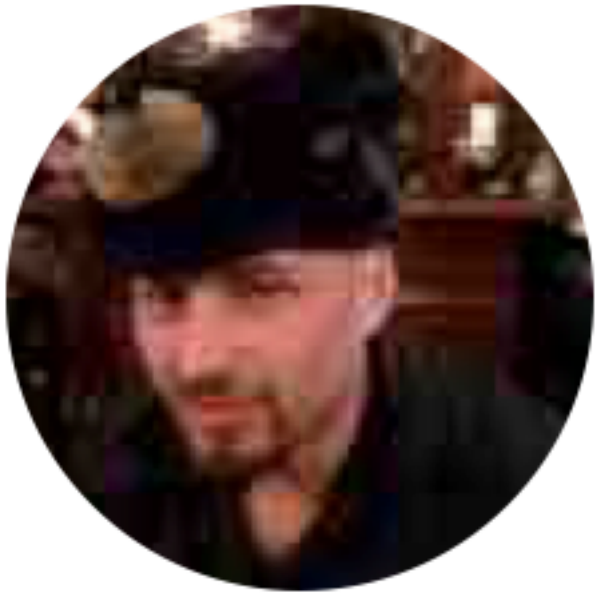
Fractional Electromagnetism from Noether's Second Theorem

Thanks to: NSF, EFRC (DOE)

Gabriele La Nave



Rev. Mod. Phys. 2019 (arXiv:1904.01023)
CIMP 2019
Adv. Th. Math. Phys. 2019

Kridsangaphong Limtragool

standard electricity and magnetism

$U(1)$ ⟷ $\psi' = e^{iq\Lambda}\psi$ $\quad [q\Lambda] = 0$

$qA \to qA - q\partial_\mu \Lambda$ → $D - iqA$

$[qA] = 1$

$[A] = 1$

$$S = \int d^dx (J_\mu A^\mu + \cdots)$$

$$S \to S + \int d^dx J_\mu \partial\Lambda$$

fixes dimension of current

$$[d^dx JA] = 0$$

$$\partial_\mu J^\mu = 0$$

$$[J] = d - 1$$

Noether's Thm. I

current conservation

Are there exceptions?

Superconductivity for Particular Theorists*)

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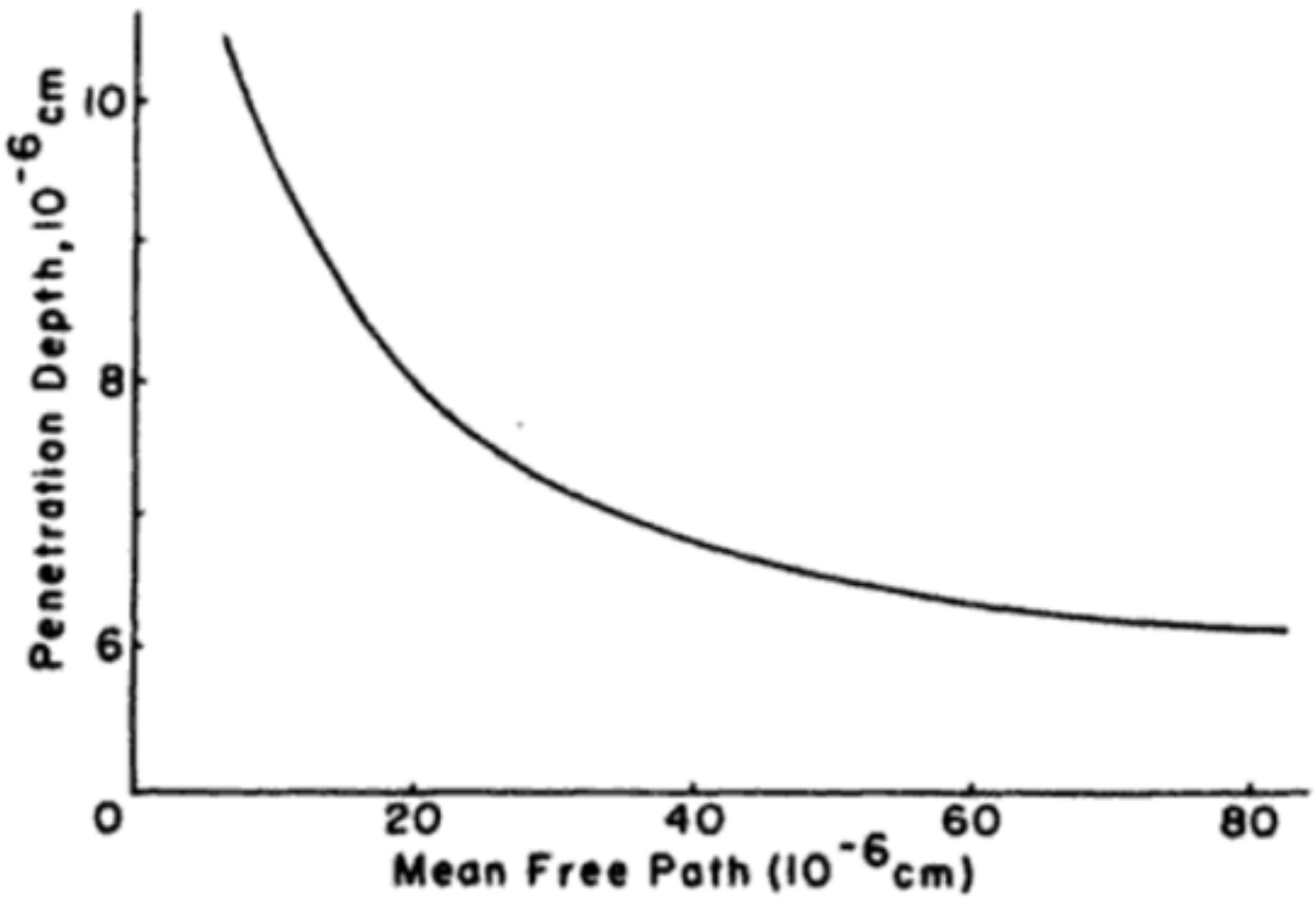
No one did more than Nambu to bring the idea of spontaneously broken symmetries to the attention of elementary particle physicists. And, as he acknowledged in his ground-breaking 1960 article "Axial Current Conservation in Weak Interactions", Nambu was guided in this work by an analogy with the theory of superconductivity, to which Nambu himself had made important contributions. It therefore seems appropriate to honor Nambu on his birthday with a little pedagogical essay on superconductivity, whose inspiration comes from experience with broken symmetries in particle theory. I doubt if anything in this article will be new to the experts, least of all to Nambu, but perhaps it may help others, who like myself are more at home at high energy than at low temperature, to appreciate the lessons of superconductivity.

Pippard's problem



$$J_s \neq \frac{-c}{4\pi\lambda^2} A$$

London Eq.

failure of local London relations

Superconductivity ala Weinberg

$$U(1) \to \mathbb{Z}_2$$

$$A_\mu - \partial_\mu \phi = 0$$

$U(1)/\mathbb{Z}_2$

$$\nabla \phi - A = 0$$

stable equilibrium

around minimum

$$L_m = L_{m0} - \frac{1}{2} \int C^{\mu\nu}(\mathbf{x}, \mathbf{x}')(\mathbf{A}_\mu(\mathbf{x}) - \partial_\mu \phi(\mathbf{x}))$$
$$\times (A_\nu(\mathbf{x}') - \partial_\nu \phi(\mathbf{x}'))\mathbf{d^3x'd^3x} + \cdots$$

Pippard Current

$$J_\mu(\mathbf{x}) = \frac{\delta \mathbf{L_m}}{\delta \mathbf{A}_\mu} = -\int \mathbf{C}^{\mu\nu}(\mathbf{x},\mathbf{x}')(\mathbf{A}_\nu(\mathbf{x}') - \partial_\nu \phi(\mathbf{x}'))\mathbf{d^3x}'$$

Pippard kernel

$$J_s = -\frac{3}{4\pi c \xi_0 \lambda} \int \frac{(\vec{r}-\vec{r}')((\vec{r}-\vec{r}')\cdot \vec{A}(\vec{r}'))e^{-(\vec{r}-\vec{r}')/\xi(\ell)}}{(\vec{r}-\vec{r}')^4} d^3\vec{r}'$$

non-local

magnetic energies

$$C\xi^3 L^3 A^2 = C\xi^3 L^5 B^2 = C\xi^3 L^2 (\underbrace{L^3 B^2}_{\text{expulsion energy}})$$

Meissner Effect $\quad C\xi^3 L^2 \gg 1$

Units of Current

$$J_\mu(\mathbf{x}) = \frac{\delta \mathbf{L_m}}{\delta \mathbf{A}_\mu} = -\int \mathbf{C}^{\mu\nu}(\mathbf{x},\mathbf{x}')(\mathbf{A}_\nu(\mathbf{x}') - \partial_\nu \phi(\mathbf{x}'))\mathbf{d^3x}'$$

$$[J] = d - d_C - d_A$$

anomalous dimension

Standard Result

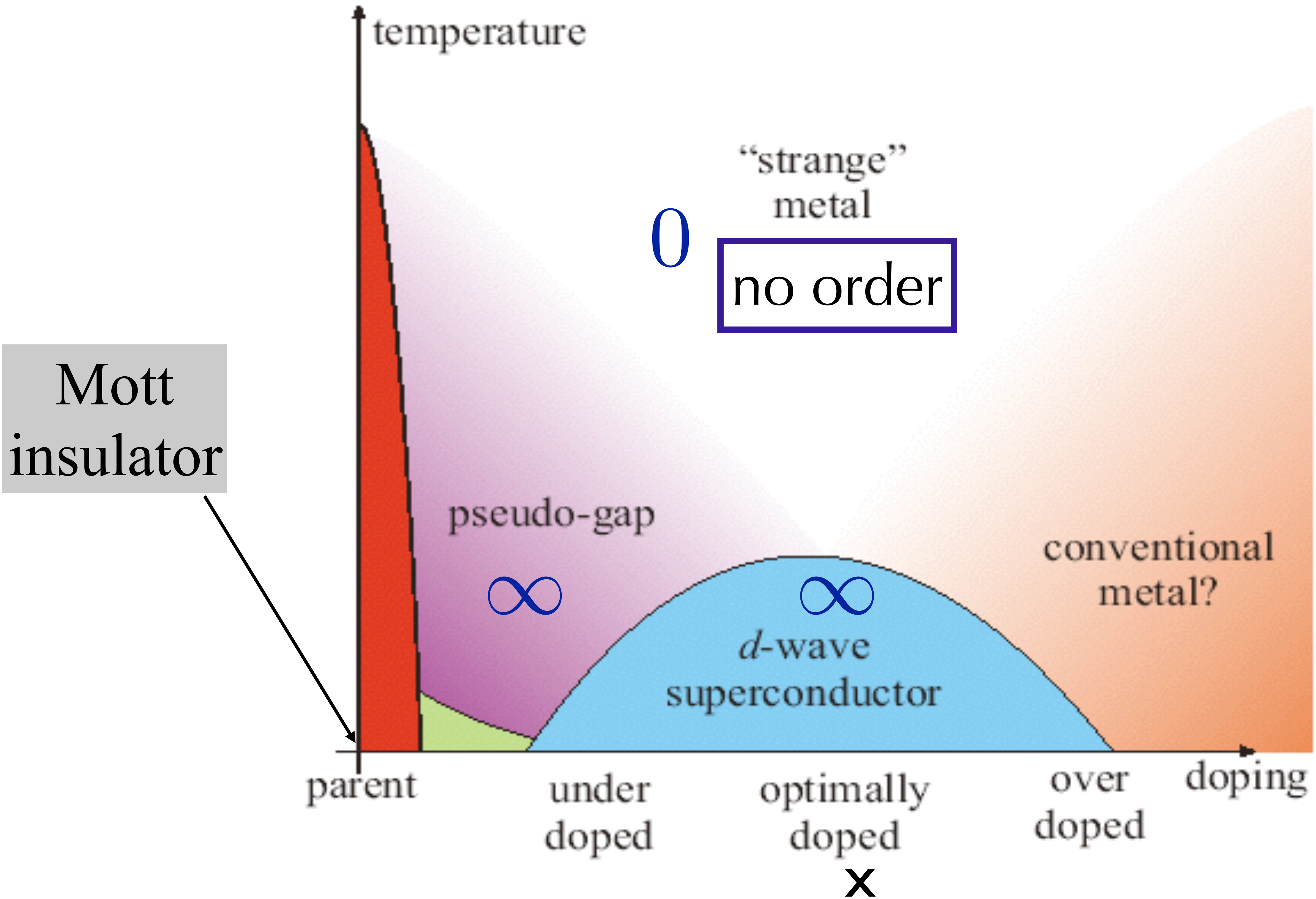
$$\delta(x_0 - y_0)[J_\mu(x), \phi(y)] = \delta^d(x-y)\delta\phi(y)$$

$$[J] = d - 1$$

Are there other examples of currents with anomalous dimensions?

underlying electricity and magnetism?

is symmetry breaking necessary?



temperature
"strange"
metal
0
no order
Mott
insulator
pseudo-gap
∞
∞
d-wave
superconductor
conventional
metal?
parent
under
doped
optimally
doped
over
doped
doping
x

strange metal: experimental facts

Quantum critical behaviour in a high-T_c superconductor

D. van der Marel[1]*, H. J. A. Molegraaf[1]*, J. Zaanen[2], Z. Nussinov[2]*, F. Carbone[1]*, A. Damascelli[3]*, H. Eisaki[3]*, M. Greven[3], P. H. Kes[2] & M. Li[2]

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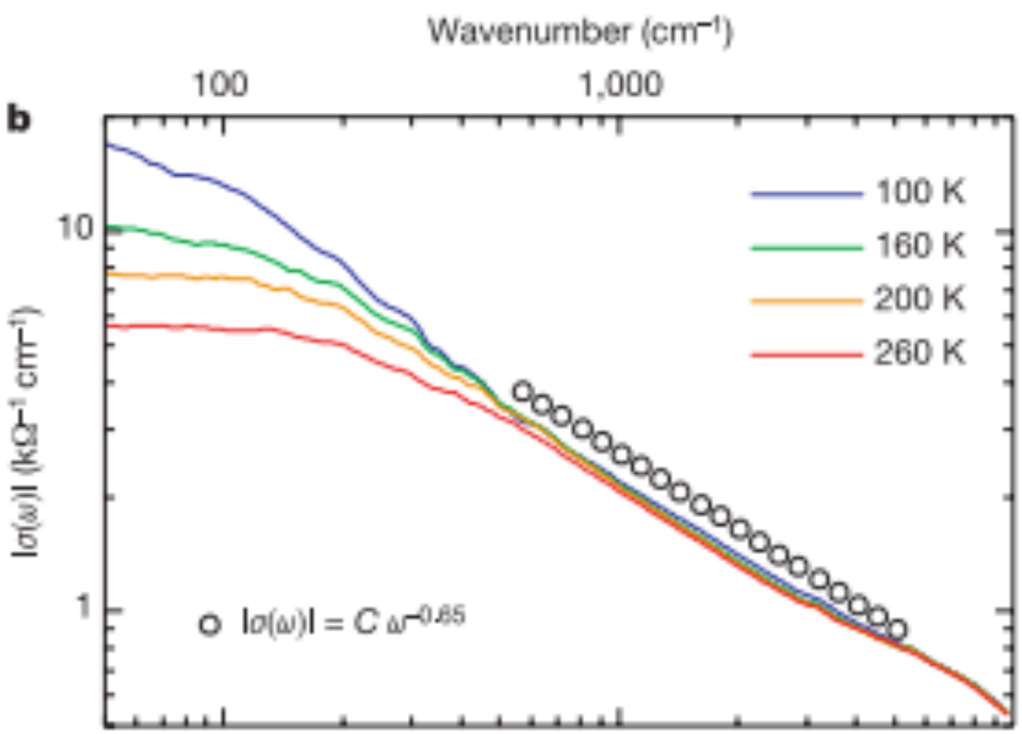
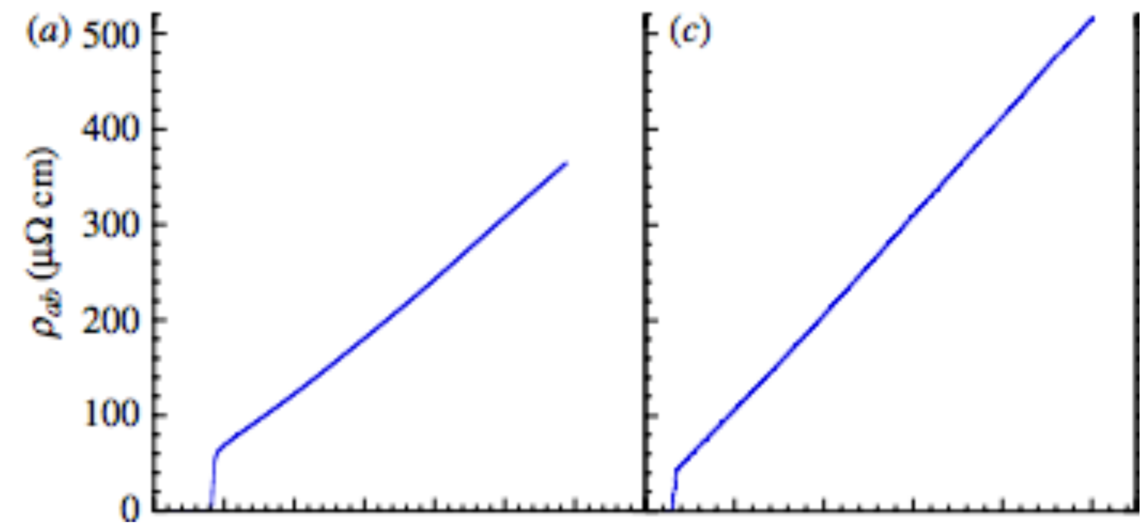


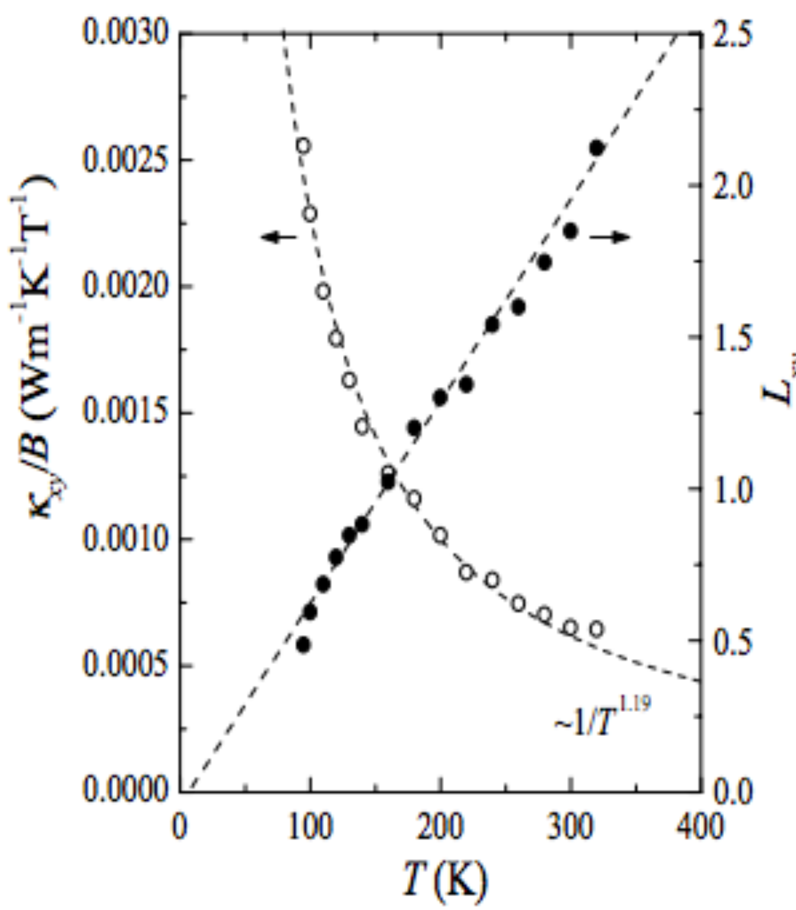
$$\sigma(\omega) = C\omega^{-\frac{2}{3}}$$

$$\frac{n\tau e^2}{m}\frac{1}{1-i\omega\tau}$$

T-linear resistivity

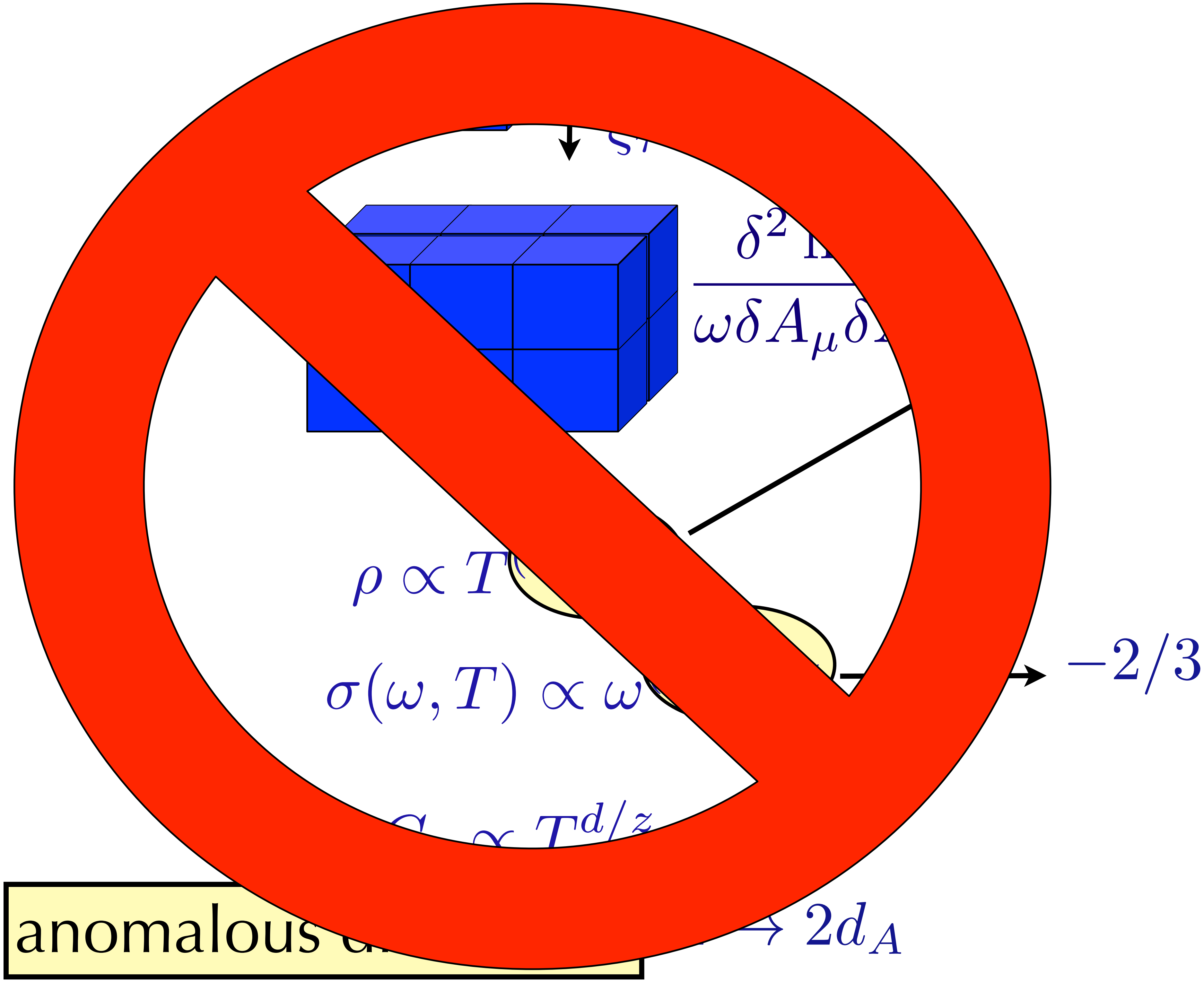


$$L_{xy} = \kappa_{xy}/T\sigma_{xy} \neq \# \propto T$$



why is the problem hard?

single-parameter scaling

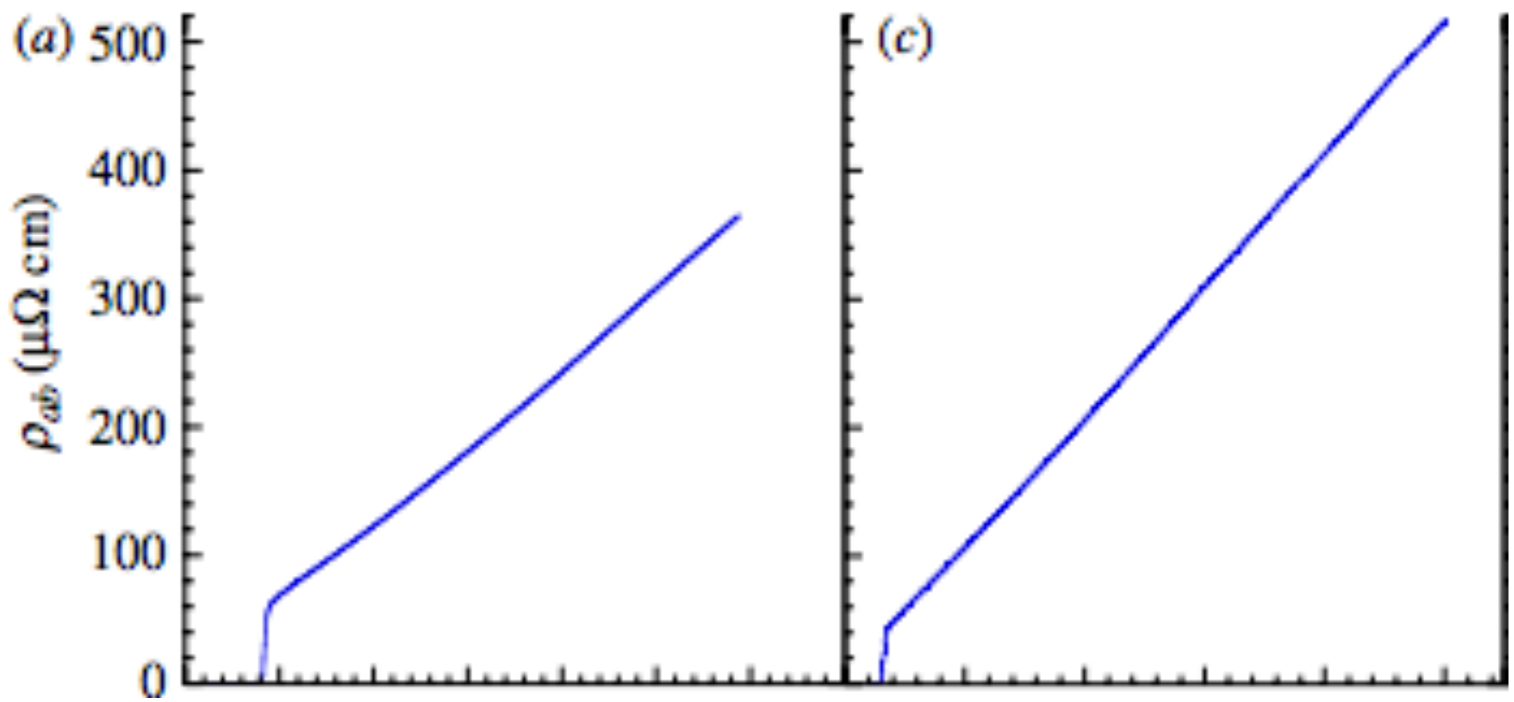


strange metal explained!

Hall Angle

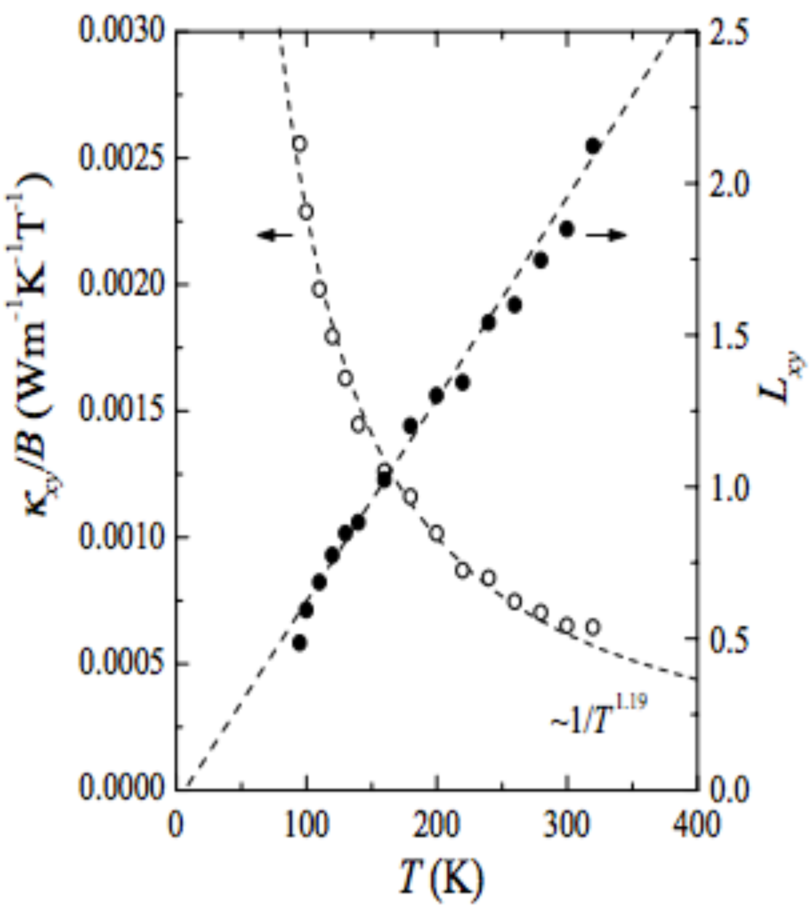
$$\cot\theta_H \equiv \frac{\sigma_{xx}}{\sigma_{xy}} \approx T^2$$

T-linear resistivity



Hall Lorenz ratio

$$L_{xy} = \kappa_{xy}/T\sigma_{xy} \neq \# \propto T$$



all explained if

$$[J_\mu] = d - \theta + \Phi + z - 1$$

$$[A_\mu] = 1 - \Phi$$

$$\Phi = -2/3$$

Hartnoll/Karch

strange metal

$$[J_\mu] = d - \theta + \Phi + z - 1$$

$$[A_\mu] = 1 - \Phi$$

$$\Phi = -2/3$$

$$[E] = 1 + z - \Phi$$

$$[B] = 2 - \Phi$$

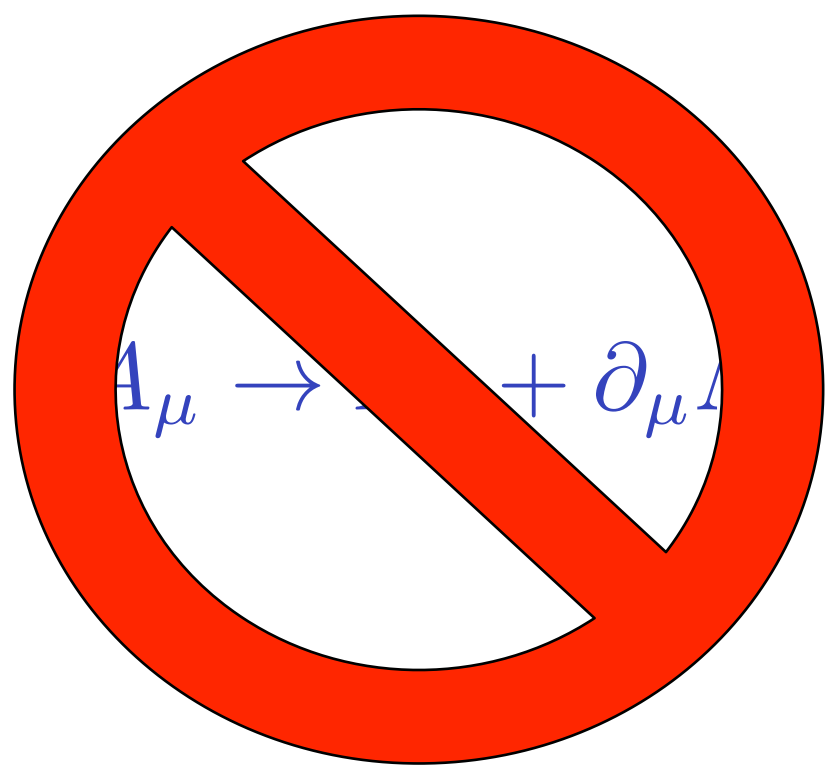
note $\pi r^2 B \neq$ flux

How is this possible - - if at all?

what is the new gauge principle?

if

$$[A_\mu] \neq 1$$



~Noether's Second Theorem

hint

$$\partial_\mu J^\mu = 0$$

current conservation

what if

$$[\partial_\mu, \hat{Y}] = 0$$

new current

$$[\tilde{J}] = d - 1 - D_Y$$

$$\partial_\mu \hat{Y} J^\mu = \partial_\mu \tilde{J}^\mu = 0$$

possible gauge transformations

$$S = -\frac{1}{4}\int d^d x F^2$$

$$S = \frac{1}{2}\int \frac{d^d k}{2\pi^d} A_\mu(k)[k^2\eta^{\mu\nu} - k^\mu k^\nu]A_\nu(k)$$

$$M_{\mu\nu}k^\nu = 0$$

zero eigenvector

$$ik_\mu \to \partial_\nu$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

family of zero eigenvalues

$$M_{\mu\nu} \underbrace{f k^{\nu}} = 0$$

generator of gauge symmetry

1.) rotational invariance

2.) A is still a 1-form

3.) $[f, k_{\mu}] = 0$

only choice

$$f \equiv f(k^2)$$

$$(\Delta)^\gamma$$

$$A_\mu \to A_\mu + (\Delta)^{\frac{(\gamma-1)}{2}} \partial_\mu \Lambda$$

$$[A_\mu] = \gamma$$

what kind of E&M has such gauge transformations?

model with anomalous dimensions

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left[\mathcal{R} - \frac{(\partial_\mu\phi)^2}{2} - \frac{Z(\phi)}{4}F^2 + V(\phi)\right]$$

$$\begin{cases} Z(\phi) \underset{\phi\to\infty}{\to} Z_0 e^{\gamma\phi} \\ V(\phi) \underset{\phi\to\infty}{\to} V_0 e^{-\delta\phi}. \end{cases}$$

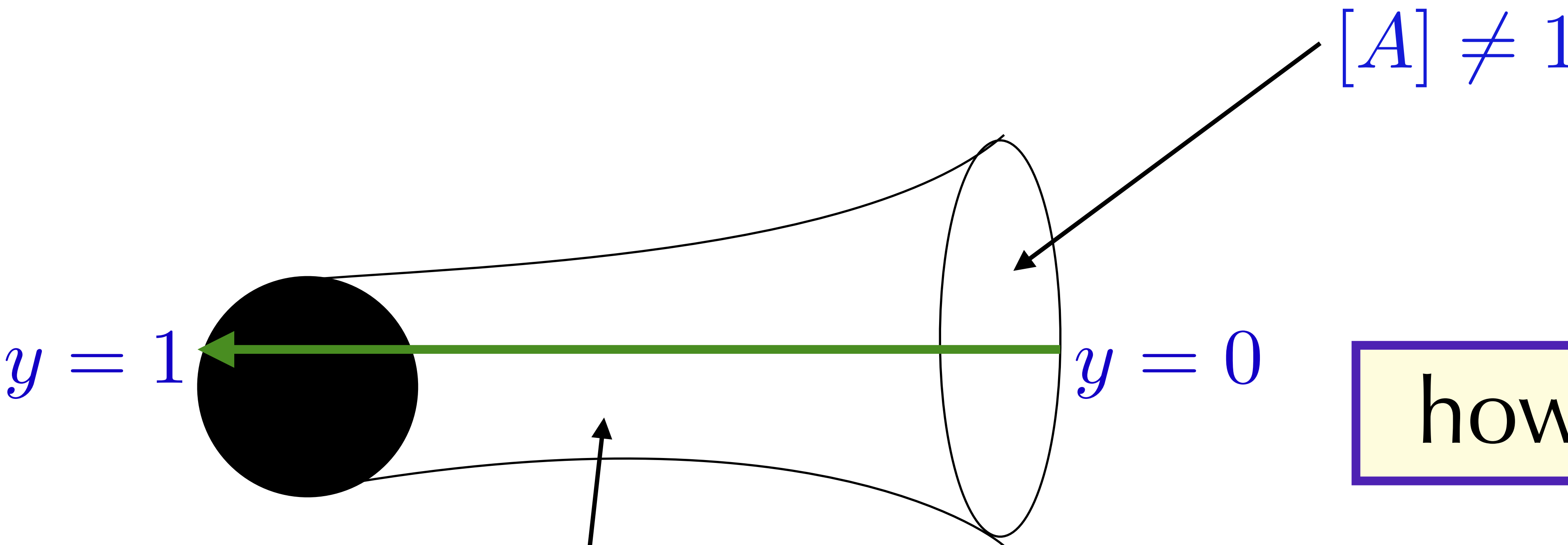
$$e^{\phi} = r^{\pm\kappa}$$

$$r^{\alpha}F^2$$

Karch:1405.2926

Gouteraux: 1308.2084

claim



how?

?

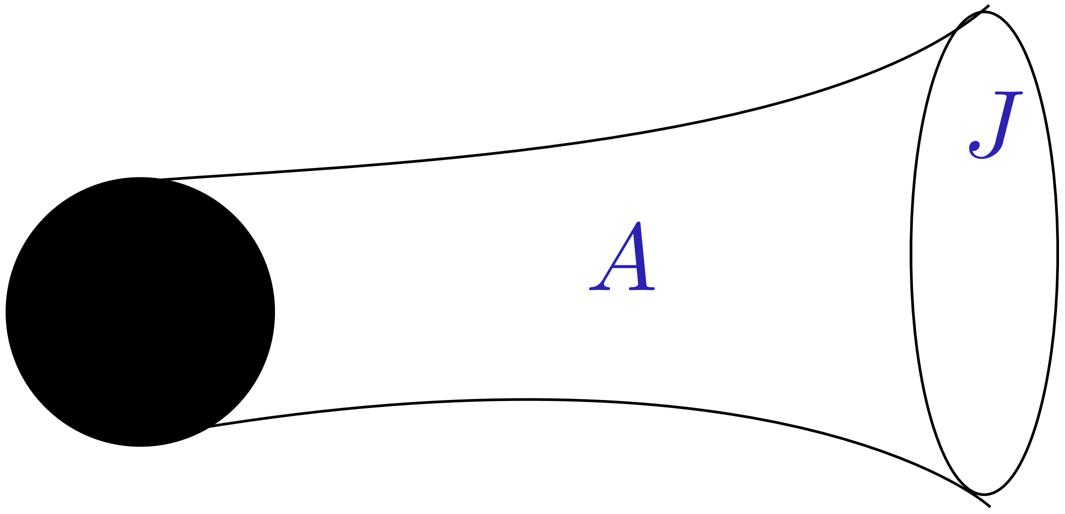
$$S = \int dV_d dy \left(y^a F^2 + \cdots\right)$$

$$F = dA$$

Karch:1405.2926
Gouteraux: 1308.2084

if holography is RG then how can it lead to an anomalous dimension?

standard case



$$\int d^d x A_{||} J$$

$$A(y=0) = A_{||}$$

bc does not satisfy

$$A(y=0) \neq A_{||} + d\Lambda$$

alternatively

$$(A + d\Lambda)_{\partial\Omega} = a + d^{||}\Lambda_{\partial\Omega}$$

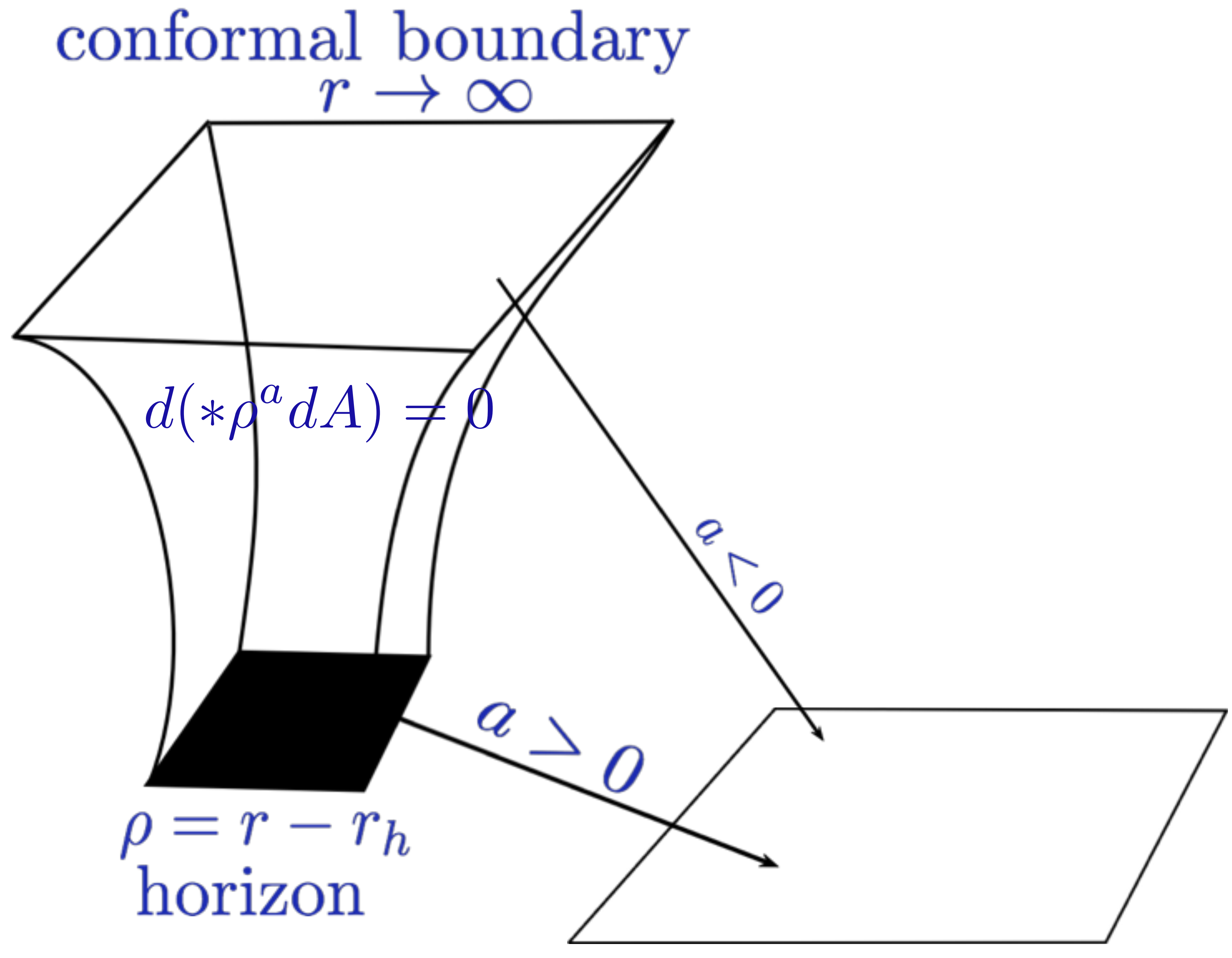
boundary theory has non-trivial gauge structure

AdS/ Lifshitz

$$\int dy/y = \infty$$

large gauge transformation

membrane paradigm



construct boundary
theory explicitly

$$S = \int dV_d dy \left(y^a F^2 + \cdots \right)$$

eom

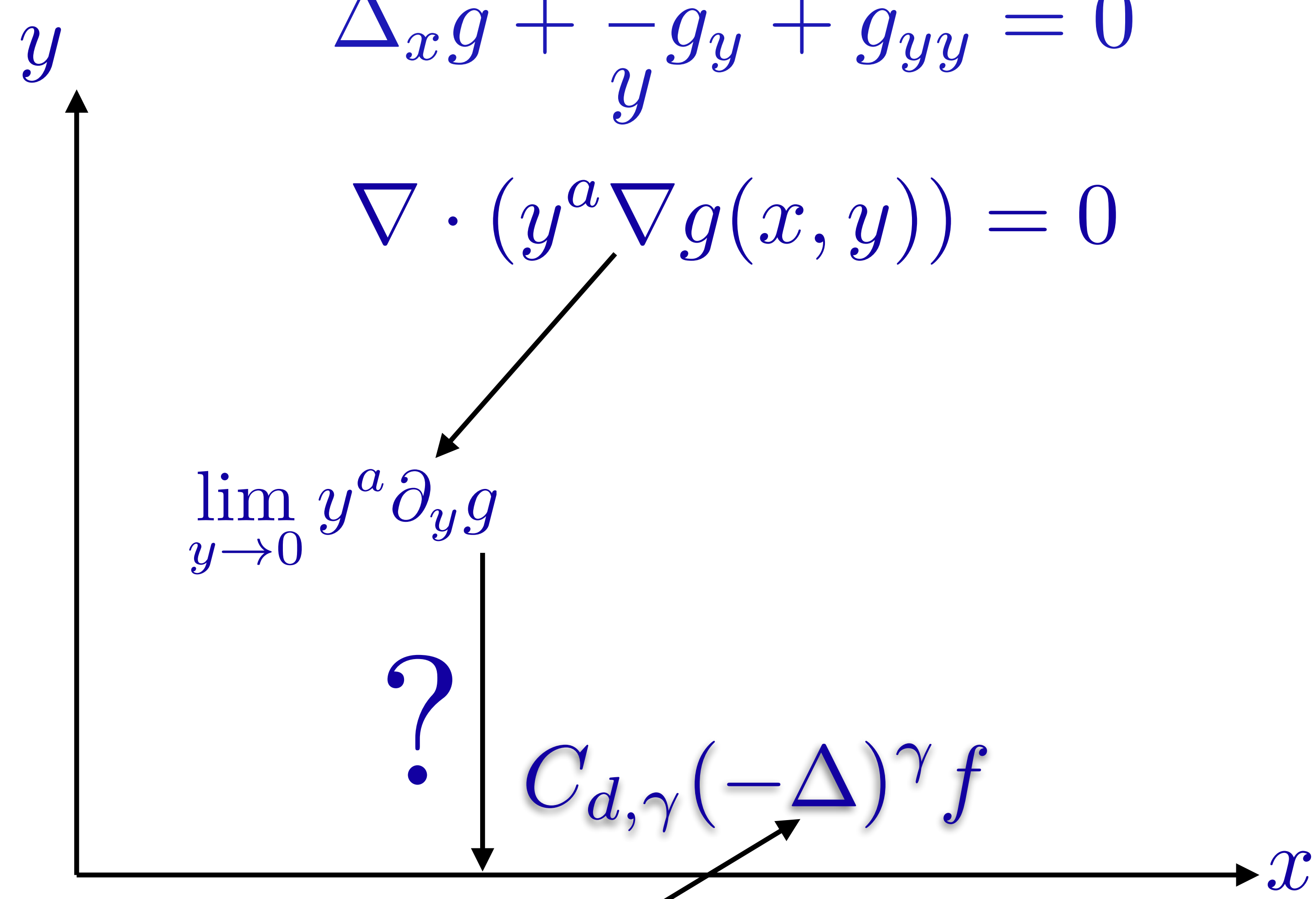
$$d(y^a \star dA) = 0$$

Caffarelli-Silvestre extension theorem (2006)

$$g(x, y = 0) = f(x)$$

$$\Delta_x g + \frac{a}{y} g_y + g_{yy} = 0$$

$$\nabla \cdot (y^a \nabla g(x, y)) = 0$$



$$\lim_{y \to 0} y^a \partial_y g$$

$$C_{d,\gamma} (-\Delta)^\gamma f$$

fractional Laplacian

$$g(z = 0, x) = f(x)$$

$$\gamma = \frac{1-a}{2}$$

closer look

$$\nabla \cdot (y^a \nabla u) = 0$$

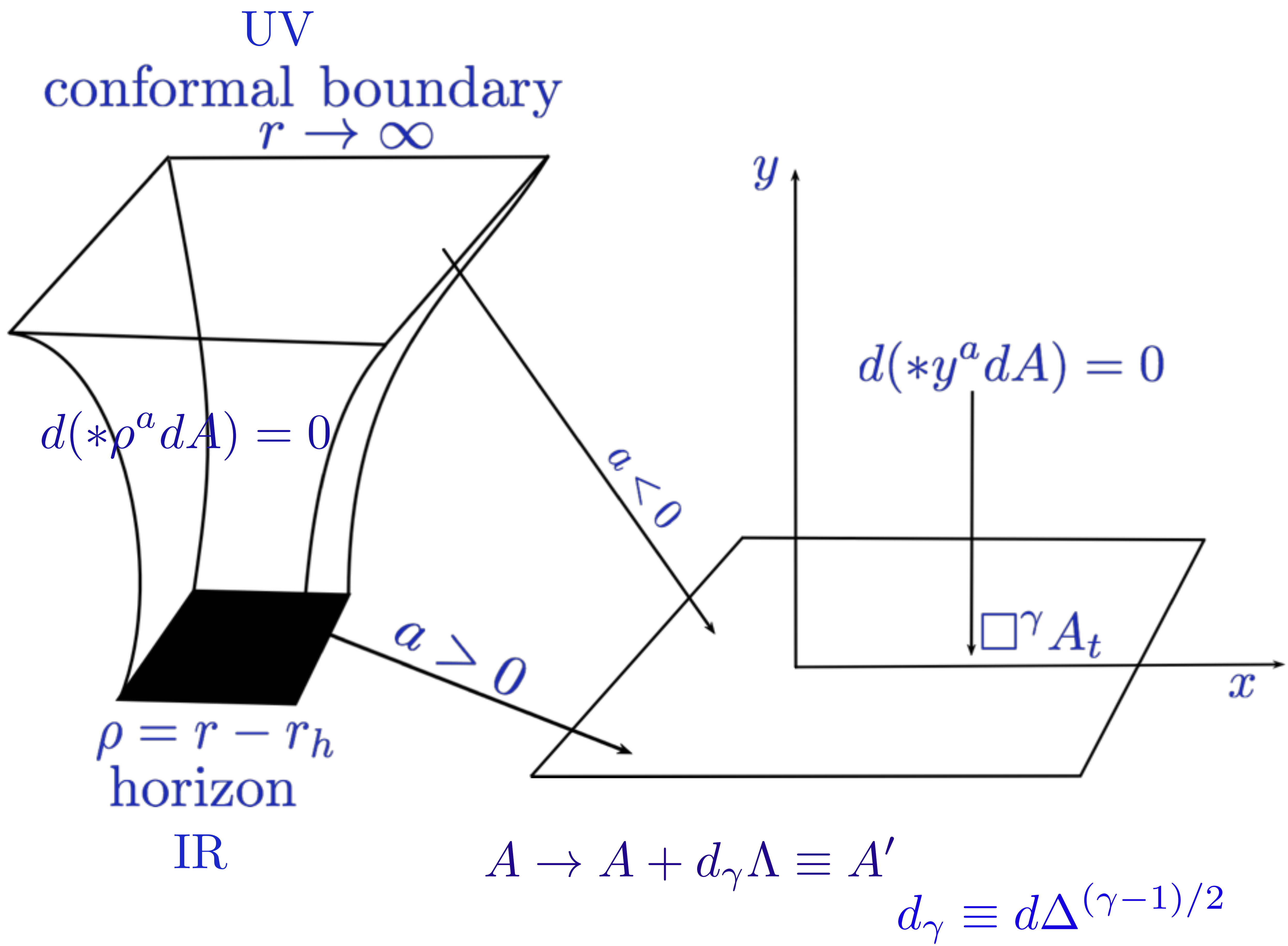
scalar field
(use CS theorem)

$$d(y^a \star dA) = 0$$

holography

similar equations

generalize CS theorem to
p-forms
GL,PP:1708.00863
(CIMP, 366, 199 (2019)))



UV
conformal boundary
$r \to \infty$
$d(*\rho^a dA) = 0$
$\rho = r - r_h$
horizon
IR
$a > 0$
$a < 0$
y
x
$d(*y^a dA) = 0$
$\Box^\gamma A_t$
$A \to A + d_\gamma \Lambda \equiv A'$
$d_\gamma \equiv d\Delta^{(\gamma-1)/2}$

boundary action:
fractional Maxwell
equations

$$\Delta^\gamma A_\perp = J$$

boundary action has
`anomalous dimension'
(non-locality)

if holography is RG then how can it lead to an anomalous dimension?

$$S = \int dV_d dy \left(y^a F^2 + \cdots \right)$$

$$[A] = 1 - a/2$$

dimension of A is fixed by the bulk theory: not really anomalous dimension define

$$F_{ij} = \partial_i^\gamma A_j - \partial_j^\gamma A_i \equiv d_\gamma A = d\Delta^{\frac{\gamma-1}{2}} A,$$

$$S = \int -\frac{1}{4} F_{ij} F^{ij}$$

integrate by parts

$$S = \int \frac{1}{2} A_i (-\Delta)^{2\gamma} A^i,$$

non-local boundary action

new gauge transformation

$$A \to A + d_\gamma \Lambda \equiv A'$$

$$d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$$

$$[A] = \gamma$$

action of gauge group

$$D_{\gamma,A}(e^\Lambda \star \phi) = e^{(\gamma-1)/2} D_{\gamma,A'} \phi$$

$$D_{\gamma,A} = (d + A)\Delta^{(\gamma-1)/2}$$

causality

from the bulk

use CS theorem

$$\lim_{y \to 0} [y^a \partial\phi(x,y), y^a \partial\phi(x',y)] = \Delta^\gamma [\phi(x,y,\phi(x',y)]$$

$$= \Delta^\gamma \delta(x - x') = 0$$

no problem
with causality

$$\partial_\mu J^\mu = 0$$

current conservation

↑ what if

$$[\partial_\mu, \hat{Y}] = 0$$

answer

?

$$[\partial_\mu, \hat{Y}] = 0$$

$$[d, \Delta^\alpha] = 0$$

$$\hat{Y} = \Delta^\alpha$$

$$J \to \Delta^\alpha J \qquad [J] = d - 1 - \alpha$$

Ward identities

$$C^{ij}(k) \propto (k^2)^\gamma \left(\eta^{ij} - \frac{k^i k^j}{k^2} \right).$$

standard Ward identity

$$k_i C^{ij}(k) = 0 \Rightarrow \partial_i C^{ij}(k) = 0$$

but

$$k^{\gamma-1} k_\mu C^{\mu\nu} = 0 \Rightarrow \partial_\mu (-\Delta)^{\frac{\gamma-1}{2}} C^{\mu\nu} = 0$$

inherent ambiguity in E&M

Noether's Second Theorem

$$\sum \psi_i \delta u_i = \delta f -$$

$$- \frac{d}{dx}\left\{\sum\left[\binom{1}{1}\frac{\partial f}{\partial u_i^{(1)}}\delta u_i + \binom{2}{1}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{1}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-1)}\right]\right\} +$$

$$+ \frac{d^2}{dx^2}\left\{\sum\left[\binom{2}{2}\frac{\partial f}{\partial u_i^{(2)}}\delta u_i + \binom{3}{2}\frac{\partial f}{\partial u_i^{(3)}}\delta u_i^{(1)} + \cdots + \binom{\kappa}{2}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_i^{(\kappa-2)}\right]\right\} +$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$+ (-1)^\kappa \frac{d^\kappa}{dx^\kappa}\left\{\sum\binom{\kappa}{\kappa}\frac{\partial f}{\partial u_i^{(\kappa)}}\delta u_1\right\} \qquad (6)$$

$$A_\mu \to A_\mu + \partial_\mu \Lambda + \partial_\mu \partial_\nu G^\nu + \cdots ,$$

$$A \to A + d_\gamma \Lambda \equiv A'$$

$$d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$$

Noether's Second Theorem and Ward Identities for Gauge Symmetries

Steven G. Avery[a], Burkhard U. W. Schwab[b]

For simplicity, we focus on the case when the transformation may be written in the form[6]

$$\delta_\lambda \phi = f(\phi)\,\lambda + f^\mu(\phi)\partial_\mu \lambda, \tag{10}$$

but it is straightforward to consider transformations, as Noether did, involving arbitrarily high derivatives of λ. (Although, the authors know of no physically interesting examples.) Let us start with

arxiv:1510.07038

family of zero eigenvalues

$$M_{\mu\nu} f k^{\nu} = 0$$

most fundamental conservation law

$$\partial^{\mu} \underbrace{(-\nabla^2)^{(\gamma-1)/2} J_{\mu}}_{J'_{\mu}} = 0$$

is there a consistent algebra
for fractional currents?

Yes

Virasoro algebra

$$L_n := -z^{n+1}\frac{\partial}{\partial z}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Witt algebra

central extension

conformal transformations on unit disk

$$\mathcal{V} \to \mathcal{W} \to 1$$

Fractional Virasoro algebra

generators

$$L_n^a = -z^{a(n+1)} \left(\frac{\partial}{\partial z}\right)^a \qquad \bar{L}_n^a := -\bar{z}^{a(n+1)} \left(\frac{\partial}{\partial \bar{z}}\right)^a$$

$$[L_n, L_m](z^{ak}) = \left(\frac{\Gamma(a(k+n)+1)}{\Gamma(a(k-1+n)+1)} - \frac{\Gamma(a(k+m)+1)}{\Gamma(a(k-1+m)+1)}\right) L_{n+m}(z^{ak})$$

$$= (A^a_{n,m}(k) \otimes L_{n+m})(z^{ak})$$

$$[L_m^a, L_n^a] = A_{m,n} L_{m+n}^a + \delta_{m,n} h(n) c Z^a$$

algebra for conformal non-local actions

$Z^2_\star(\mathcal{W}_a, \mathcal{H})/B^2_\star(\mathcal{W}_a, \mathcal{H})$

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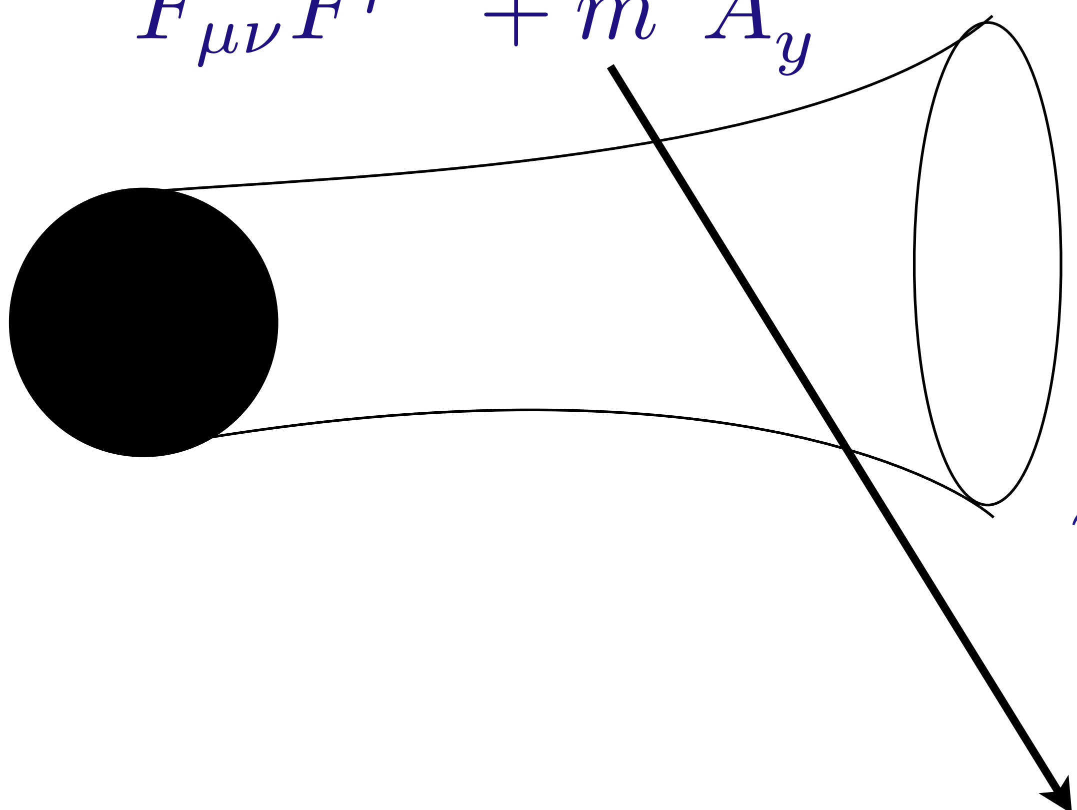
is there a hidden broken symmetry?

application: gauge fields with anomalous dimensions

$F_{\mu\nu}F^{\mu\nu} + m^2 A_y^2$

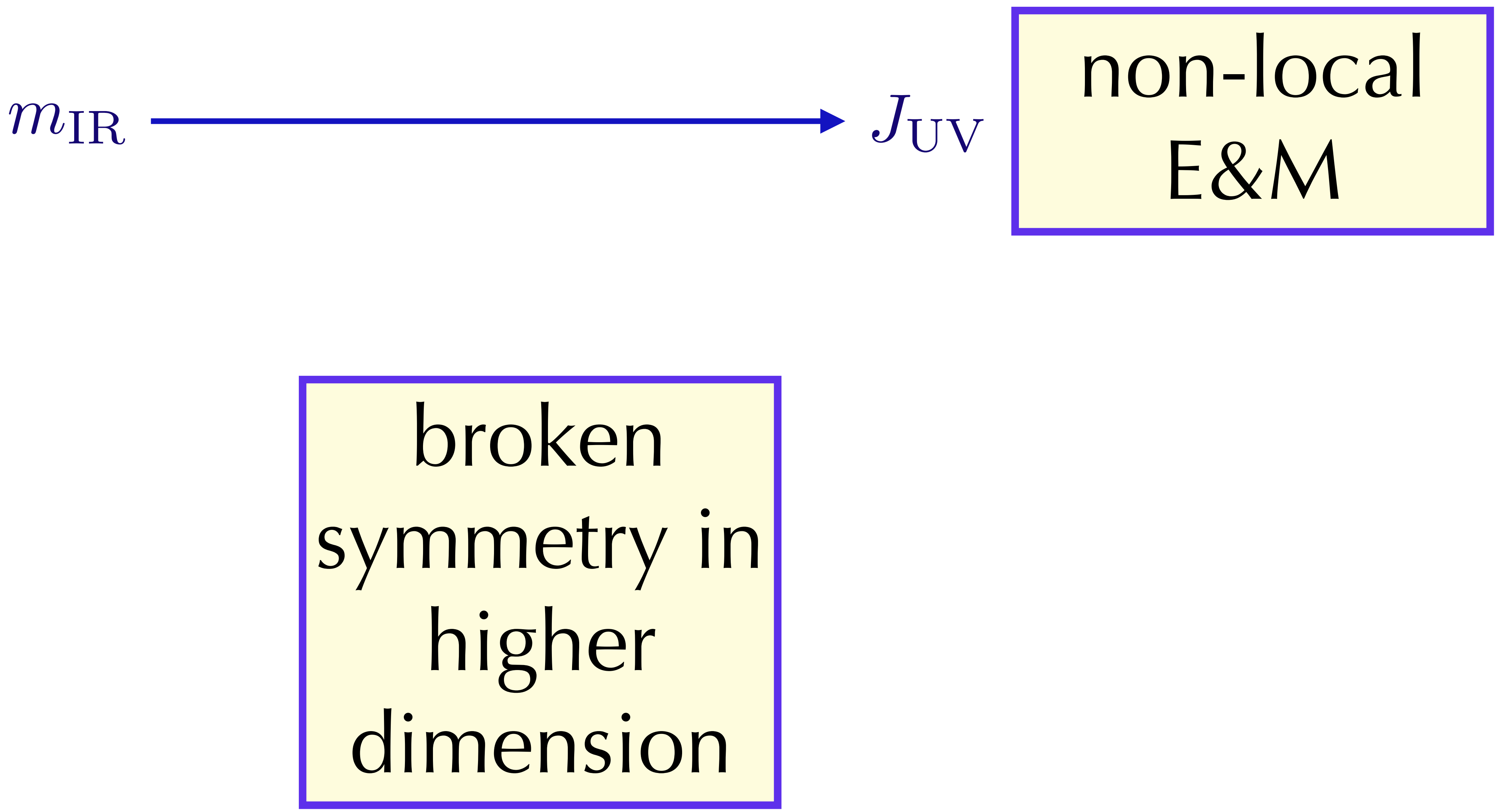


$A_\mu^\perp (-\Delta)^\gamma A^{\perp\mu}$

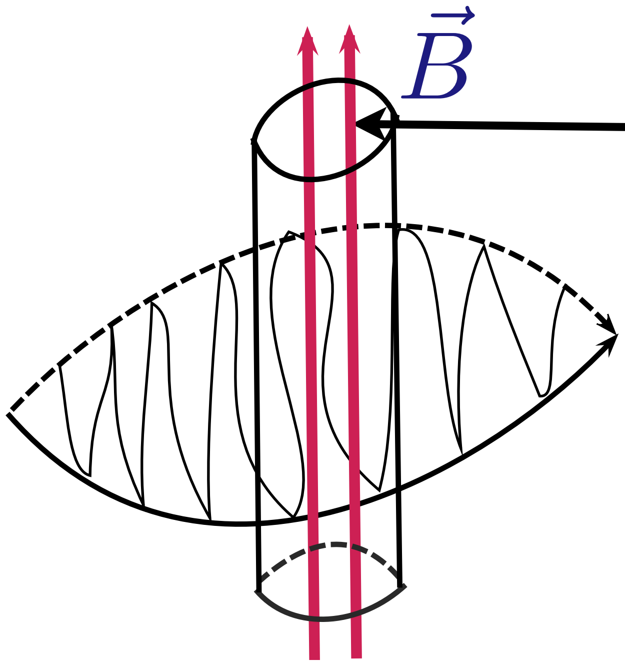
$\gamma = \sqrt{d^2 + m^2 - 1/2}$

dynamical `Higgs' mode

additional length scale


m_{IR}
J_{UV}
non-local
E&M
broken
symmetry in
higher
dimension

experiments?



magnetic flux

$\pi r^2 B$

should be dimensionless

$$[B] = 2 - \Phi = 2 + 2/3 \neq 2$$

what's the resolution?

correct dimensionless quantity

$$a_i \equiv [\partial_i, I_i^\alpha{}_\alpha A_i] = \partial_i I_i^\alpha{}_\alpha A_i$$

$\to \Delta^{-\alpha}$

what's the relationship?

$$\oint_{\partial\Sigma} a \qquad\qquad \oint_{\partial\Sigma} A$$

$$\text{Norm} \oint_{\partial\Sigma} a = \frac{1}{\Gamma(3/2-\gamma)} \oint_{\partial\Sigma} A$$

not an integer

obstruction theorem to charge quantization (NST)

$$A_\mu \to A_\mu + \partial_\mu \Lambda + \partial_\mu \partial_\nu G^\nu + \cdots ,$$

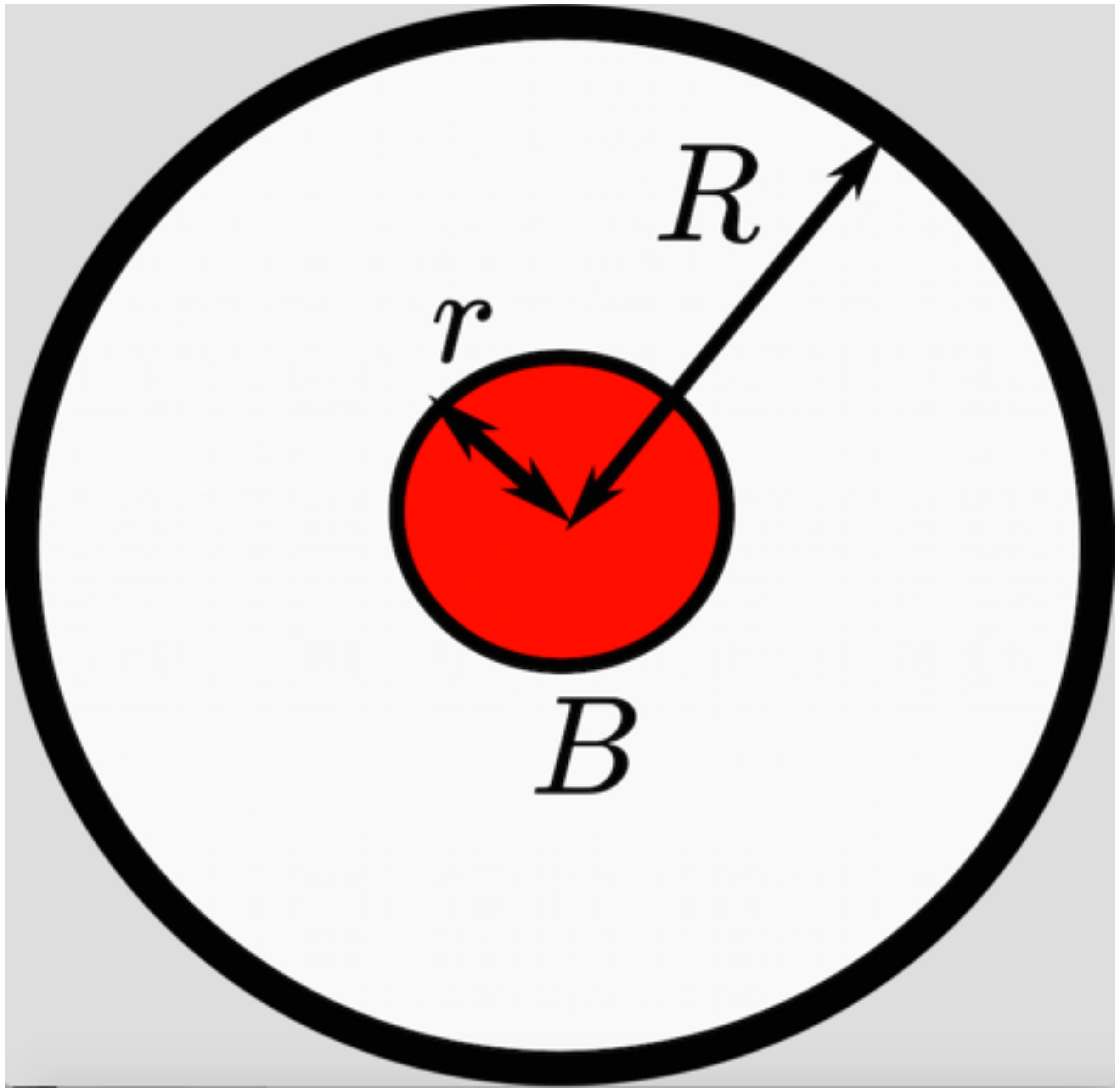
$$A \to A + d_\gamma \Lambda \equiv A'$$

$$d_\gamma \equiv (\Delta)^{\frac{\gamma-1}{2}} d$$

J'_μ

charge ill-defined (new landscape problem)

New Aharonov-Bohm Effect



$$\Delta\phi_{\rm D} = \frac{e}{\hbar}\pi r^2 B R^{2\alpha-2}\left(\frac{\sqrt{\pi}2^{1-\alpha}\Gamma(2-\alpha)\Gamma(1-\frac{\alpha}{2})}{\Gamma(\alpha)\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\sin^2\frac{\pi\alpha}{2}\,{}_2F_1(1-\alpha,2-\alpha;2;\frac{r^2}{R^2})\right)$$

is the correction large?

$$\alpha = 1 + 2/3 = 5/3$$

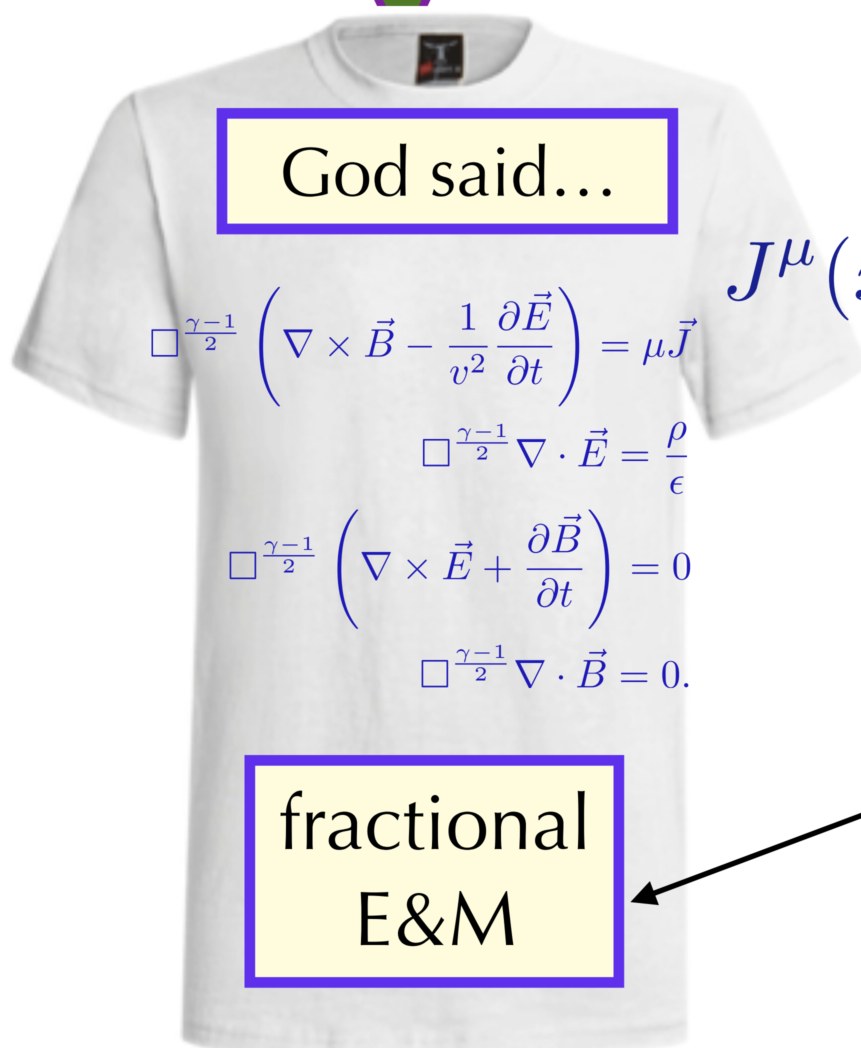
$$\Delta\Phi_R = \frac{eB\ell^2}{\hbar} L^{-5/3}/(0.43)^2$$

yes!

if in the strange metal

$$[A_\mu] = d_A \neq 1$$

Pippard Kernel



$$J^\mu(x) = -\int d^d x' C_{\mu\nu}(|x - x'|) A^\nu$$

$$[J] \neq d - 1$$

$$[A] \neq 1$$

fractional E&M

in SC!

$$\omega = ck$$

$U(1) \to \mathbb{Z}_2$