Spectral Form Factor as an OTOC Averaged over the Heisenberg Group

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Spectral Form Factor

- The spectral form factor (SFF) is

$$g_2(\beta, t) \equiv \frac{R_2(\beta, t)}{R_2(\beta, 0)}, \tag{1}$$

where

$$R_2(\beta, t) \equiv |\mathrm{Tr}\big(Z(\beta, t)\big)|^2 \tag{2}$$

is the unnormalized two-point SFF, β is the inverse temperature, H is the Hamiltonian of the system, and

$$Z(\beta, t) \equiv \exp(-\beta H - iHt). \tag{3}$$

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- The Sachdev-Ye-Kitaev (SYK) model provides the consistent universal dynamical form with the random matrix theory.
- The issue of information loss can be probed by the late time study in the SFF from the violation of bound.

Reference of the Spectral Form Factor

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- J. S. Cotler *et al.*, "Black Holes and Random Matrices," JHEP **1705**, 118 (2017) Erratum: [JHEP **1809**, 002 (2018)] [arXiv:1611.04650 [hep-th]].
- O. Bohigas, M. J. Giannoni and C. Schmit, "Characterization of chaotic quantum spectra and universality of level fluctuation laws," Phys. Rev. Lett. **52**, 1 (1984).

OTOC

- The out-of-time ordered correlation function (OTOC) is defined by the square of commutator of two operators in a bosonic system

$$C_4(t) \equiv \frac{\mathrm{Tr}\big(\rho W(t)V(0)W(t)V(0)\big)}{\mathrm{Tr}\rho}, \tag{4}$$

where $\rho \equiv \exp(-\beta H)$.

Reference of the OTOC

- A. I. Larkin and Yu. N. Ovchinnikov, "Quasiclassical Method in the Theory of Superconductivity," JETP **28**, 1200 (1969).

Regularization

- It has been shown that the unregularized OTOC does not share the universal Lyapunov exponent with the regularized OTOC due to the sensitivity of the infrared regulator. In the SYK model at the large-q limit, the universal Lyapunov exponent can be captured by the regularized OTOC. Hence the regularized OTOC should be better for the universal meaning. The regularized OTOC is

$$C_{\mathrm{r}4}(t) \equiv \frac{\mathrm{Tr}\left(\rho^{1/4} W(t) \rho^{1/4} V(0) \rho^{1/4} W(t) \rho^{1/4} V(0)\right)}{\mathrm{Tr}\rho}. \qquad (5)$$

Reference of the Regularization

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Reference of the Regularization

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Reference of the Observation

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SFF and OTOC in Qubit Models

- Consider a quantum system in an L-dimensional Hilbert space. Recall the average over $L \times L$ unitary matrices with the Haar measure is

$$\int dA\, A^j_k A^{\dagger\, l}_m = \frac{1}{L}\delta^j_m \delta^l_k\,. \tag{6}$$

  The integral over A is over all possible unitary operators on the Hilbert space.

- In terms of the regularized two-point OTOC $O(t) \equiv \mathrm{Tr}\big(A(0)\sqrt{\rho}A^\dagger(t)\sqrt{\rho}\big)/L$, it is clear that

$$\begin{aligned}\int dA\; O(t) &= \frac{1}{L}\int dA\;\mathrm{Tr}(A\sqrt{\rho}e^{-iHt}A^\dagger e^{iHt}\sqrt{\rho}) \\ &= R_2(\beta/2, t)\,.\end{aligned} \tag{7}$$

Heisenberg Group Averaging

- A general element of the Heisenberg group is specified by the variables, q_1, q_2, as follows $U(q_1, q_2) \equiv \exp(iq_1 X + iq_2 P)$. By direct computation, we find

$$\int_{-\infty}^{\infty} \frac{dq_1}{2\pi} \int_{-\infty}^{\infty} dq_2 \, \langle x_1 | U(q_1, q_2) | x_2 \rangle \langle y_1 | U^\dagger(q_1, q_2) | y_2 \rangle = \delta(x_2 - y_1)\, \delta(x_1 - y_2) \, . \quad (8)$$

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- This already implies

$$\begin{aligned} & \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} \frac{dq_2}{2\pi} \int_{-\infty}^{\infty} dx \, O(x, t, q_1, q_2) \\ = \; & \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_1 \, \langle x_1 | e^{-iHt} | x_1 \rangle \langle x | e^{iHt} | x \rangle \, . \end{aligned} \tag{9}$$

Non-Interacting Scalar Field Theory

- Rewrite this computation in terms of oscillators since this generalizes easily to non-interacting scalar field theory, which is an assembly of non-interacting oscillators. Using

$$a = \frac{(P - i\omega X)}{\sqrt{2\omega}}, \qquad a^\dagger = \frac{(P + i\omega X)}{\sqrt{2\omega}}, \tag{10}$$

the unitary operators that we have considered are given by

$$U(q_1, q_2) = e^{a\left(iq_2\sqrt{\frac{\omega}{2}} - \frac{q_1}{\sqrt{2\omega}}\right)} e^{a^\dagger\left(iq_2\sqrt{\frac{\omega}{2}} + \frac{q_1}{\sqrt{2\omega}}\right)} e^{\frac{q_1^2}{4\omega} + \frac{q_2^2\omega}{4}} .$$

Non-Interacting Scalar Field Theory

Now consider a non-interacting scalar field theory, in a box (with the periodic boundary condition), so that momenta $\vec{k}$ are discrete, with an oscillator for every $\vec{k}$.

Non-Interacting Scalar Field Theory

Now consider a non-interacting scalar field theory, in a box (with the periodic boundary condition), so that momenta $\vec{k}$ are discrete, with an oscillator for every $\vec{k}$. The Hamiltonian is

$$H_{\mathrm{NS}} = \frac{1}{V}\sum_{\vec{k}} \frac{1}{2}\tilde{a}^{\dagger}(\vec{k})\tilde{a}(\vec{k}), \tag{11}$$

where V is the volume of the box. The $\tilde{a}^{\dagger}$ and $\tilde{a}$ are the usual creation and annihilation operators in the box, and they satisfy the commutation relation $[\tilde{a}(\vec{k}_1), \tilde{a}^{\dagger}(\vec{k}_2)] = 2V\omega_{\vec{k}_1}\delta_{\vec{k}_1\vec{k}_2}$, where $\omega^2_{\vec{k}_1} \equiv |\vec{k}_1|^2 + m^2$ with m the mass of the non-interacting scalar field. Hence we can perform the field redefinition $\tilde{a}(\vec{k}) \equiv \sqrt{2V\omega(\vec{k})}a(\vec{k})$ and apply the result of the harmonic oscillator to the non-interacting scalar field theory.

Coherent State

- We consider the exactly solvable model from the two-photon non-degenerate Jaynes-Cummings (JC) model with the rotating wave approximation, which ignores the oscillating fast term.

Coherent State

- The effective Hamiltonian is

$$H_{\mathrm{JC}} \equiv N_1 + N_2 + M, \tag{12}$$

where

$$N_j = \omega_j \left(a_j^\dagger a_j + \frac{(\sigma_z + 1)}{2} \right) \tag{13}$$

and

$$M \equiv \frac{\Delta(\sigma_z + 1)}{2} + g_a(a_1 a_2 \sigma^+ + a_1^\dagger a_2^\dagger \sigma^-), \tag{14}$$

where

$$\sigma^+ \equiv \frac{\sigma_x + i\sigma_y}{2}, \qquad \sigma^- \equiv \frac{\sigma_x - i\sigma_y}{2}. \tag{15}$$

Coherent State

- The coherent states that we use are:

$$a_1|\alpha_1\alpha_2\rangle = \alpha_1|\alpha_1\alpha_2\rangle, \qquad a_2|\alpha_1\alpha_2\rangle = \alpha_2|\alpha_1\alpha_2\rangle,$$

  and

$$|\alpha_1\alpha_2\rangle = \exp\big(-(|\alpha_1|^2+|\alpha_2|^2)/2\big)\exp\big(\alpha_1 a_1^\dagger + \alpha_2 a_2^\dagger\big)|0,0\rangle.$$

  Completeness of the coherent states is

$$\int \frac{d^2\alpha_1}{\pi}\int\frac{d^2\alpha_2}{\pi}\,|\alpha_1\alpha_2\rangle\langle\alpha_1\alpha_2| = 1\,. \qquad (16)$$

Coherent State

- In terms of the unitary operator

$$U(q_1, q_2, r_1, r_2) = \exp(iq_1 X_1 + iq_2 P_1 + ir_1 X_2 + ir_2 P_2), \quad (17)$$

we compute the regularized two-point OTOC (repeated indices a, b are summed over 1,2)

$$\begin{aligned} & C(t) \\ = \;& \langle \alpha_1 \alpha_2 | U(q_1, q_2, r_1, r_2) [e^{-\beta H_{\mathrm{JC}}/2 - iH_{\mathrm{JC}} t}]_{aa} U(q_1, q_2, r_1, r_2)^\dagger \\ & \times [e^{-\beta H_{\mathrm{JC}}/2 + iH_{\mathrm{JC}} t}]_{bb} | \alpha_1 \alpha_2 \rangle \,, \end{aligned} \quad (18)$$

where $[\cdots]_{aa}$ is the matrix element of the row-a and the column-a with the repeated summation.

Coherent State

- Direct computation gives

$$
\begin{aligned}
&\langle \alpha_1\alpha_2 | U(q_1,q_2,r_1,r_2) | \gamma_1^1\gamma_2^1\rangle \\
= \;& e^{\bar{\alpha}_1\left(iq_2\sqrt{\frac{\omega_1}{2}}+\frac{q_1}{\sqrt{2\omega_1}}\right)} e^{\bar{\alpha}_2\left(ir_2\sqrt{\frac{\omega_2}{2}}+\frac{r_1}{\sqrt{2\omega_2}}\right)} \\
&\times e^{\gamma_1^1\left(iq_2\sqrt{\frac{\omega_1}{2}}-\frac{q_1}{\sqrt{2\omega_1}}\right)+\gamma_2^1\left(ir_2\sqrt{\frac{\omega_2}{2}}-\frac{r_1}{\sqrt{2\omega_2}}\right)} \\
&\times e^{-\frac{|\alpha_1|^2+|\alpha_2|^2+|\gamma_1^1|^2+|\gamma_2^1|^2}{2}+\bar{\alpha}_1\gamma_1^1+\bar{\alpha}_2\gamma_2^1} e^{-\frac{q_1^2}{4\omega_1}-\frac{q_2^2\omega_1}{4}-\frac{r_1^2}{4\omega_2}-\frac{r_2^2\omega_2}{4}} .
\end{aligned}
$$

This matrix element is common for any two-particle problem - it is the coherent state expectation value of an element of the two-particle Heisenberg group. The integrations that we need to perform over coherent state parameters are Gaussian integrals, which is a nice simplification that will always be present.

Coherent State

In general, we will not be able to carry things out exactly. Nevertheless, given that t is a large parameter, the final integration naturally lends themselves to saddle point evaluations.

Large-N Matrix QM

Concretely, consider the model

$$H_{\mathrm{QMN}} = \frac{P^j P^j}{2} + \mu^2 \frac{X^j X^j}{2} + g\frac{(X^j X^j)^2}{4}, \tag{19}$$

where $j = 1, 2, \cdots, N$, and g is the coupling constant.

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where $j = 1, 2, \cdots, N$, and g is the coupling constant. Using the simplifications of the large-N, we replace this Hamiltonian with the approximate form (σ is a constant.)

$$H_{\mathrm{QMNM}} = \frac{P^j P^j}{2} + \mu^2 \frac{X^j X^j}{2} + \lambda\sigma \frac{X^j X^j}{2}. \tag{20}$$

The 't Hooft coupling constant $\lambda \equiv gN$ is fixed as we scale $N \to \infty$, and we determine $\sigma = \sum_{j=1}^{N} \langle X^j X^j \rangle / N$ from the two-point function. The large-N theory is harmonic oscillators but now with a modified frequency.

Large-N Matrix QM

The SFF is

$$g_2(\beta, t) = \left(\frac{1 + e^{-2\sqrt{\mu^2+\lambda\sigma}\beta} - 2e^{-\sqrt{\mu^2+\lambda\sigma}\beta}}{1 + e^{-2\sqrt{\mu^2+\lambda\sigma}\beta} - 2\cos(\sqrt{\mu^2+\lambda\sigma}\,t)e^{-\sqrt{\mu^2+\lambda\sigma}\beta}} \right)^N .$$

Large-N Matrix QM

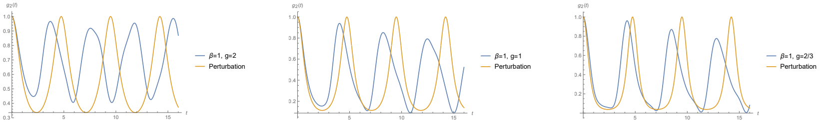

Figure: We fix the inverse temperature $\beta = 1$ while choosing the 't Hooft coupling constant $\lambda = gN = 2$. The lattice sizes are 8 in $N=1$ and 4 in $N=2$, 3. The numbers of lattice points are 128 in $N=1$ and 32 in $N=2$, 3. We compute the two-point spectral form factor $g_2(t)$ from 16 low-lying eigenenergy modes for $N=1$, 2, and 3 in the left, middle, and right figures respectively. The numerical solution in $N=3$ matches the large-N perturbation quantitatively.

Conclusion

- We link the spectral statistics to the OTOC through the Heisenberg group averaging in bosonic QM and QFT.

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- We link the spectral statistics to the OTOC through the Heisenberg group averaging in bosonic QM and QFT.
- The late time limit is also the classical limit. Therefore, we apply our study to coherent state, which is a quantum state closest to a classical regime, and large-N matrix QM. It is useful for understanding the late time behavior of the SFF.
- Because the uncertainty principle forbids the infinitesimal perturbation, the OTOC cannot have the late time chaos. The link between the spectral statistics and OTOC gives the late time chaos to the OTOC.