Nonlocality and Entropy

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Non-locality

Recent work of P. Phillips and the author (see Philip's talk and Comm. Math. Phys. xxx,2019 Colloq. RMP, xxx 2019) *non-local EM*

- the strange metal
- holography and the symmetry breaking mechanism

$$L = D_{\gamma,A}\phi(D_{\gamma,A}\phi)^* - m^2\phi^*\phi - F_\gamma^{\mu\nu}F_{\mu\nu_\gamma}, \tag{1}$$

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$$L = D_{\gamma,A}\phi(D_{\gamma,A}\phi)^* - m^2\phi^*\phi - F_\gamma^{\mu\nu}F_{\mu\nu_\gamma}, \qquad (1)$$

where $D_{\gamma,A}\phi = \left(\partial_\mu + ie\Box^{(1-\gamma)/2}A^\mu\right)\Box^{(1-\gamma)/2}\phi$ and $F_\gamma^{\mu\nu} = \partial_\mu\Box^{(\gamma-1)/2}A_\nu - \partial_\nu\Box^{(\gamma-1)/2}A_\mu$ and can be interpreted as the commutator $[D_A, D_{\gamma,A}]$.

Goal

- Entropy non-locality
- Hilbert space non-locality
- Their connection in examples
- Holography?

Entropy

$\mathbb{R}^{d-1} = \{t = const\}$, $\mathbb{R}^{d-1} = A \cup \bar{A}$ and$\Sigma = \partial A$. ρ_A= reduced density matrix

$$S = -\text{Tr}\rho_A \log \rho_A. \tag{2}$$

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Eq. 2 is hard to compute with. Better use geometric entropy a la Callan Wilczek and use the replica trick

$$S_N = -(\partial_N - 1) \log \text{Tr}\rho_A^N \tag{3}$$

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For a Gaussian theory on $\mathbb{R}^d$, get flat cone, C_δ with deficit angle $\delta = 2\pi(1 - N)$.

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For a Gaussian theory on $\mathbb{R}^d$, get flat cone, C_δ with deficit angle $\delta = 2\pi(1 - N)$.
The quantity of interest is then (not-normalized)

$$S_\delta = -(2\pi\partial_\delta + 1)\log Z_\delta \tag{4}$$

and the limit to obtain the entanglement entropy is $\delta \to 0$.

Entropy and locality

In fact, *local* quantum field theories (with a UV fixed point) EE S scales as the area of the entangling surface: for a local d dimensional field theory the leading UV divergence

$$S \sim \kappa_{d-2} \left(\frac{1}{\epsilon}\right)^{d-2} + \ldots \tag{5}$$

where $1/\epsilon$ is a characteristic length scale of the entangling surface and κ_{d-2} is a function defined on the entangling surface (cf Casini and Huerta arxiv:0905.2562).

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So we can formulate

Criterion of nonlocality- Entropy method

We say that a QFT is Entropy non-local if the ground state entropy does NOT satisfy an area Law

Holographic Entanglement Entropy and non-locality

In Li and Takayanagi (arxiv:1010.3700) holographic theory for flat space might look like on the sphere at ∞.

$$S_A \propto Volume \tag{6}$$

proportional to the volume, NOT to the Area of A. The boundary theory must be non-local.

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Given by

$$S_{boundary} = \int d\Omega_d \; \phi f(-\Delta)\phi \tag{7}$$

with f of the form $f(x) = e^{x^\gamma}$, and NOT of the form $f(x) = x^\gamma$ (which obeys an area law)

Entanglement Entropy and non-locality

In Arxiv:1311.1643, by Shiba and Takayanagi *ground state entanglement entropy* (EE) for a slightly different theory

$$H = \int d^{d-1}x \left(\frac{1}{2}(\partial_t \phi)^2 + B_0 \phi e^{A_0(-\Delta)^\gamma} \phi \right) \tag{8}$$

They compactify the space $\mathbb{R}^{d-2}$ into a torus with radius Ra (a the lattice constant and R is the size of torus in the lattice space.) With $\Omega = \left\{-\frac{La}{2} \leq x_1 \leq \frac{La}{2}, x_i \in [0, Ra], \text{ for } i \geq 2\right\}$ they show

$$S_\Omega = \begin{cases} C_1 ALR^{d-2} & L << A\text{(volume law)} \\ C2A^2R^{d-2} & L >> A\text{(area law)} \end{cases} \tag{9}$$

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Using c-MERA they propose that this should be holographic dual to

$$ds^2 = A_0^2 \frac{dz^2}{z^{2(2\gamma+1)}} + \frac{1}{z^2} \sum_{i=1}^{d-1} dx_i^2 + g_{tt} dt^2 \tag{10}$$

The fake non-local theory

Back to Li-Takayanagi

$$Z[J] = \int \mathcal{D}\phi \, e^{i \int d^d x \, [\frac{1}{2}\phi(-\Delta + m^2)^\gamma \phi + J\phi]}$$

We calculate

$$Z[J] = \frac{1}{\det(-\Delta)^\gamma} \, e^{-i \int_{\mathbb{R}^{2n}} d^d x d^d y \, J(x) G_\gamma(x-y) J(y)} = \frac{1}{\det(-\Delta)^\gamma} e^{iW(J)} \tag{11}$$

where

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where

$$W(J) = -\frac{1}{2} \int_{\mathbb{R}^{2n}} d^d x d^d y \, J(x) G_\gamma(x-y) J(y)$$

and $G_\gamma(x-y)$ is the *fractional* propagator

$$(-\Delta + m^2)^\gamma G_\gamma(x-y) = \delta^d(x-y)$$

Local or nonlocal?

Can see this theory is **not truly non-local**: after the field redefinition

$$\psi = (-\Delta + m^2)^{\frac{1-\gamma}{2}}\phi$$

this is consistent with the Area law result of Li-Takayanagi

A Truely non-local theory

New model of non-local theory, which we dub the *true non-local theory* (in contrast with $f(x) = x^{2\gamma} + m^2$)

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one has

$$Z_\gamma[J] = \frac{1}{\det\left((-\Delta)^\gamma + m^2\right)} e^{iW_\gamma(J)} \qquad (12)$$

where

$$W_\gamma(J) = -\frac{1}{2}\int_{\mathbb{R}^{2n}} d^d x d^d y \, J(x) D_\gamma(x-y) J(y)$$

with

$$\left((-\Delta)^\gamma + m^2\right) D_\gamma(x-y) = \delta^d(x-y)$$

True non-local theory continued

An easy expansion (for $m \neq 0$) reveals

$$\frac{1}{2}J(x)D_\gamma(x-y)J(y) = \frac{1}{2}\sum_{k=0}^{\infty} m^{2k} J(x)G_{\gamma k}(x-y)J(y) \tag{13}$$

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where

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at least up to (possibvly) a finite dimensional vector space of the Hilbert space The propagators $G_{\gamma k}(x-y)$ have a *local behavior* as seen previously

First test of non-locality

Criterion of nonlocality:

Criterion of nonlocality- Hilbert space method

We say that a QFT is truly non-local if there is no transformation of Hilbert spaces (even possibly defined away from a finite dimensional vector space) which manifest the theory as a finite sum of local theories

Checking the Entropy of the Fake non-local theory

The calculation of Li and Takayanagi
if effective action

$$F = \int_{\epsilon^{2\gamma}}^{\infty} \frac{ds}{s} \mathrm{Tr} e^{s\Delta^{\gamma}}. \quad (14)$$

$A = d - 1$-dim. slab of length L. Then, the cutoff scale is given by $\epsilon = 1/L$.

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The get Area Law

$$\begin{aligned} S_A &\sim \kappa_{d-2} \int_{L^{-2\gamma}}^{\infty} \frac{ds}{s} s^{-\frac{d-2}{2\gamma}} \\ &\sim \kappa_{d-2} L^{d-2}. \end{aligned} \tag{15}$$

Entropy of the true non local theory

By contrast,

$$I = \int_{\mathbb{R}^d} d^d x \phi(-\Delta^\gamma + m^2)\phi. \tag{16}$$

its entropy does **not** obey an *area law*, if $d > 2$.

The leading order divergence: non-area law

For small ϵ (and $d > 2$) the leading order divergence is (for $m \neq 0$) [1]

$$S = \kappa_{d-2} \int_{\epsilon^{2\gamma}}^{\infty} \frac{ds}{s} s^{-\frac{d-2+2\gamma}{2\gamma}} e^{-sm^2}$$

$$\sim \kappa_{d-2} m^{\frac{d-2+2\gamma}{\gamma}} \Gamma\left(-\frac{2\gamma+d-2}{2\gamma}, m^2\epsilon^{2\gamma}\right) \qquad (17)$$

$$\sim \kappa_{d-2} \left(\frac{1}{\epsilon}\right)^{d-2+2\gamma} + \ldots$$

keep only terms with ϵ for small ϵ, corresponding to the UV limit. One can now immediately notice the volume law when $\gamma = 1/2$. Furthermore, the non-local theory **never follows an area law** since $0 < \gamma < 1$.

[1] Recall the *incomplete Gamma function* $\Gamma(s, x) := \int_x^{+\infty} dt\, t^{s-1} e^{-t}$ is asymptotic to $-\frac{1}{s} x^s$ as $x \to 0$ for $Re(s) < 0$

Conclusions

We have discussed two notions of non locality

- The Hilbert space one
- The entanglement entropy one

We conjecture them to be equivalent under reasonable conditions (e.g. theories admitting a UV fixed point) We therefore put forth that theories like our fractional EM theory subject to a Higgs mechanism gives rise to truly non-local theory (in both the Hilbert space sense and Entropy sense) which could appear as a non-local holographic dual.
In current work with our student Cunwei Fan we are studying transition from Area to non-Area law for Lovelock theories near the vacuum AdS