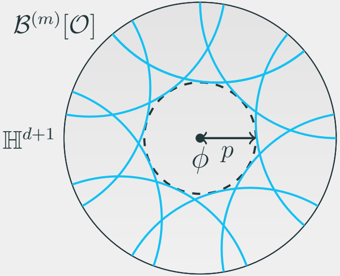
2. Reconstruct the AdS scalar field from the blocks

- $\hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}]$:

  The Radon transform of the **AdS scalar field** ϕ when $\mathcal{O}$ scalar

- Reproduce the (Euclidean) **HKLL formula** :

$$\phi(Y) = \phi(\hat{\phi}) = \int d^d x \, K(Y|x) \, \mathcal{O}(x)$$



Overview of our results

3. Study the kinematics and impliment of spinning defect in CFT:

(1) calculating several correlators of bulk and defect local operators

(2) exploring the OPE of spinning conformal defect

(3) considering the correlators of two spinning conformal defects

- deduced to thoese of **scalar** defects by recursion relation.

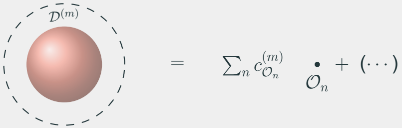
Defect OPE blocks

Defect OPE blocks

We expect the defect OPE of the form

$$\mathcal{D}^{(m)}(P_\alpha) = \langle \mathcal{D}^{(m)}(P_\alpha) \rangle \left[\sum_n c^{(m)}_{\mathcal{O}_n} R^{\Delta_n} \mathcal{O}_n(C) + (\text{descendants}) \right]$$

$(R$: radius, C : center vector, P_α : vector to fix the defect$)$



- The descendant terms are fixed by the primary $\mathcal{O}_n$ and the conformal symmetry,

$$\mathcal{D}^{(m)}(P_\alpha) = \sum_n \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n]$$

- The defect OPE block is in the irreducible rep. of $\mathcal{O}_n$:

$$\langle \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n]\, \mathcal{O}_n(X) \rangle = \langle \mathcal{D}^{(m)}(P_\alpha)\, \mathcal{O}_n(X) \rangle \,.$$

Projectors and shadows

- Want to characterize the defect OPE blocks by their irreps
- Spectral decomposition by the irreps of the conformal group:

$$\mathbf{1} = \sum_n |\mathcal{O}_n|$$

- $|\mathcal{O}_n|$: Projector onto the conformal multiplet of the primary $\mathcal{O}_n$ [Ferrara-Grillo-Parisi-Gatto 72, $\cdots$, Simmons-Duffin 12]

  For a scalar operator,

$$|\mathcal{O}_\Delta| = \frac{1}{\mathcal{N}_\Delta} \int D^d X \, |\mathcal{O}_\Delta(X)\rangle \langle \tilde{\mathcal{O}}_{d-\Delta}(X)|$$

  $\tilde{\mathcal{O}}_{d-\Delta}$: the shadow operator of $\mathcal{O}_\Delta$

  Similary, the projector for **spin l operator** is investigated

Integral representation of defect OPE blocks

- Expand the defect by the projectors:

$$\langle \mathcal{D}^{(m)}(P_\alpha)\,\cdots\rangle = \sum_{\Delta} \langle \mathcal{D}^{(m)}(P_\alpha)|\mathcal{O}_\Delta|\cdots\rangle + \text{(other irrep.)}$$

$$= \sum_{\Delta} \frac{1}{\mathcal{N}_\Delta} \int D^d X \,\langle \mathcal{D}^{(m)}(P_\alpha)\,\mathcal{O}_\Delta(X)\rangle\,\langle \tilde{\mathcal{O}}_{d-\Delta}(X)\,\cdots\rangle$$

$$+ \text{(other irrep.)}$$

- Can read off the block contribution:

The integral rep. of the defect OPE block

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_\Delta] = \frac{1}{\mathcal{N}_\Delta} \int D^d X\, \tilde{\mathcal{O}}_{d-\Delta}(X)\,\langle \mathcal{D}^{(m)}(P_\alpha)\,\mathcal{O}_\Delta(X)\rangle$$

Constraint equations

There are two types of equations the defect OPE block satisfies

The conformal Casimir equation

$$\left(L^2(P_\alpha) + \mathcal{C}_{\Delta,l}\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0$$

- $L^2(P_\alpha) \equiv \frac{1}{2} L_{AB}(P_\alpha)\, L^{AB}(P_\alpha)$: quadratic Casimir operator
- $\mathcal{C}_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$: the eigenvalue

"Trivial" equations for scalar primaries $\mathcal{O}_\Delta$

$$C_{ABCD}(P_\alpha)\, \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_\Delta] = 0$$

- $C_{ABCD}(P_\alpha) \equiv \frac{1}{2} L_{A[B}(P_\alpha) L_{CD]}(P_\alpha)$: $\frac{d(d^2-1)(d+2)}{24}$ quadratic operators

Moduli space of conformal defects

- The moduli space has a coset structure:

$$\mathcal{M}^{(d,m)} = \frac{SO(d+1,1)}{SO(m) \times SO(d+1-m,1)}$$

- The quadratic Casimir operator is the Laplacian on $\mathcal{M}^{(d,m)}$

$$-L^2(P_\alpha) = \Box_{\mathcal{M}^{(d,m)}}$$

- The defect OPE block is a **scalar field on $\mathcal{M}^{(d,m)}$**

Klein-Gordon equation on $\mathcal{M}^{(d,m)}$

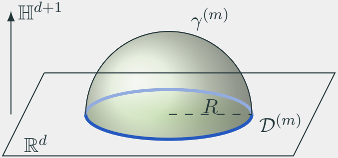
$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0 \ , \quad M^2 = \mathcal{C}_{\Delta,l}$$

Reconstruction of AdS scalar fields

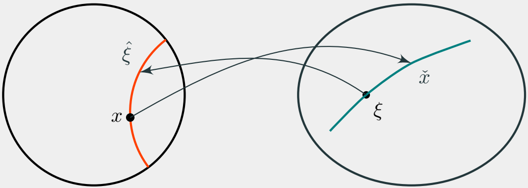
Conformal defects and submanifolds in AdS

- Associated to a given defect $\mathcal{D}^{(m)}$ is a **unique** submanifold $\gamma^{(m)}$ in AdS s.t. $\partial\gamma^{(m)} = \mathcal{D}^{(m)}$
- Their moduli spaces are equivalent:

$$\mathcal{M}^{(d,m)} = \frac{\text{Isom}(\text{AdS}_{d+1})}{\text{Stab}(\gamma^{(m)} \in \text{AdS}_{d+1})}$$



- there is a map called **Radon transform** between Euclidean AdS ($= \mathbb{H}^{d+1}$) and $\mathcal{M}^{(d.m)}$



Radon transform

From $\mathbb{H}^{d+1}$ to $\mathcal{M}^{(d,m)}$:

$$\hat{\phi}(\xi) = \int_{x\in\xi} d\nu(x)\,\phi(x)$$

- ξ : a codim-m submanifold in $\mathbb{H}^{d+1}$
- $\phi(x)$: a function on $\mathbb{H}^{d+1}$

From $\mathcal{M}^{(d,m)}$ to $\mathbb{H}^{d+1}$:

$$\check{f}(x) = \int_{\xi\in\check{x}} d\mu(\xi)\, f(\xi)$$

- ξ : a codim-m submanifold through x
- $f(\xi)$: a function on $\mathcal{M}^{(d,m)}$

Intertwining property

$$\left(\Box_{\mathbb{H}^{d+1}} - M^2\right)\,\phi = 0 \quad \Leftrightarrow \quad \left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right)\,\hat{\phi} = 0$$

Reconstruct an AdS field from DOPE block

- we can identify defect OPE block as the Radon transform of **an AdS scalar field ϕ**
- Inversion formula for the Radon transform allows us to reconstruct ϕ from **defect OPE block** [Helgason 10]
- Equivalent to the bulk reconstruction formula

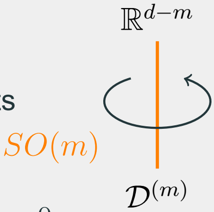
$$\phi(Y) = \int D^d X \, K_\Delta(Y|X) \, \mathcal{O}_\Delta(X)$$

with the Euclidean version of the HKLL kernel [Hamilton-Kabat-Lifschytz-Lowe 06]

$$(\Box_{\mathbb{H}^{d+1}} - M^2) \, K_\Delta(Y|X) = 0 \,, \quad M^2 = \Delta(d - \Delta)$$

Spinning conformal defect

Spinning defects and recursion relation



- Conformal defects can carry spin under $SO(m)$.
- We adapt index-free notation for spinning defects introducing auxiliary transverse vector $\hat{W}$,

$$\mathcal{D}^{(m)}_s(\hat{W}) \equiv \mathcal{D}^{(m)}_{I_1 \cdots I_s} \hat{W}^{I_1} \cdots \hat{W}^{I_s}, \quad \hat{W} \circ \hat{W} = 0$$

- In the same way as local operator case, we find **recursion relation** for one-point function,

$$\langle \mathcal{D}^{(m)}_s \, \mathcal{O}_\Delta(X) \rangle = \mathfrak{D}_{s-s_0}(\hat{W}) \, \langle \mathcal{D}^{(m)}_{s_0} \, \mathcal{O}_\Delta(X) \rangle$$

$\mathfrak{D}_{s-s_0}(\hat{W})$: $s - s_0$-th order differential operator acting on P_α

Application of Recursion relation

- Spinning defect OPE blocks

$$\mathcal{D}_s^{(m)}(\hat{W}) = \sum_n \mathcal{B}_s^{(m)}[\mathcal{O}_n, \hat{W}]$$

$$\Rightarrow \mathcal{B}_s^{(m)}[\mathcal{O}_{\Delta,l}, \hat{W}] = \mathfrak{D}_{s-s_0}(\hat{W})\, \mathcal{B}_{s_0}^{(m)}[\mathcal{O}_{\Delta,l}]$$

- Two-point function of spinnning defects

$$\langle \mathcal{D}_{s_1}^{(m_1)}(\hat{W}_1)\, \mathcal{D}_{s_2}^{(m_2)}(\hat{W}_2) \rangle = \sum_n \langle \mathcal{B}_{s_1}^{(m_1)}[\mathcal{O}_n, \hat{W}_1]\, \mathcal{B}_{s_2}^{(m_2)}[\mathcal{O}_n, \hat{W}_2] \rangle$$

$$\Rightarrow \langle \mathcal{D}_{s_1}^{(m_1)}(\hat{W}_1)\, \mathcal{D}_{s_2}^{(m_2)}(\hat{W}_2) \rangle|_{\text{spin-}l} = \mathfrak{D}_{s_1}(\hat{W}_1)\, \mathfrak{D}_{s_2}(\hat{W}_2)\, \langle \mathcal{D}^{(m_1)}\, \mathcal{D}^{(m_2)} \rangle|_{\text{spin-}l}$$

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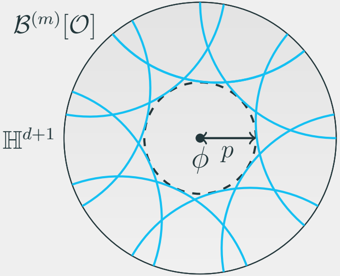

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Future direction

Integrable system

The Casimir equation for the defect conformal block is shown to be equivalent to the Schrödinger equation of the Calogero-Sutherland model

[Isachenkov-Liendo-Linke-Schomerus 18,...]

- relation with our formalism?

Holographic dual

Can extend the construction to higher spin fields in AdS?

- No known Radon transform beyond a scalar field in AdS
- Spinning defects to incorporate spins