Quantum Lyapunov spectrum of the Sachdev-Ye-Kitaev model

YITP-YIPQS Workshop, YITP, Kyoto University

27 May 2019

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SCIENCE
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Collaborators in this work

arXiv:1809.01671 (JHEP04(2019)082)
arXiv:1902.11086

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(University of Maryland)

Plan of the talk

- Characterization of many-body quantum chaos
- The Sachdev-Ye-Kitaev model
- **The quantum Lyapunov spectrum**
- **The singular values of two-point correlators**
- Summary

Plan of the talk

Characterization of many-body quantum chaos

The Sachdev-Ye-Kitaev model

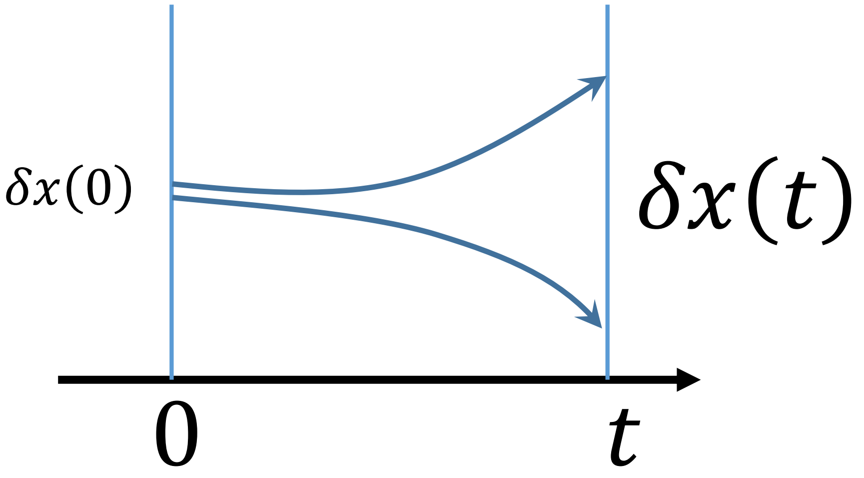
The quantum Lyapunov spectrum

The singular values of two-point correlators

Summary

Chaos in deterministic classical dynamics

- Sensitivity to initial conditions: exponential growth of initial perturbation



"butterfly effect"

Bounded, nonperiodic dynamics with nonlinearity

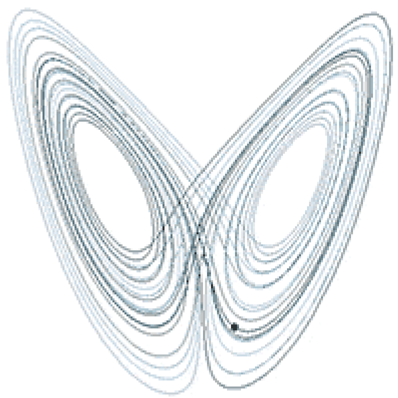
What happens in quantum mechanics?

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)



by Dan Quinn
(on Wikimedia Commons)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

The Hidden Heroines of Chaos

By Joshua Sokol

May 20, 2019

Two women programmers played a pivotal role in the birth of chaos theory. Their previously untold story illustrates the changing status of computation in science.

Building 24, MIT

Acknowledgments. The writer is indebted to Dr. Barry Saltzman for bringing to his attention the existence of nonperiodic solutions of the convection equations. Special thanks are due to Miss Ellen Fetter for handling the many numerical computations and preparing the graphical presentations of the numerical material.

Acknowledgement

The writer is greatly indebted to Mrs. Margaret Hamilton for her assistance in performing the many numerical computations which were necessary in this work.

The Statistical Prediction of Solutions of Dynamic Equations

By EDWARD N. LORENZ

Massachusetts Institute of Technology, Cambridge, U.S.A.

Reprinted from the Proceedings of the International Symposium on Numerical Weather Prediction in Tokyo November 7–13, 1960

Published by the Meteorological Society of Japan *March 1962*

DISCUSSION

Bolin: Did you change the initial condition just slightly and see how much different results were in the forecasting in this way?

A: As a matter of fact, we tried out that once with the same equation to see what could happen. We changed one of the 12 variables by a factor of a small fraction of 1%, a change which would be considered to be smaller than observational error. We found that this error grew and continued to grow at a slow exponential rate. After 1 or 2 months, it is still pretty small so the map looked about right but it is comparable with the observational error. But after 6 months there is no resembrance at all between the 2 maps, which implied that at least for this particular set of equations there is a limit to how far you can forecast. Thus by these dynamic methods if you assume you have any observational error whatever to begin with, eventually the error predominates. Each of the series has the same statistics.
But they were simply different series after about 6 months.

How to characterize quantum chaos?

$\hat{H}$ = const.

$$i\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \qquad |\psi(t)\rangle = \hat{\mathrm{T}}\exp\left[-i\int_0^t \hat{H}(t')dt\right]|\psi(t=0)\rangle = \exp\left(-i\hat{H}t\right)|\psi(t=0)\rangle$$

Linear dynamics

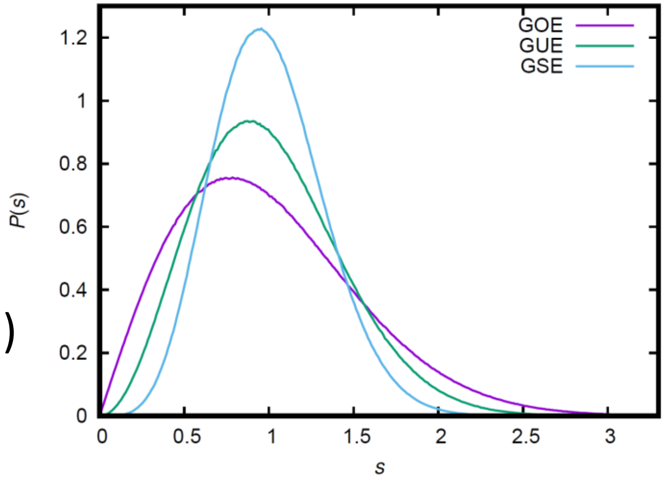
Unitary time evolution

- Long time: energy level statistics

Correlation between levels, as in random matrices

$P(s)$: normalized level separation distribution

Uncorrelated: Poisson (e^{-s})



cf. Bohigas-Giannoni-Schmit conjecture

- Short time: out-of-time correlator

Classically,

$$\{x_i(t), p_j(0)\}_{\mathrm{PB}}^{\;2} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}}t} \text{ for large } t$$

Quantum version:

$$\text{OTOC: } C_T(t) = \left\langle \left|\left[\hat{W}(t), \hat{V}(t=0)\right]\right|^2 \right\rangle$$

$$= \left\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\right\rangle + \cdots$$

Numerically ➔ limited to very small systems

➔ Hard to see exponential time dependence

Characterization of quantum many-body chaos

- Random-matrix like energy level correlation

- Exponential Lyapunov growth of out-of-time-order correlators (OTOC)

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \sim C + \#\, e^{2\lambda_{\mathrm{L}} t}$$

Example: the Sachdev-Ye-Kitaev model

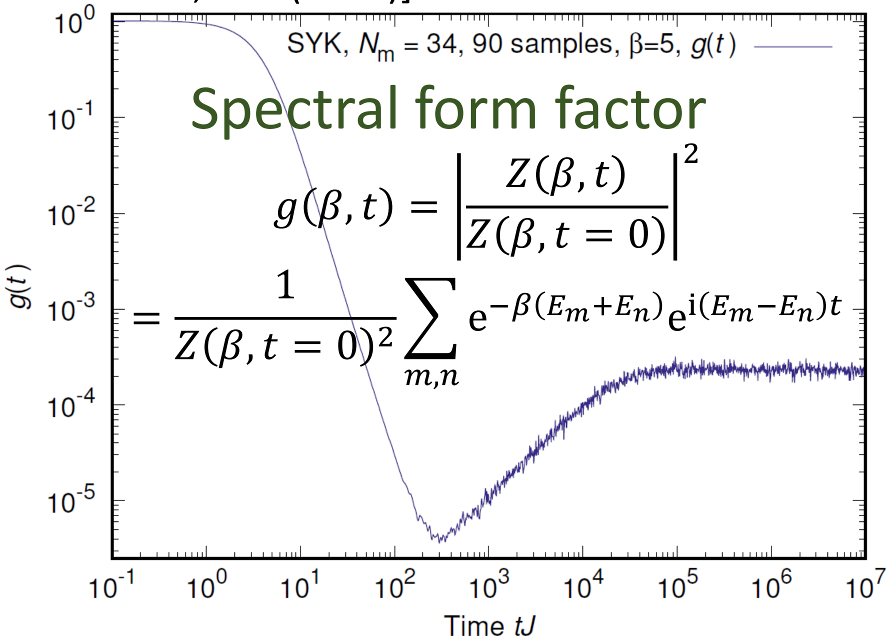
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

J_{abcd} : Gaussian random ($\langle {J_{abcd}}^2\rangle = J^2 = 1$)

[Kitaev 2015]

[Cotler, MT et al., JHEP **1705**, 118 (2017)]



Spectral form factor

$$g(\beta,t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} e^{-\beta(E_m+E_n)} e^{i(E_m-E_n)t}$$

N mod 8	RMT
0	GOE
2	GUE
4	GSE
6	GUE

Lyapunov exponent

$$\lambda_{\mathrm{L}} = \frac{2\pi k_{\mathrm{B}} T}{\hbar} \text{ in low } T \text{ limit}$$

(Maldacena-Shenker-Stanford chaos bound)

Plan of the talk

Characterization of many-body quantum chaos

The Sachdev-Ye-Kitaev model

The quantum Lyapunov spectrum

The singular values of two-point correlators

Summary

The Sachdev-Ye-Kitaev model

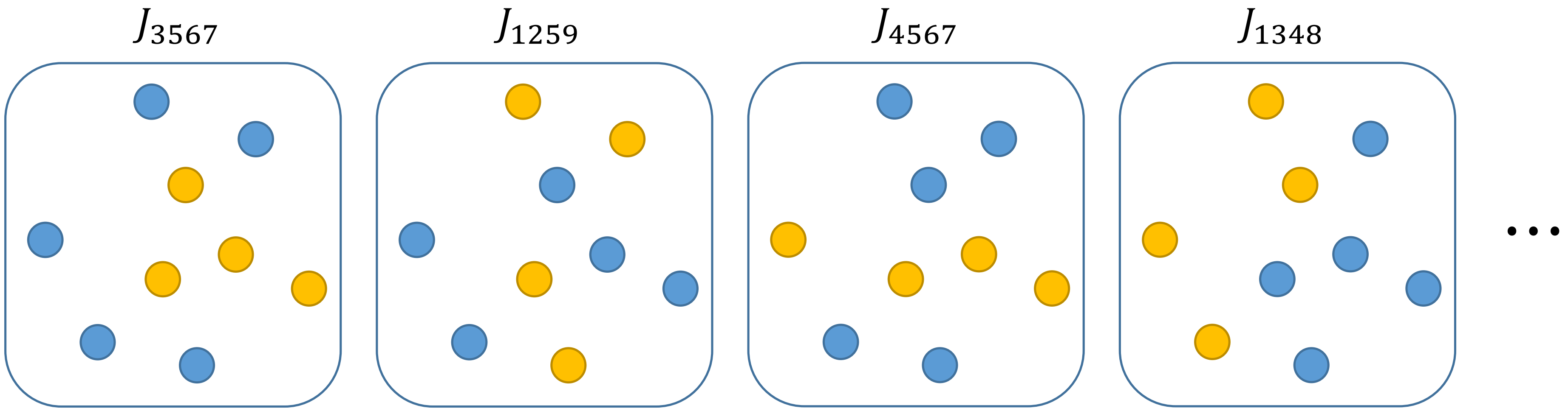
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

cf. Sachdev-Ye model (1993)

$\hat{\chi}_{a=1,2,\ldots,N}$: N Majorana fermions ($\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$)

[A. Kitaev, talks at KITP (2015)]

J_{abcd} : Gaussian random couplings ($\overline{J_{abcd}{}^2} = J^2 = 1,\ \overline{J_{abcd}} = 0$)

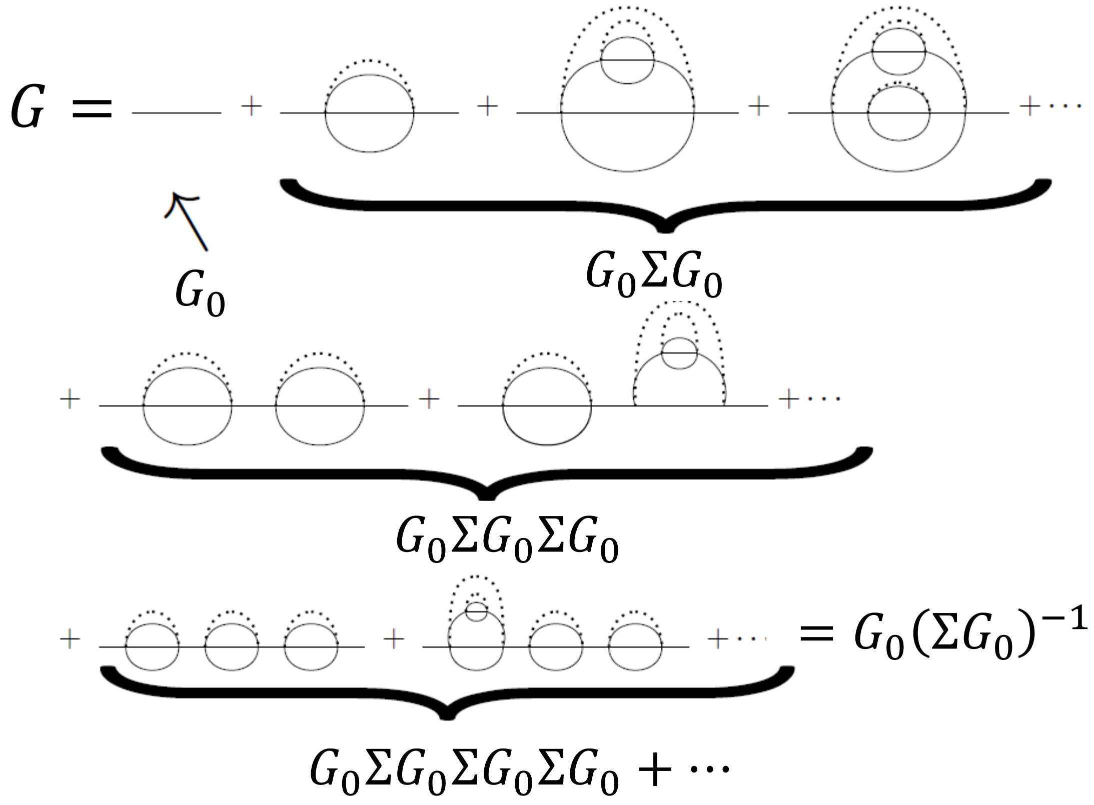


The SYK model

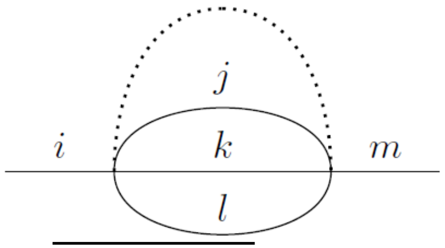
Figures from [I. Danshita, MT, and M. Hanada: Butsuri **73**(8), 569 (2018)]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

Analytically solvable in $N \gg 1$ limit

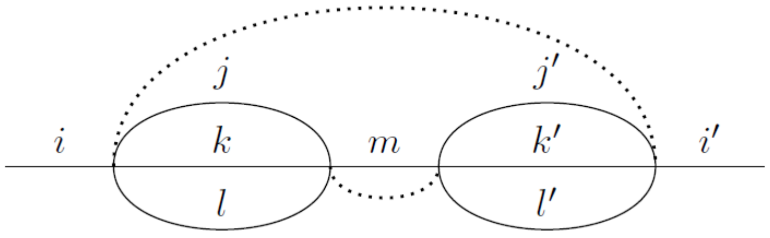


$O(1)$



$\overline{J_{ijkl} J_{jklm}} = \delta_{im}$

$O(N^{-2})$



Only "melon-type" diagrams survive

$$G(i\omega)^{-1} = \boxed{i\omega} - \Sigma(i\omega) \qquad \Sigma = J^2 G^3$$

The SYK model

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

Analytically solvable in $N \gg 1$ limit

$$G(1 - \Sigma G_0) = G_0$$

$$G^{-1} = {G_0}^{-1} - \Sigma$$

$$G(i\omega)^{-1} = \boxed{i\omega} - \Sigma(i\omega)$$

$$\Sigma = J^2 G^3$$

Low energy ($\omega, T \ll J$): ignore $i\omega$ and we have

$$\int dt G(t_1, t)\Sigma(t, t_2) = -\delta(t_1, t_2)$$

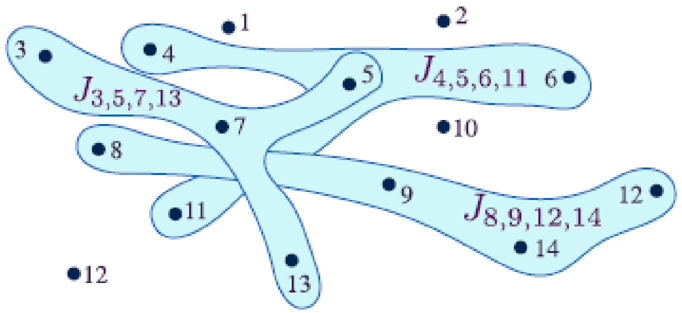
$$J^2 \int dt G(t_1, t) G(t, t_2)^3 = -\delta(t_1, t_2)$$

$$G(t) = -\left(\frac{1}{4\pi J^2}\right)^{1/4} \frac{\mathrm{sgn}(t)}{\sqrt{t}}$$

- Emergent conformal symmetry
- Satisfies the "chaos bound" $\lambda_{\mathrm{L}} \le \frac{2\pi k_{\mathrm{B}} T}{\hbar}$ in the $T \rightarrow 0$ limit

Holographic connection to gravity

$$H = \frac{1}{(2N)^{3/2}} \sum_{i,j,k,\ell=1}^{N} J_{ij;k\ell}\, c_i^\dagger c_j^\dagger c_k c_\ell$$



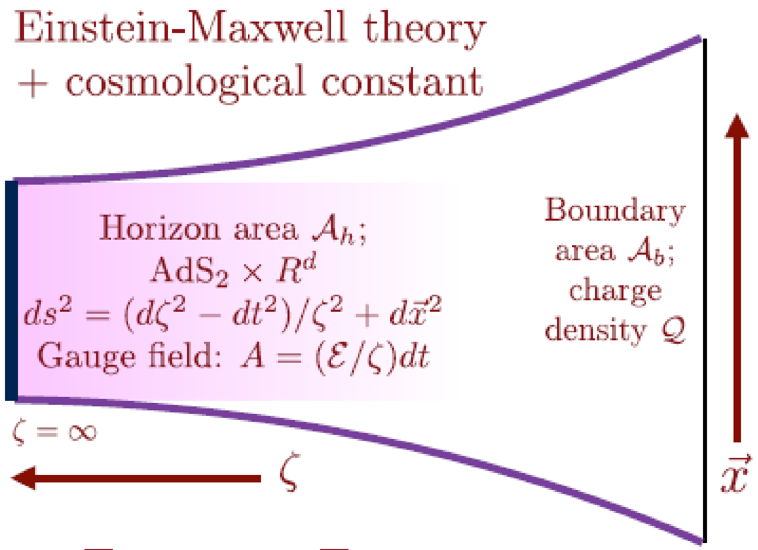
$$\mathcal{Q} = \frac{1}{N} \sum_i \left\langle c_i^\dagger c_i \right\rangle .$$

$$-\left\langle c_i(\tau) c_i^\dagger(0) \right\rangle \sim \begin{cases} -\tau^{-1/2}, & \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2}, & \tau < 0. \end{cases}$$

Known "equation of state" determines $\mathcal{E}$ as a function of $\mathcal{Q}$

Microscopic zero temperature entropy density $\mathcal{S}$ obeys

$$\frac{\partial \mathcal{S}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$



$$\mathcal{L} = \overline{\psi}\Gamma^\alpha D_\alpha \psi + m\overline{\psi}\psi$$

$$-\left\langle \psi(\tau)\overline{\psi}(0) \right\rangle \sim \begin{cases} -\tau^{-1/2}, & \tau > 0 \\ e^{-2\pi\mathcal{E}}\, |\tau|^{-1/2}, & \tau < 0. \end{cases}$$

"Equation of state" relating $\mathcal{E}$ and $\mathcal{Q}$ depends upon the geometry of spacetime far from the AdS_2

Black hole thermodynamics (classical general relativity) yields

$$\frac{\partial \mathcal{S}_{BH}}{\partial \mathcal{Q}} = 2\pi\mathcal{E}$$

[S. Sachdev, Phys. Rev. X **5**, 041025 (2015)]

Proposal for experimental realization

s: molecular levels

$$\hat{H}_{\mathrm{m}} = \sum_{s=1}^{n_s} \left\{ \nu_s \hat{m}_s^\dagger \hat{m}_s + \sum_{i,j} g_{s,ij} \left(\hat{m}_s^\dagger \hat{c}_i \hat{c}_j - \hat{m}_s \hat{c}_i^\dagger \hat{c}_j^\dagger \right) \right\}.$$

$|\nu_s| \gg |g_{s,ij}|$

Modified SYK model:

$$\hat{H}_{\mathrm{eff}} = \sum_{s,i,j,k,l} \frac{g_{s,ij} g_{s,kl}}{\nu_s} \hat{c}_i^\dagger \hat{c}_j^\dagger \hat{c}_k \hat{c}_l.$$

[I. Danshita, M. Hanada, MT: arXiv:1606.02454; PTEP **2017**, 083I01 (2017)]
(also a proceedings manuscript arXiv:1709.07189)

Setup:

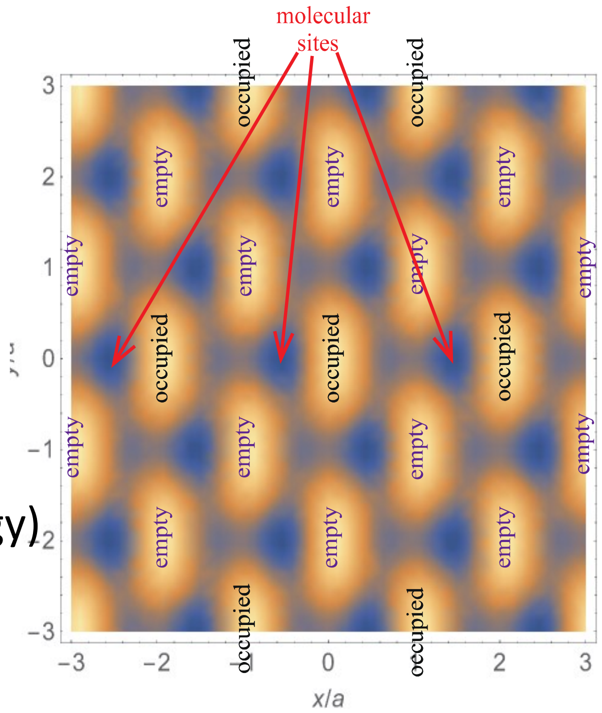
A double-well optical lattice

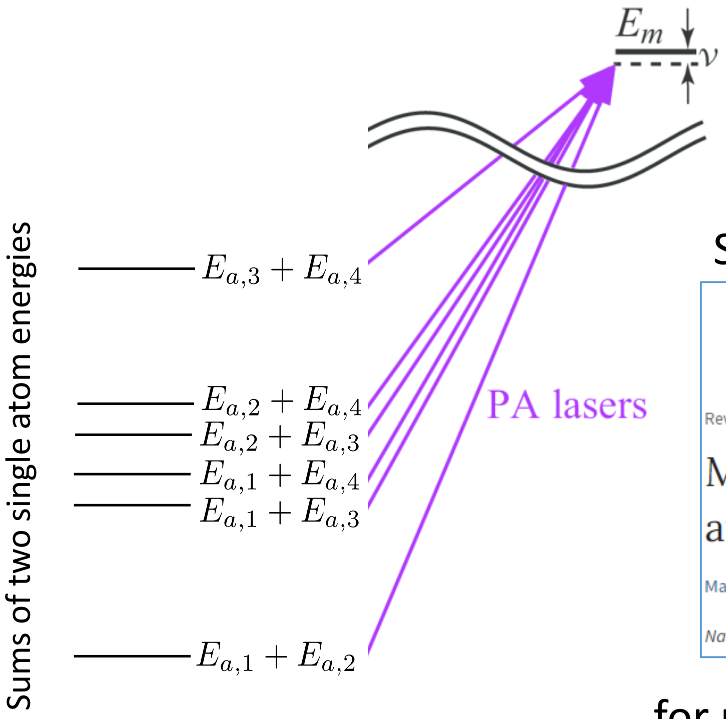
(no degeneracy in the band levels)

with

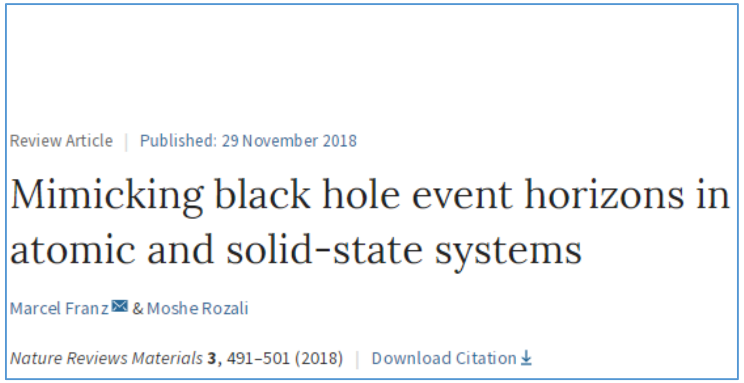
^{6}Li

(large recoil energy)





See


Review Article | Published: 29 November 2018

Mimicking black hole event horizons in atomic and solid-state systems

Marcel Franz & Moshe Rozali

Nature Reviews Materials **3**, 491–501 (2018) | Download Citation

for review including other proposals
e.g. using topological insulator, graphene

Sachdev-Ye-Kitaev model

N Majorana- or Dirac- fermions randomly coupled to each other

[Majorana version]

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

[A. Kitaev: talks at KITP (Feb 12, Apr 7 and May 27, 2015)]

[Dirac version]

$$\hat{H} = \frac{1}{(2N)^{3/2}} \sum_{ij;kl} J_{ij;kl} \hat{c}_i^{\dagger} \hat{c}_j^{\dagger} \hat{c}_k \hat{c}_l$$

[A. Kitaev's talk]

[S. Sachdev: PRX **5**, 041025 (2015)]

- Solvable in the large *N* limit, Sachdev-Ye "spin liquid" ground state
- Nearly conformal symmetric at low temperature ("emergent …")
- Holographically corresponds to a quantum black hole?
- Experimental schemes have been proposed
- Realizes the Maldacena-Shenker-Stanford chaos bound $\lambda_{\mathrm{L}} = 2\pi k_{\mathrm{B}} T / \hbar$

Generalizations: *q*-fermion interactions "SYK$_q$", supersymmetric SYK, lattice of SYK lands; etc.

Classification and random matrix theory

$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1 \le a < b < c < d \le N} J_{abcd} \hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$$

SPT phase classification for class BDI:

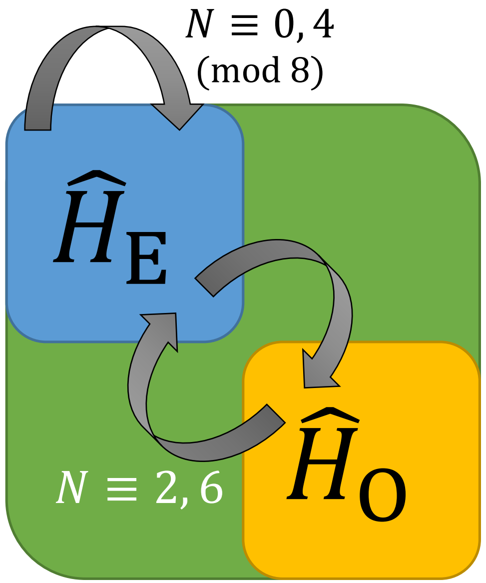
$\mathbb{Z} \to \mathbb{Z}_8$ due to interaction

[L. Fidkowski and A. Kitaev, PRB 2010, PRB 2011]

Introduce $N/2$ complex fermions $\hat{c}_j = \frac{(\hat{\chi}_{2j-1} + \mathrm{i}\hat{\chi}_{2j})}{\sqrt{2}}$

$\hat{\chi}_a \hat{\chi}_b \hat{\chi}_c \hat{\chi}_d$ respects the complex fermion parity

Even ($\hat{H}_{\mathrm{E}}$) and odd ($\hat{H}_{\mathrm{O}}$) sectors: $L = 2^{N/2-1}$ dimensions



N mod 8	0	2	4	6
η	-1	+1	+1	-1
$\hat{X}^2$	**+1**	+1	**-1**	-1
$\hat{X}$ maps H_{E} to	H_{E}	H_{O}	H_{E}	H_{O}
Class	**AI**	**A+A**	**AII**	**A+A**
Gaussian ensemble	**GOE**	**GUE**	**GSE**	**GUE**

$$\hat{X} = \hat{K} \prod_{j=1}^{N/2} (\hat{c}_j^\dagger + \hat{c}_j)$$

$$\hat{X} \hat{c}_j \hat{X} = \eta \hat{c}_j^\dagger$$

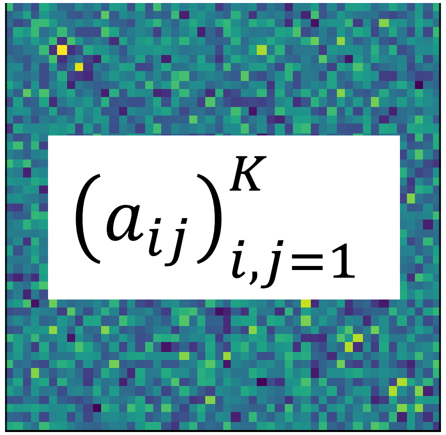
[You, Ludwig, and Xu, PRB 2017]

[Fadi Sun and Jinwu Ye, 1905.07694] for SYK$_q$, supersymmetric SYK

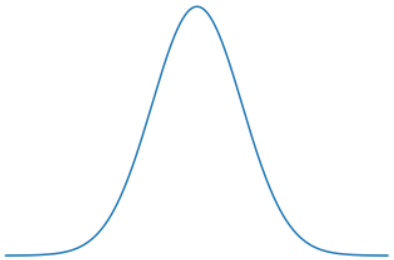
Sparse, but energy spectral statistics strongly resemble that of the corresponding (dense) Gaussian ensemble

[Cotler, …, MT, JHEP 2017]

Gaussian random matrices



$a_{ij} = a_{ji}^*$



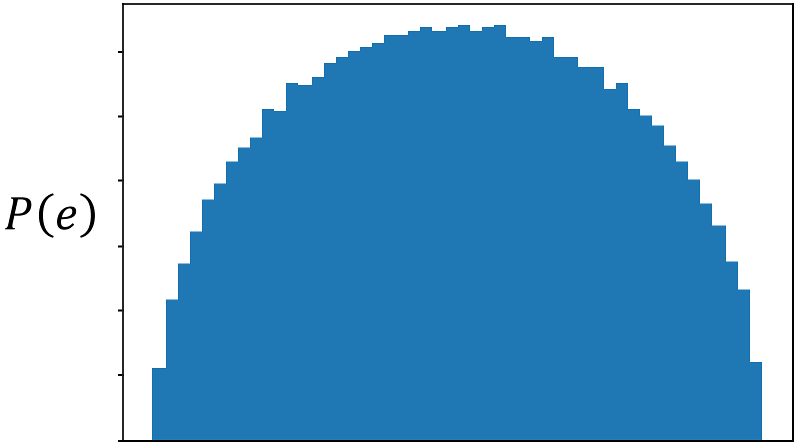
Gaussian distribution

$$\text{Density} \propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}\left|a_{ij}\right|^2\right)$$

[F. J. Dyson, J. Math. Phys. **3**, 1199 (1962)]

Real (β=1): Gaussian Orthogonal Ensemble (GOE)
Complex (β=2): G. Unitary E. (GUE)
Quaternion (β=4): G. Symplectic E. (GSE)

Eigenvalue distribution: semi-circle law



$e_1 \leq e_2 \leq \cdots \leq e_L$
Δ: averaged level separation near e_j

Joint distribution

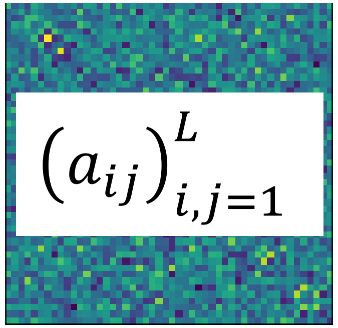
Level repulsion

$$p(e_1, e_2, \ldots, e_K) \propto \prod_{1\leq i<j\leq K}\left|e_i - e_j\right|^{\beta}\prod_{i=1}^{K} e^{-\beta K e_i{}^2/4}$$

- $P(s)$: Distribution of normalized level separation $s = \frac{e_{j+1}-e_j}{\Delta(\bar{e})}$

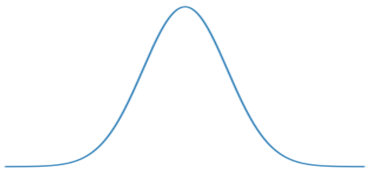
GOE/GUE/GSE: $P(s) \propto s^{\beta}$ at small s, has e^{-s^2} tail
Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

Gaussian random matrices



$(a_{ij})_{i,j=1}^{L}$

$a_{ij} = a_{ji}^*$



Gaussian distribution

Density $\propto e^{-\frac{\beta K}{4}\mathrm{Tr}H^2} = \exp\left(-\frac{\beta K}{4}\sum_{i,j}^{K}|a_{ij}|^2\right)$

Real (β=1): Gaussian Orthogonal Ensemble (GOE)
Complex (β=2): G. Unitary E. (GUE)
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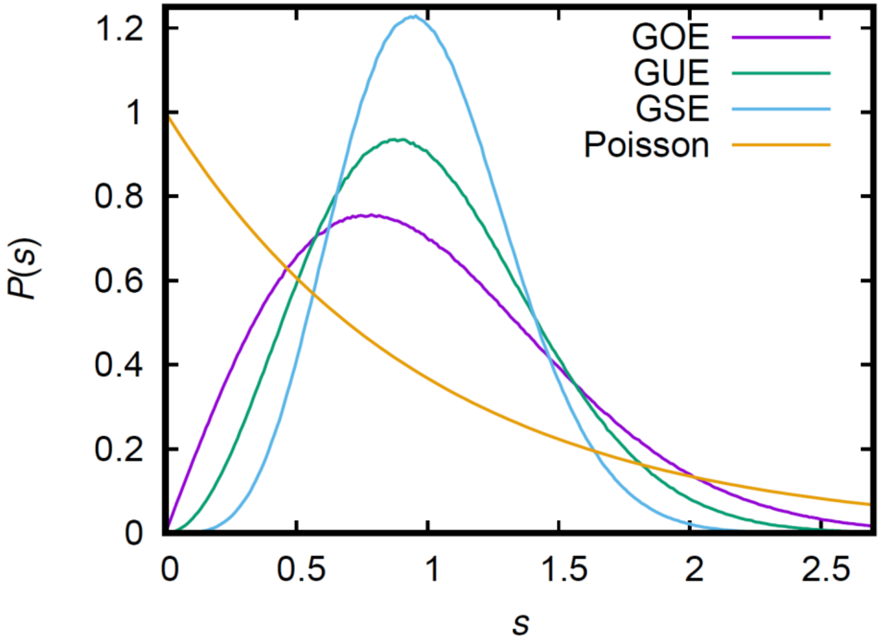

Level repulsion

Joint distribution

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- $P(s)$: Distribution of normalized level separation $s = \frac{e_{j+1}-e_j}{\Delta(\bar{e})}$

  GOE/GUE/GSE: $P(s) \propto s^\beta$ at small s, has e^{-s^2} tail
  Uncorrelated: $P(s) = e^{-s}$ (Poisson distribution)

- $\langle r \rangle$: Average of neighboring gap ratio

$$r = \frac{\min(e_{i+1} - e_i\,,\ e_{i+2} - e_{i+1})}{\max(e_{i+1} - e_i\,,\ e_{i+2} - e_{i+1})}$$

	Uncorrelated	GOE	GUE	GSE
$\langle r \rangle$	2log 2 – 1 = 0.38629…	0.5307(1)	0.5996(1)	0.6744(1)

[Y. Y. Atas *et al.* PRL 2013]

➔ SYK model results: indistinguishable from corresponding Gaussian ensemble

Density of states

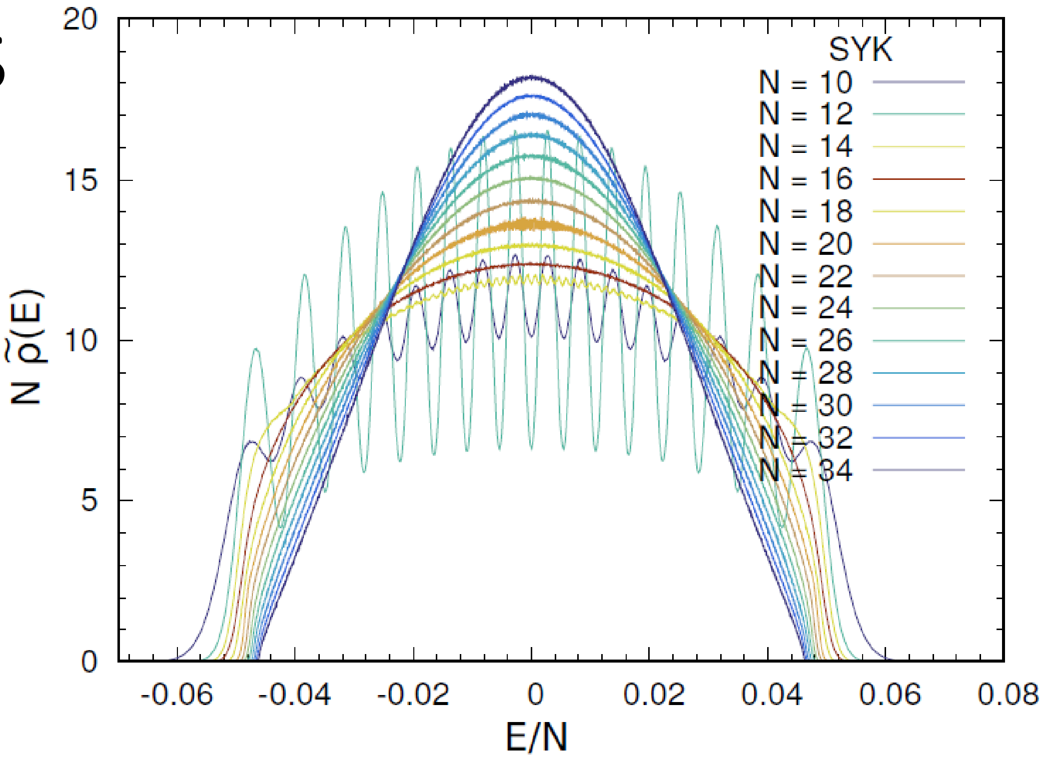


Figure 15. Normalized density of states $\tilde{\rho}(E)$ for the SYK model with $N = 10, 12, \ldots, 34$. The bin width is $10^{-3}J$. Notice that the energy is measured in units of NJ. The numbers of samples are 21600000 ($N = 10$), 10800000 ($N = 12$), 5400000 ($N = 14$), 1200000 ($N = 16$), 600 000 ($N = 18$), 240 000 ($N = 20$), 120 000 ($N = 22$), 48 000 ($N = 24$), 10 000 ($N = 26$), 3 000 ($N = 28$), 1 000 ($N = 30$), 516 ($N = 32$), 90 ($N = 34$).

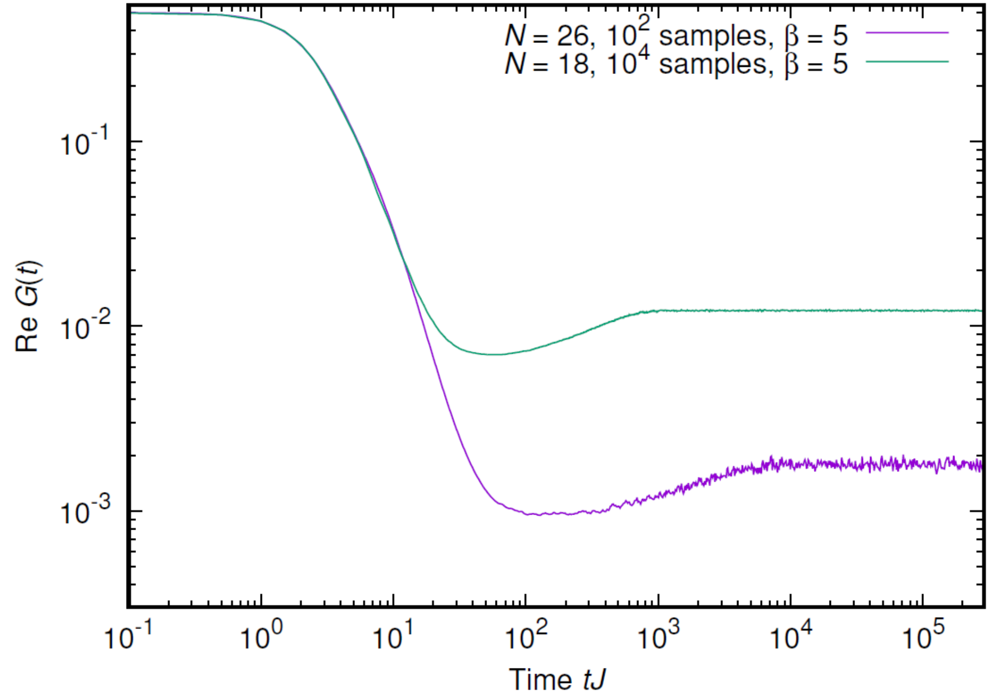
[Cotler, MT et al., JHEP05(2017)118]

[Cotler, MT et al., JHEP05(2017)118]

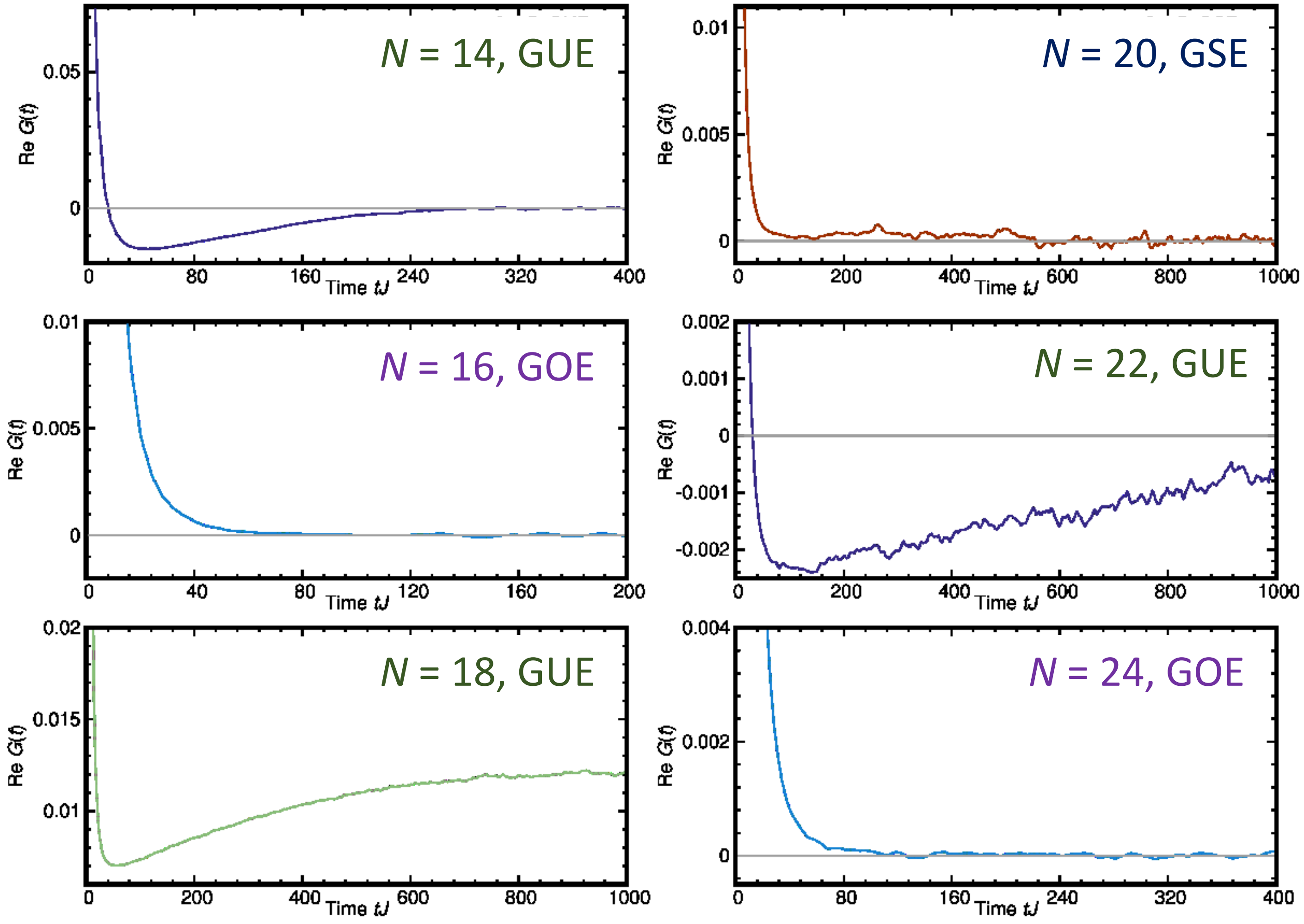
Correlation function

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta)}\sum_{m,n} e^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 e^{i(E_m-E_n)t}$$

Dip-ramp-plateau structure for $N \equiv 2 \pmod 8$



N mod 8	0	2	4	6
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
$\langle \mathrm{even}|\chi|\mathrm{odd}\rangle$		finite		0
Gaussian ensemble	GOE	GUE	GSE	GUE

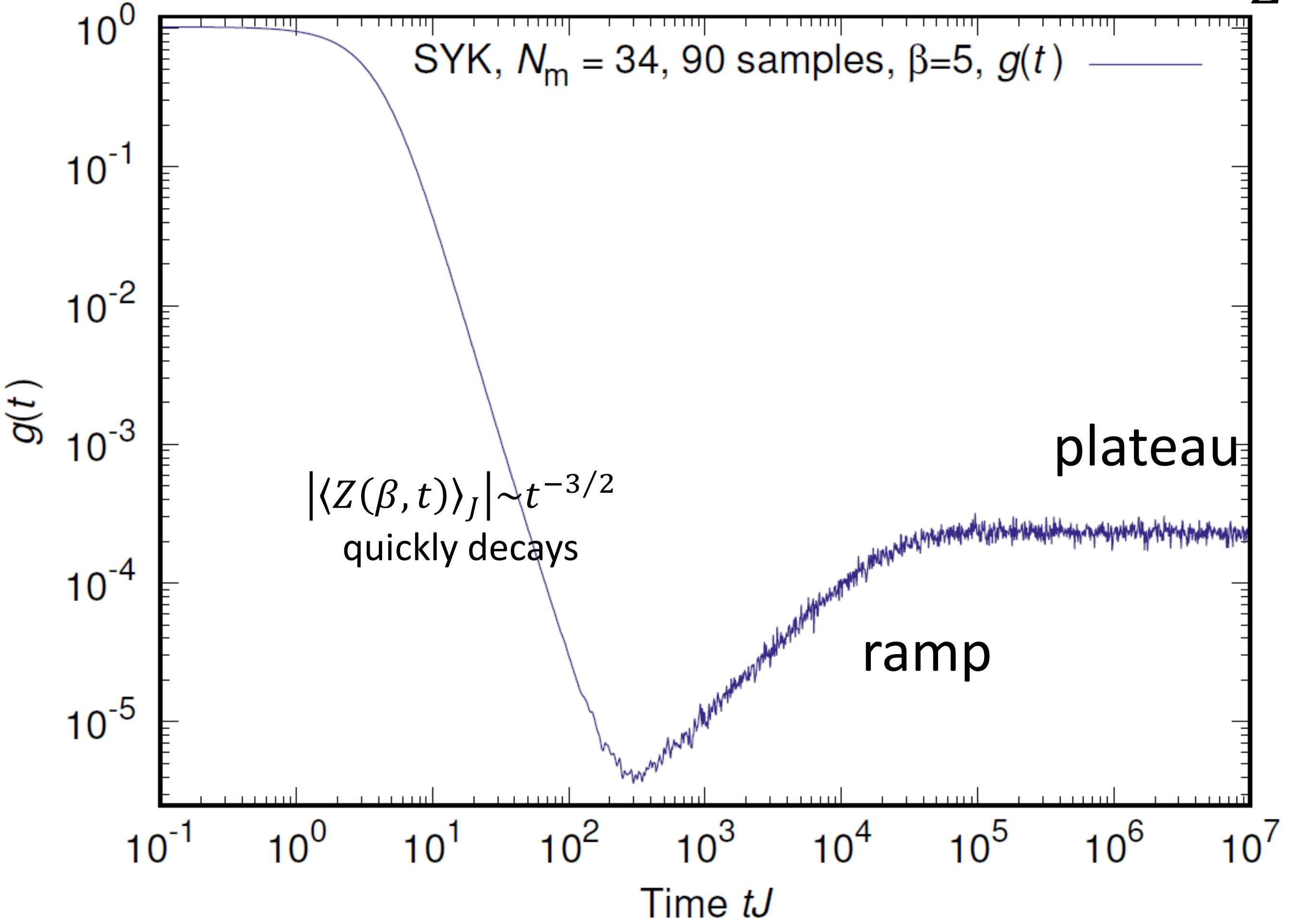


Spectral form factor

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta, t=0)} \sum_{m,n} \mathrm{e}^{-\beta E_m} |\langle m|\hat{\chi}_a|n\rangle|^2 \mathrm{e}^{\mathrm{i}(E_m - E_n)t}$$

$$g(\beta, t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2 = \frac{1}{Z(\beta,t=0)^2} \sum_{m,n} \mathrm{e}^{-\beta(E_m+E_n)} \mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

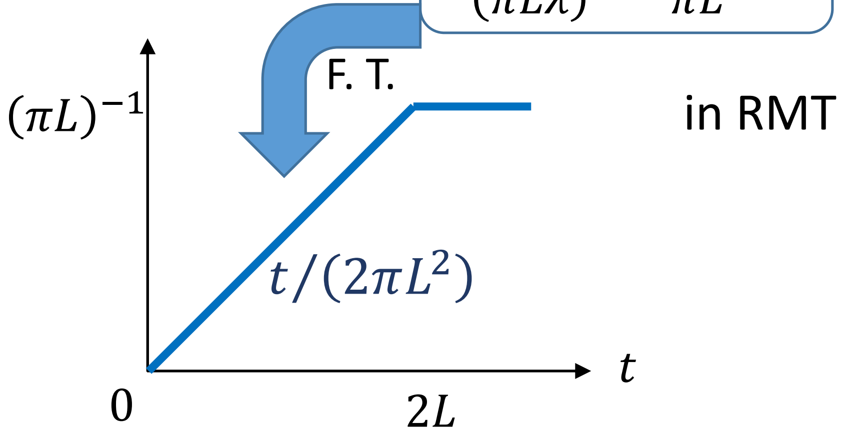
$$Z(\beta,t) = Z(\beta + \mathrm{i}t) = \mathrm{Tr}\left(\mathrm{e}^{-\beta\hat{H} - \mathrm{i}\hat{H}t}\right)$$



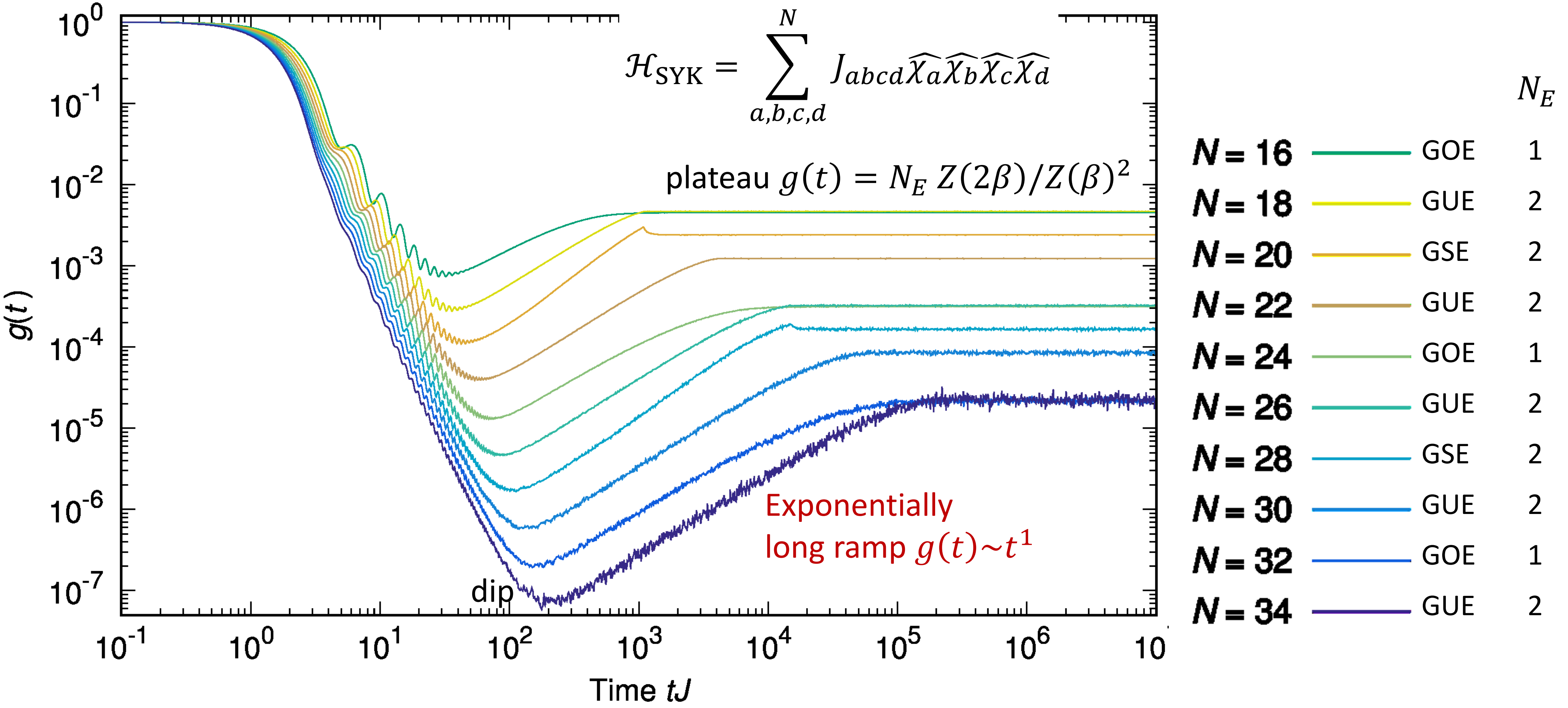
$$g_c(\beta,t) = \frac{\langle |Z(\beta,t)|^2\rangle_J - |\langle Z(\beta,t)\rangle_J|^2}{\langle Z(\beta)\rangle_J^2}$$

$$\sim \iint d\lambda_1 d\lambda_2 \langle \delta\rho(\lambda_1)\delta\rho(\lambda_2)\rangle e^{\mathrm{i}t(\lambda_1-\lambda_2)}$$

$$R(\lambda) = \langle \delta\rho(\lambda_1)\delta\rho(\lambda_1-\lambda)\rangle = -\frac{\sin^2 L\lambda}{(\pi L\lambda)^2} + \frac{1}{\pi L}\delta(\lambda)$$



N dependence of the spectral form factor



[Cotler, MT et al., JHEP05(2017)118]; for dip time dependence on N see [Gharibyan-Hanada-Shenker-MT, JHEP07(2018)124].

Characterization of quantum many-body chaos

- Random-matrix like energy level correlation

- Exponential Lyapunov growth of out-of-time-order correlators (OTOC)

$$\langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle \sim C + \#\, e^{2\lambda_\mathrm{L} t}$$

Example: the Sachdev-Ye-Kitaev model

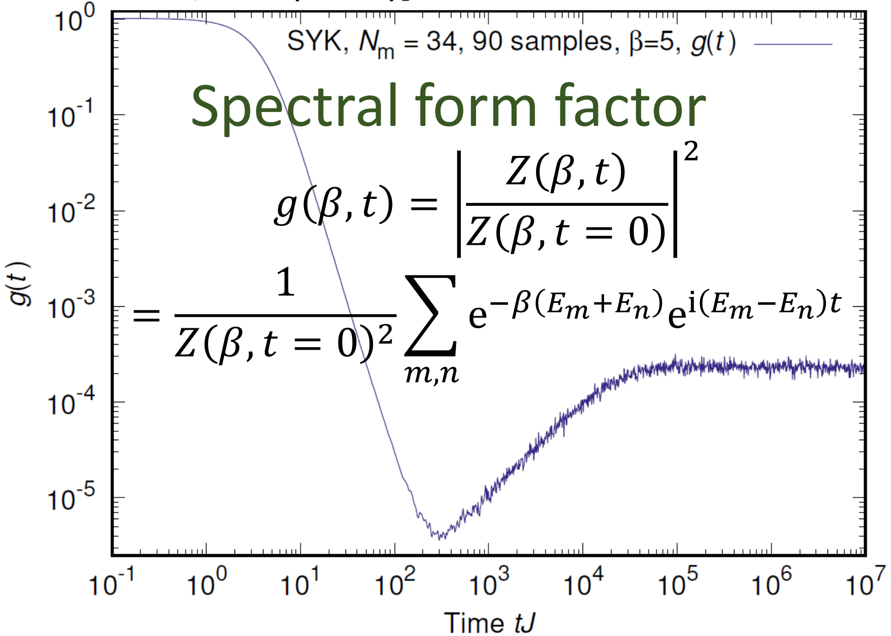
$$\hat{H} = \frac{\sqrt{3!}}{N^{3/2}} \sum_{1\le a<b<c<d\le N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d$$

$\{\hat{\chi}_a, \hat{\chi}_b\} = \delta_{ab}$

J_{abcd} : Gaussian random ($\langle {J_{abcd}}^2 \rangle = J^2 = 1$)

[Kitaev 2015]

[Cotler, MT et al., JHEP **1705**, 118 (2017)]



Spectral form factor

$$g(\beta,t) = \left|\frac{Z(\beta,t)}{Z(\beta,t=0)}\right|^2$$

$$= \frac{1}{Z(\beta,t=0)^2}\sum_{m,n} \mathrm{e}^{-\beta(E_m+E_n)}\mathrm{e}^{\mathrm{i}(E_m-E_n)t}$$

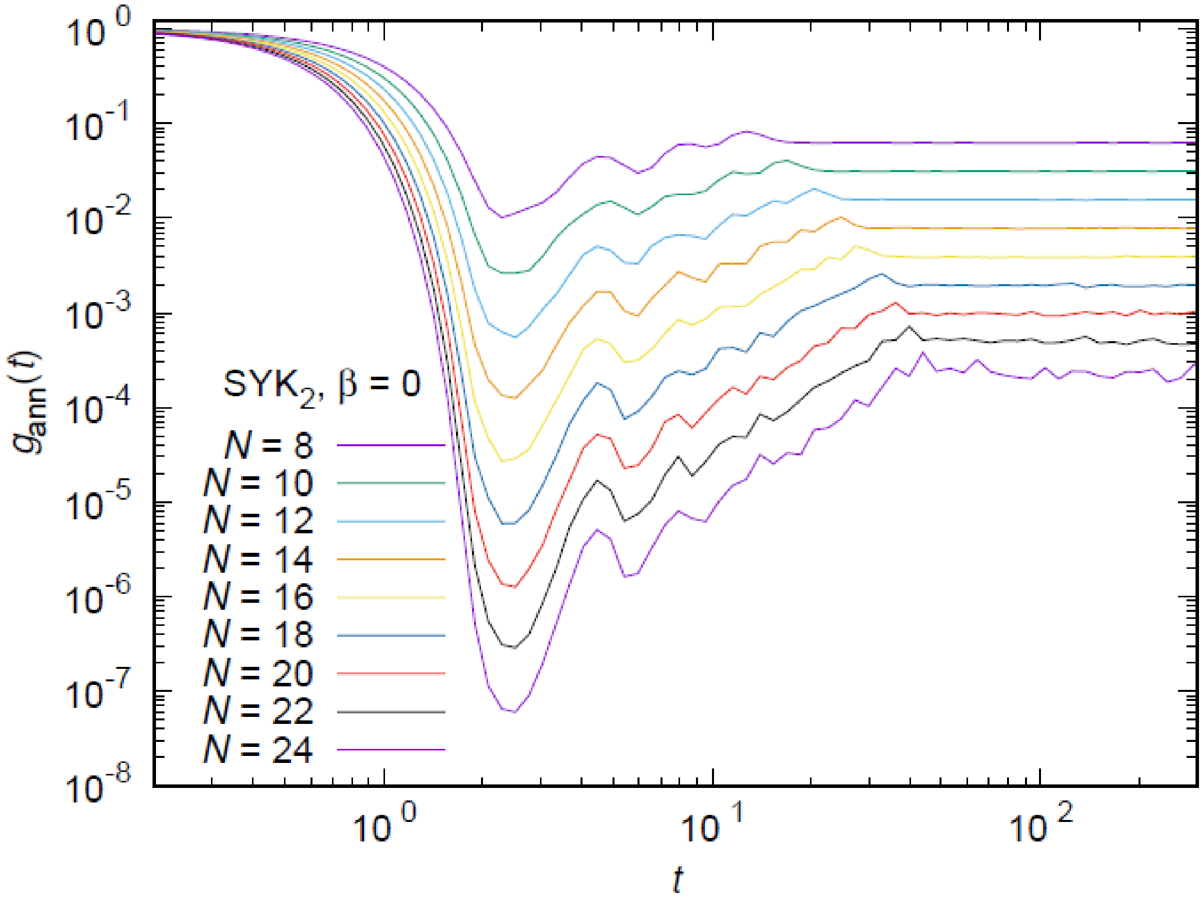
N mod 8	RMT
0	GOE
2	GUE
4	GSE
6	GUE

Lyapunov exponent

$$\lambda_\mathrm{L} = \frac{2\pi k_\mathrm{B} T}{\hbar}$$ in low T limit

(Maldacena-Shenker-Stanford chaos bound)

Note: dip-ramp-plateau structure does not require chaos



"Randomness and Chaos in Qubit Models"
Pak Hang Chris Lau, Chen-Te Ma,
Jeff Murugan, and MT, Phys. Lett. B in press
(arXiv:1812.04770)

Plan of the talk

Characterization of many-body quantum chaos

The Sachdev-Ye-Kitaev model

The quantum Lyapunov spectrum

The singular values of two-point correlators

Summary

We propose two new characterizations of quantum chaos

- Quantum Lyapunov spectrum: Quantum version of finite-time Lyapunov spectrum

$\hat{M}_{ab}(t)$: (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

$\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$ for singular values $\{s_k(t)\}_{k=1}^{N}$ of $N \times N$ matrix $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$.

JHEP04(2019)082 (arXiv:1809.01671)

- Two-point correlations:

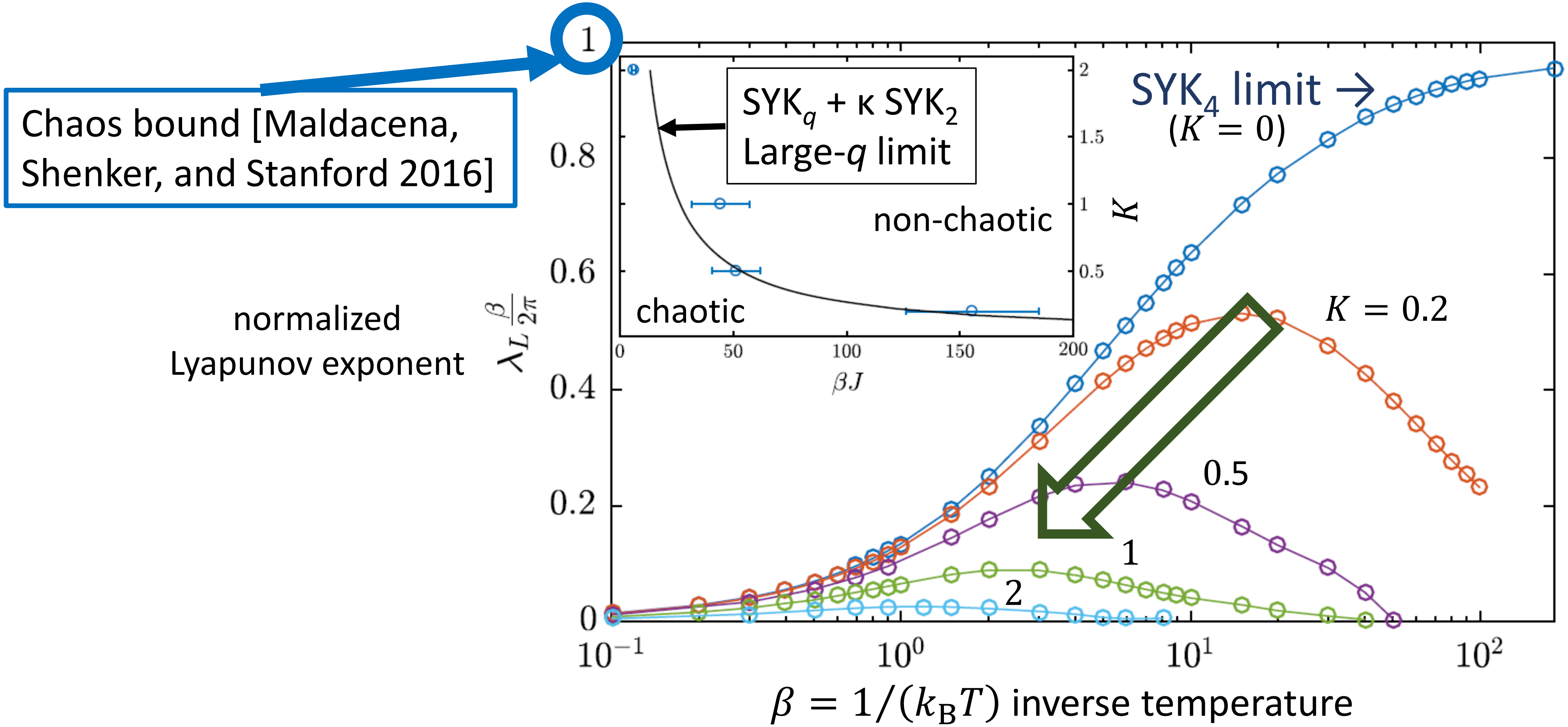
$G_{ab}^{(\phi)} = \langle\phi|\hat{O}_a(t)\hat{O}_b(0)|\phi\rangle$ as matrix, log (singular values)

arXiv:1902.11086

Modified SYK model: Large-N calculation for OTOC

$$\hat{H} = \overbrace{\sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d}^{\text{SYK}_4} + i \overbrace{\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b}^{\text{SYK}_2}$$

K_{ab}: standard deviation $\frac{K}{\sqrt{N}}$



A. M. Garcia-Garcia, B. Loureiro, A. Romero-Bermudez, and MT, PRL **120**, 241603 (2018)

Deviation from the chaos bound as SYK$_2$ component is introduced

Quantum Lyapunov spectrum

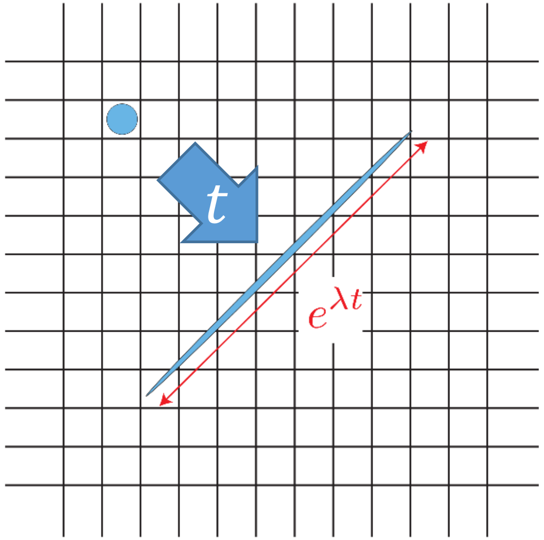
OTOCs have been intensively studied:

$$F(t) = \langle \hat{W}^\dagger(t) \hat{V}^\dagger(0) \hat{W}(t) \hat{V}(0) \rangle$$

- Measurement protocols
  - [B. Swingle, G. Bentsen, M. Schleier-Smith, P. Hayden, PRB **94**, 040302 (2016)] and experimental proposal papers for the SYK model
- Experimental measurements
  - trapped ions [M. Gärttner et al. Nat. Phys. **13**, 781 (2017) 1608.08938]
  - NMR [J. Li et al. PRX **7**, 031011 (2017) 1609.01246]
- Quantum information (scrambling, …)
- Many-body localization
- Fluctuation-dissipation theorem
  - [N. Tsuji, T. Shitara, and M. Ueda, PRE **97**, 012101 (2018)]

Q. Which operators should we use?

Lyapunov growth of phase space



Coarse-grained phase space

- Just one direction?
- If more than one, what are relations between λ?

[M. Hanada, H. Shimada, and M. Tezuka, PRE **97**, 022224 (2018)]

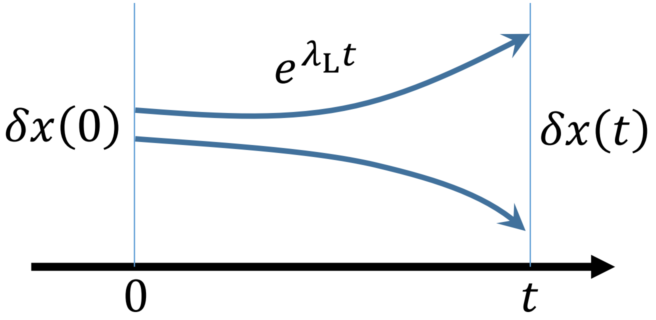
Observation for classical chaos

Classical system with K degrees of freedom

Deviation at t initial infinitesimal deviation

$$\delta x_i(t) = M_{ij}\delta x_j(0)$$

$$M_{ij} = \frac{\delta x_i(t)}{\delta x_j(0)} = \{x(t), p(0)\}_{\mathrm{PB}}$$



(Usually $t \to \infty$ limit is taken for obtaining λ_L)

$$L = \left(\frac{\delta x_i(t)}{\delta x_j(0)}\right)^2 \qquad \{x(t), p(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x(t)}{\partial x(0)}\right)^2 \to e^{2\lambda_L t}$$

We consider finite t

> Singular values of M_{ij}: $\{a_k(t)\}_{k=1}^{K}$
> Time-dependent Lyapunov spectrum
> $$\left\{\lambda_k(t) = \frac{\log a_k(t)}{t}\right\}_{k=1,2,\ldots,K}$$
> obeys random matrix-like statistics

in several chaotic systems

- Logistic map
- Lorenz attractor
- D0 brane matrix model (without fermions)

Quantum Lyapunov spectrum

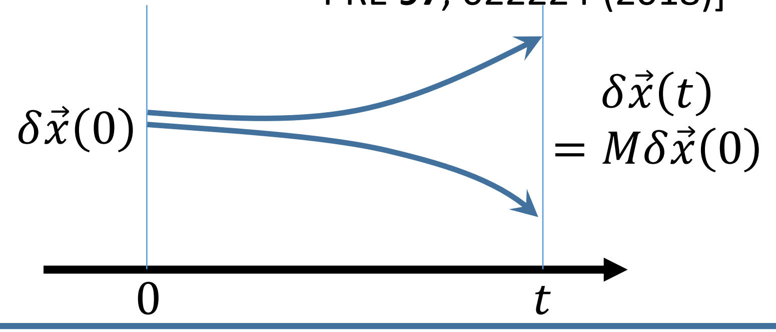
Gharibyan, Hanada, Swingle, and MT,
JHEP04(2019)082 (arXiv:**1809.01671**)

Finite-time **classical Lyapunov spectrum**: obeys RMT statistics for chaos

[Hanada, Shimada, and MT: PRE **97**, 022224 (2018)]

Singular values of $M_{ij} = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)$ **at finite t**: $\{s_k(t)\} = \{e^{\lambda_k t}\}$

$$L = \{x_i(t), p_j(0)\}_{\mathrm{PB}}{}^2 = \left(\frac{\partial x_i(t)}{\partial x_j(0)}\right)^2 \to e^{2\lambda_{\mathrm{L}} t} \text{ for large } t$$



OTOC: $C_T(t) = \left\langle \left| [\hat{W}(t), \hat{V}(t=0)] \right|^2 \right\rangle = \langle \hat{W}^\dagger(t)\hat{V}^\dagger(0)\hat{W}(t)\hat{V}(0)\rangle + \cdots$

Quantum Lyapunov spectrum: Define $\hat{M}_{ab}(t)$ as (anti)commutator of $\hat{O}_a(t)$ and $\hat{O}_b(0)$

$$\hat{L}_{ab}(t) = \left[\hat{M}(t)^\dagger \hat{M}(t)\right]_{ab} = \sum_{j=1}^{N} \hat{M}_{ja}(t)^\dagger \hat{M}_{jb}(t)$$

For $N \times N$ matrix $\langle \phi | \hat{L}_{ab}(t) | \phi \rangle$, obtain singular values $\{s_k(t)\}_{k=1}^{N}$.

The Lyapunov spectrum is defined as $\left\{\lambda_k(t) = \frac{\log s_k(t)}{2t}\right\}$.

Quantum Lyapunov spectrum for SYK model + modification

$$\hat{H} = \sum_{1\le a<b<c<d}^{N} J_{abcd}\hat{\chi}_a\hat{\chi}_b\hat{\chi}_c\hat{\chi}_d + i\sum_{1\le a<b}^{N} K_{ab}\hat{\chi}_a\hat{\chi}_b$$

J_{abcd}: s. d. $= \frac{\sqrt{6}}{N^{3/2}}$

K_{ab}: s. d. $= \frac{K}{\sqrt{N}}$

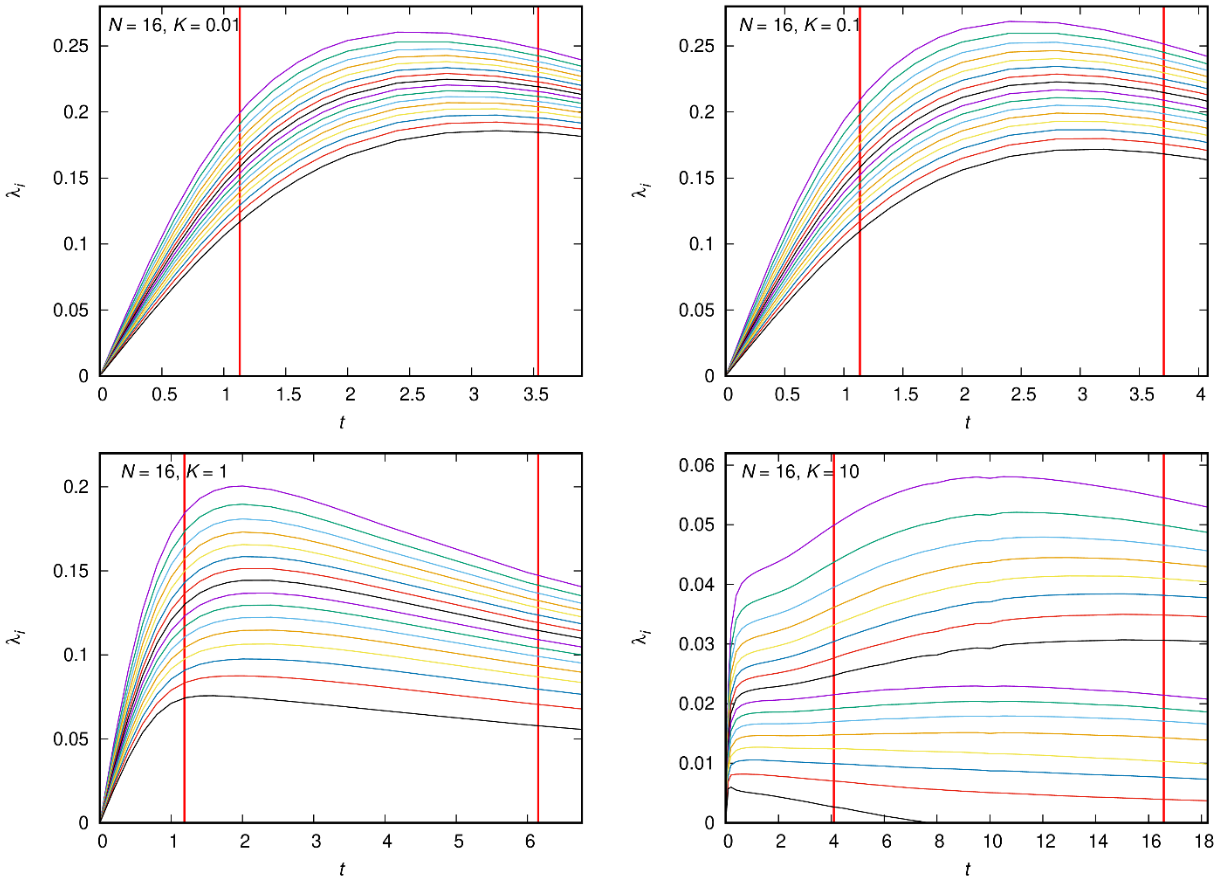
- Define $\hat{L}_{ab}(t) = \sum_{j=1}^{N} \hat{M}_{ja}(t)\hat{M}_{jb}(t)$ for time-dependent anticommutator $\hat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$.
- Obtain the singular values $\{a_k(t)\}_{k=1}^{K}$ of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$
- Quantum Lyapunov spectrum: $\left\{\lambda_k(t) = \frac{\log a_k(t)}{2t}\right\}_{k=1,2,\ldots,K}$ (also dependent on state ϕ)

Other possibilities: see Rozenbaum-Ganeshan-Galitski, 1801.10591; Hallam-Morley-Green: 1806.05204

Gharibyan, Hanada, Swingle, and MT,
JHEP04(2019)082 (arXiv:**1809.01671**)

Full Lyapunov spectrum

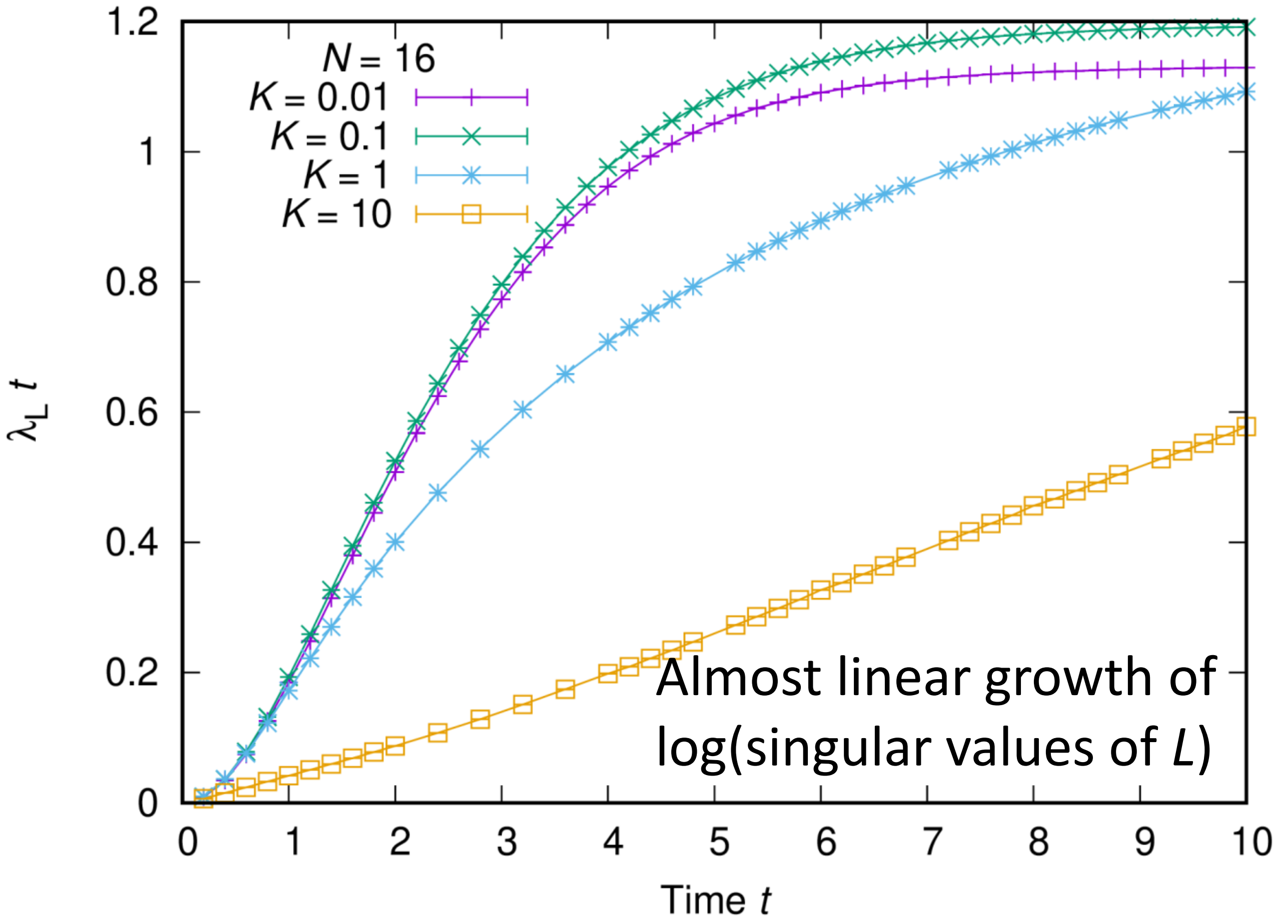
Sample- and state-averaged



Close to constant between red lines
(20 % and 80 % of the saturated value of $\lambda_N t$)

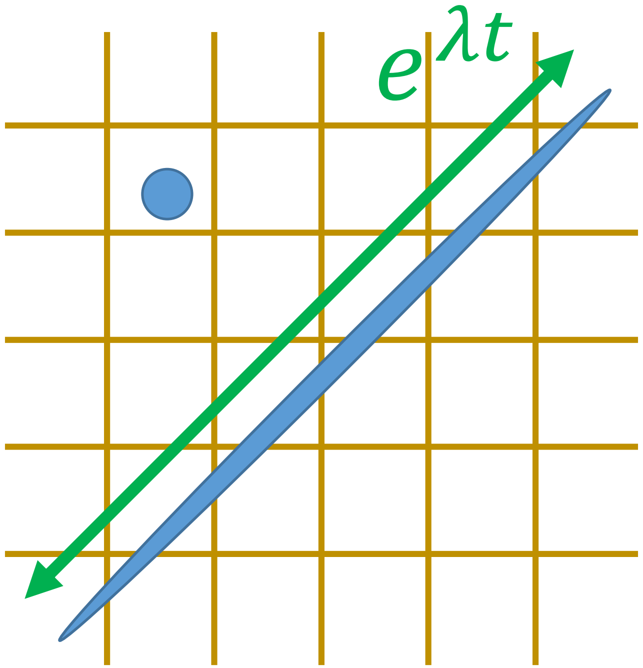
Gharibyan, Hanada, Swingle, and MT,
JHEP04(2019)082 (arXiv:**1809.01671**)

Growth of (largest Lyapunov exponent)*time



Kolmogorov-Sinai entropy

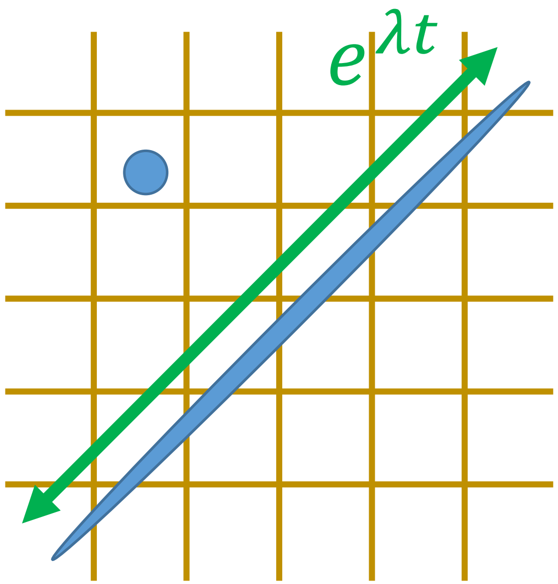
Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t



Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

Kolmogorov-Sinai entropy vs entanglement entropy production

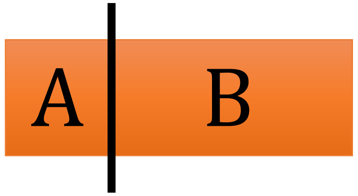
Coarse-grained entropy
= log(# of cells covering the region)
~ (sum of positive λ) t



Kolmogorov-Sinai entropy h_{KS}
= (sum of positive λ)
= entropy production rate

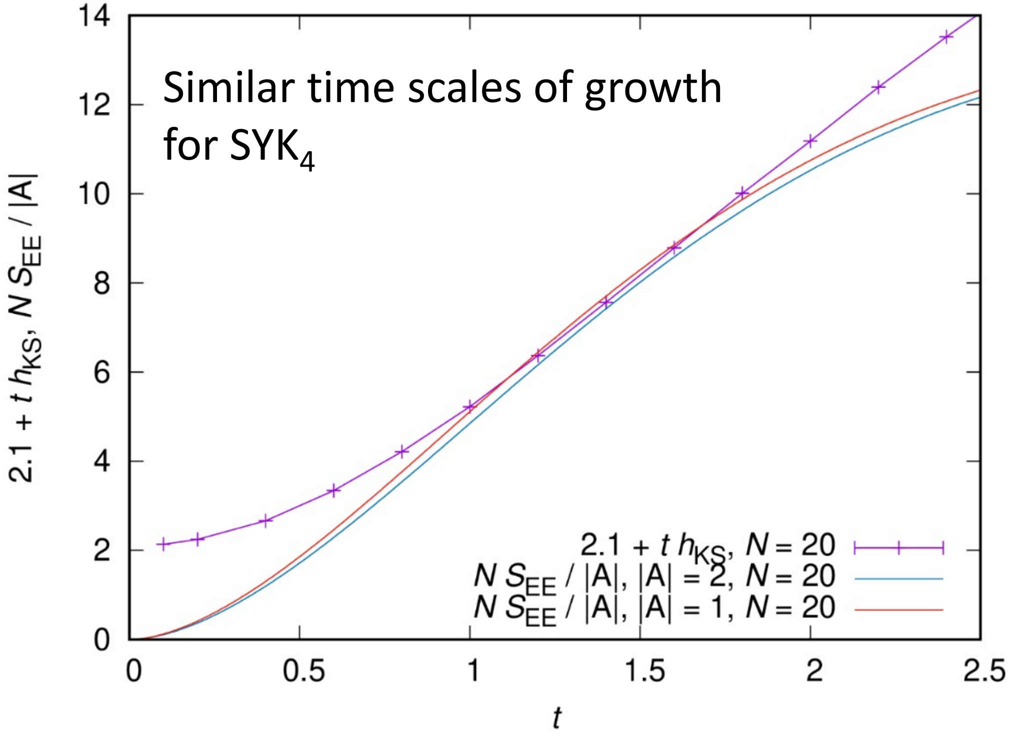
Initial state with $S_{\mathrm{EE}} = 0$:
$|\psi(t=0)\rangle = |\underbrace{000\ldots000}_{N/2}\rangle$ in the complex fermion basis

$$\hat{c}_j = \frac{(\chi_{2j-1} + \mathrm{i}\chi_{2j})}{\sqrt{2}}$$



$$\rho_{\mathrm{A}}(t) = \mathrm{Tr}_{\mathrm{B}}\, \rho(t),\ \rho(t) = |\psi(t)\rangle\langle\psi(t)|$$

$$S_{\mathrm{EE}}(t) = -\mathrm{Tr}\, \rho_{\mathrm{A}}(t) \log\big(\rho_{\mathrm{A}}(t)\big)$$



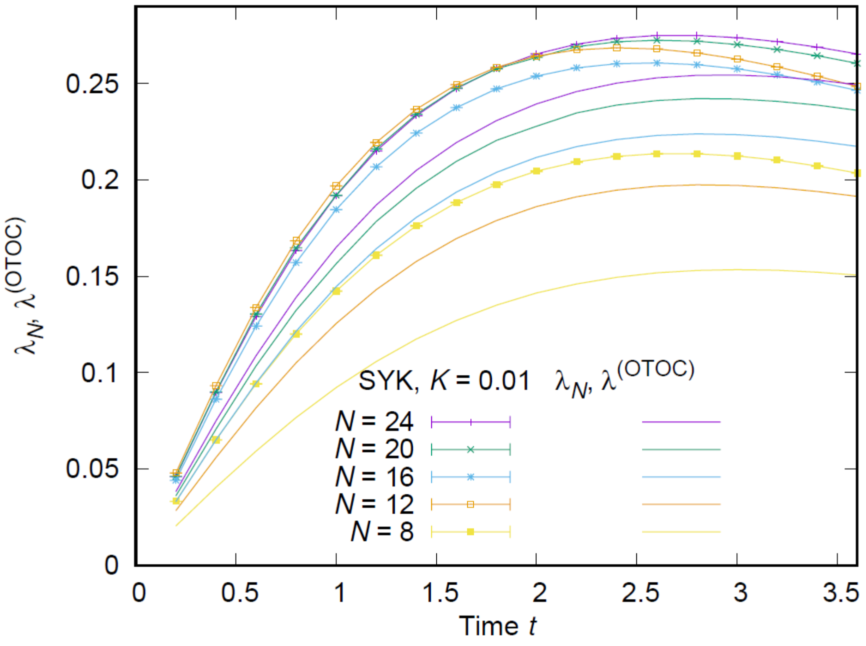
arXiv:**1809.01671**

Fastest entropy production?

SYK$_4$ limit

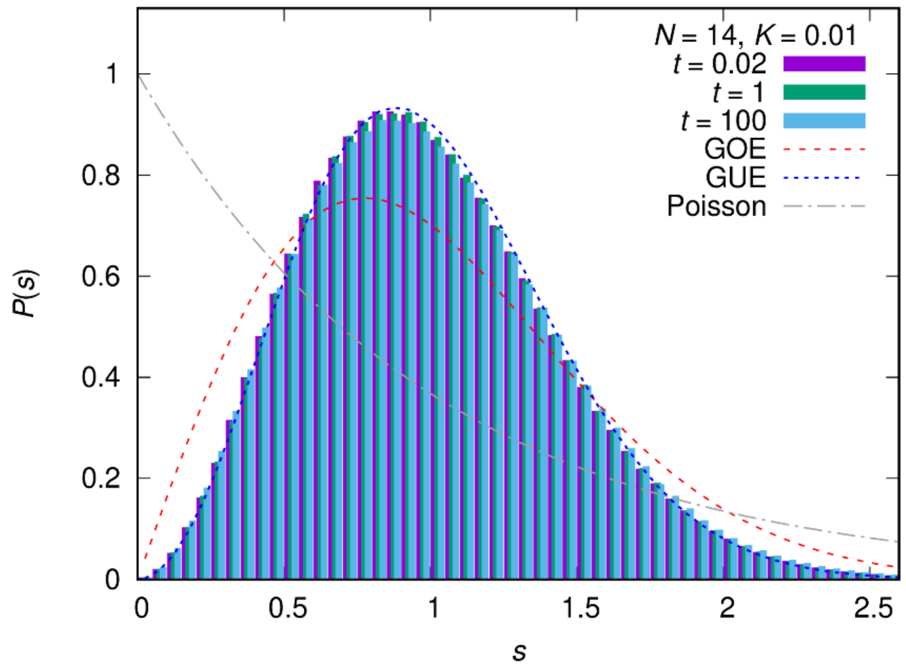
- λ_N and $\lambda_{\mathrm{OTOC}} = \frac{1}{2t}\log\left(\frac{1}{N}\sum_{i=1}^{N} e^{2\lambda_i t}\right)$ approach each other; difference decreases as $1/N$
- Same for λ_N and λ_1:

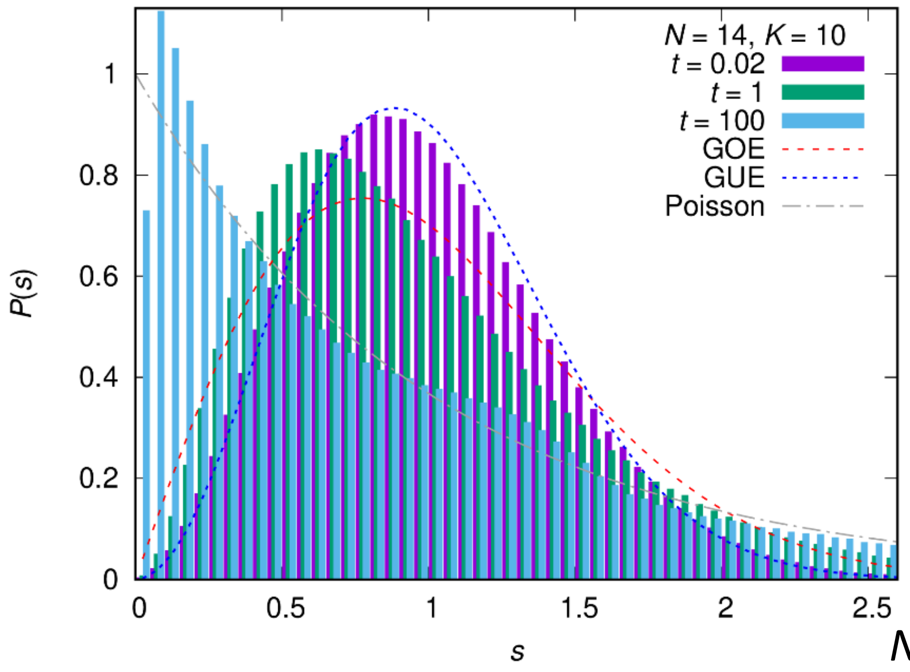
  all exponent → single peak
- All saturate the MSS bound at strong coupling (low T) limit
- Growth rate of entanglement entropy $\sim h_{\mathrm{KS}} =$ sum of positive (all) λ_i



➔ [conjecture] SYK model: not only the fastest scramblers, but also fastest entropy generators

Spectral statistics of quantum Lyapunov spectrum: SYK





arXiv:**1809.01671**

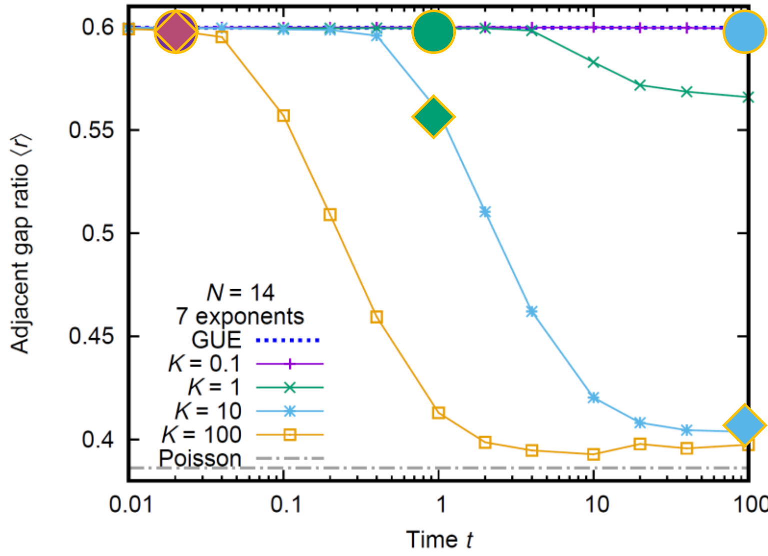
Energy eigenstates

$N/2$ larger exponents

$K = 0.01$ (●):

Remains GUE for long time

Exponents are nearly constant until the singular values of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$ saturate: Lyapunov growth



$K = 10$ (◆):

Approaches Poisson

$\langle r \rangle$: average of

$$\frac{\min(\epsilon_{i+1}-\epsilon_i\,,\epsilon_{i+2}-\epsilon_{i+1})}{\max(\epsilon_{i+1}-\epsilon_i\,,\epsilon_{i+2}-\epsilon_{i+1})}$$

(fixed-i unfolding: unfold each gap $\lambda_{i+1}-\lambda_i$ using its average)

The case of the random field XXZ model

$$\hat{H} = \sum_i^N \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_i^N h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

Many-body localization transition at $W = W_c \sim 3.6$

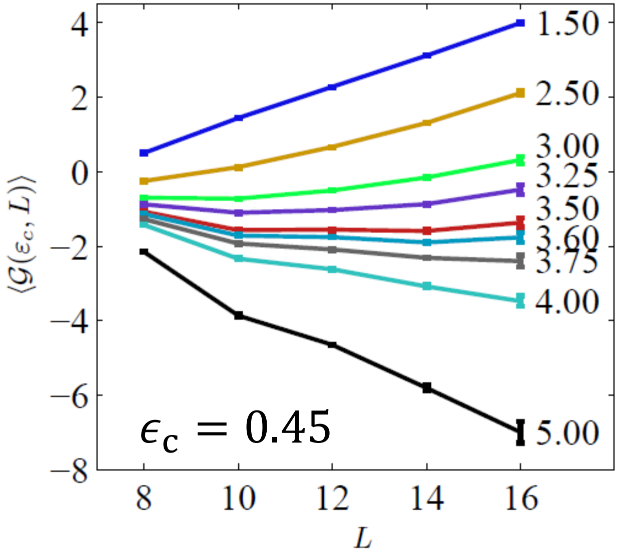
(though recently disputed; e.g. $W_c \geq 5$ proposed in E. V. H. Doggen et al., [1807.05051] using large systems with time-dependent variational principle & machine learning)

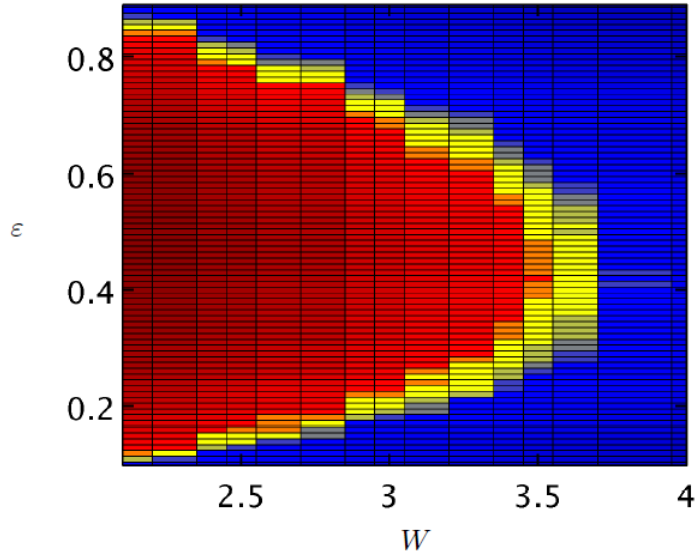
e.g. M. Serbyn, Z. Papic, and D. A. Abanin, Phys. Rev. X **5**, 041047 (2015) (arXiv:1507.01635)

Matrix element of local perturbation

$$\mathcal{G}(\varepsilon, L) = \ln \frac{|V_{n,n+1}|}{E'_{n+1} - E'_n}$$

Energy separation of neighboring energy eigenstates





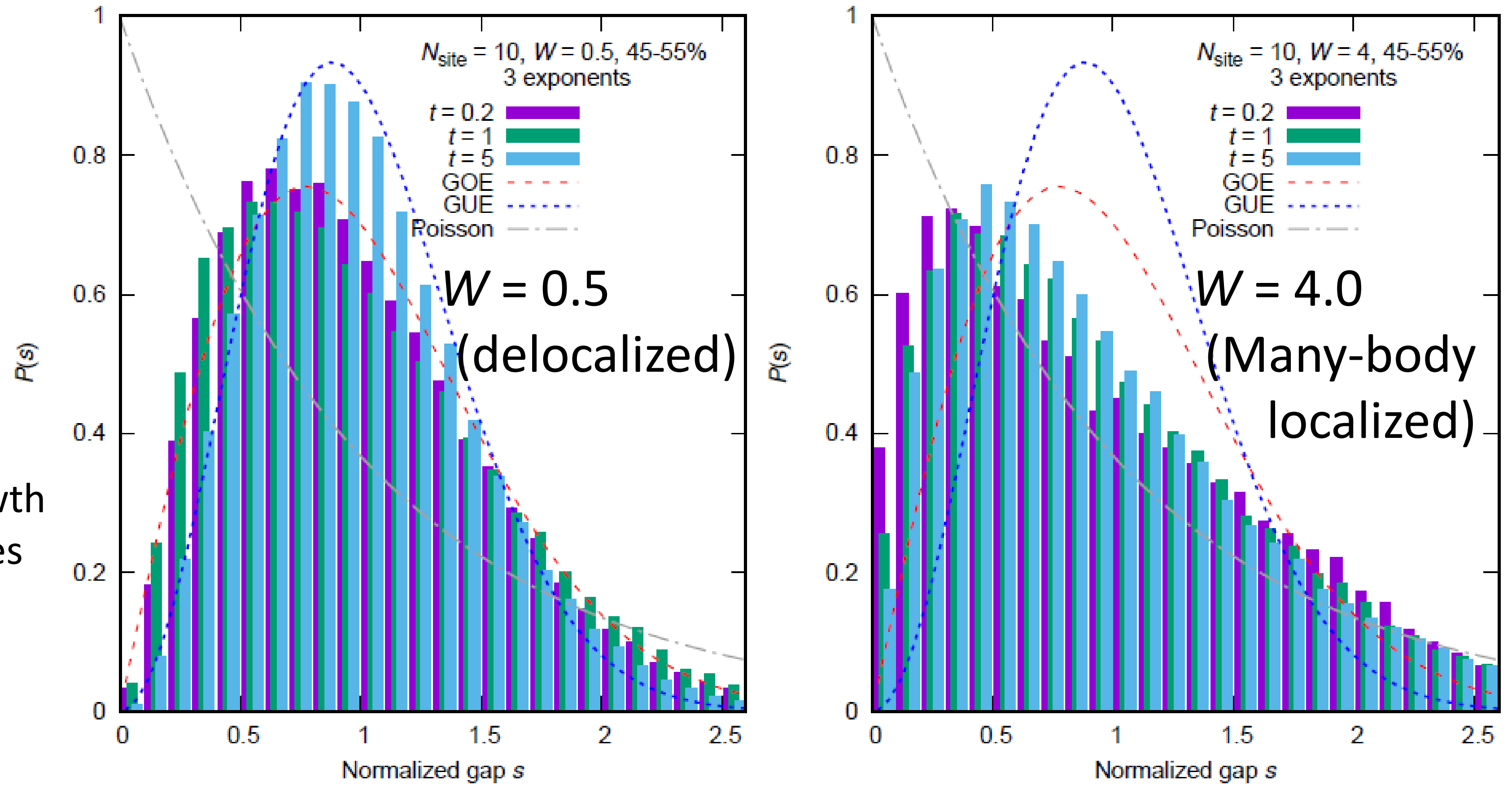
Spectral statistics of QLS for random field XXZ

arXiv:**1809.01671**

$$\hat{H} = \sum_{i}^{N} \hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_{i+1} + \sum_{i}^{N} h_i \widehat{S_i^z}$$

h_i: uniform distribution $[-W, W]$

$$\hat{M}_{ab}(t) = \left[\widehat{S_a^+}(t), \widehat{S_b^-}(0)\right]$$



- Exponential growth of the singular values is not observed, but the statistics approach GUE

Quantum Lyapunov spectrum distinguishes chaotic and non-chaotic phases

Plan of the talk

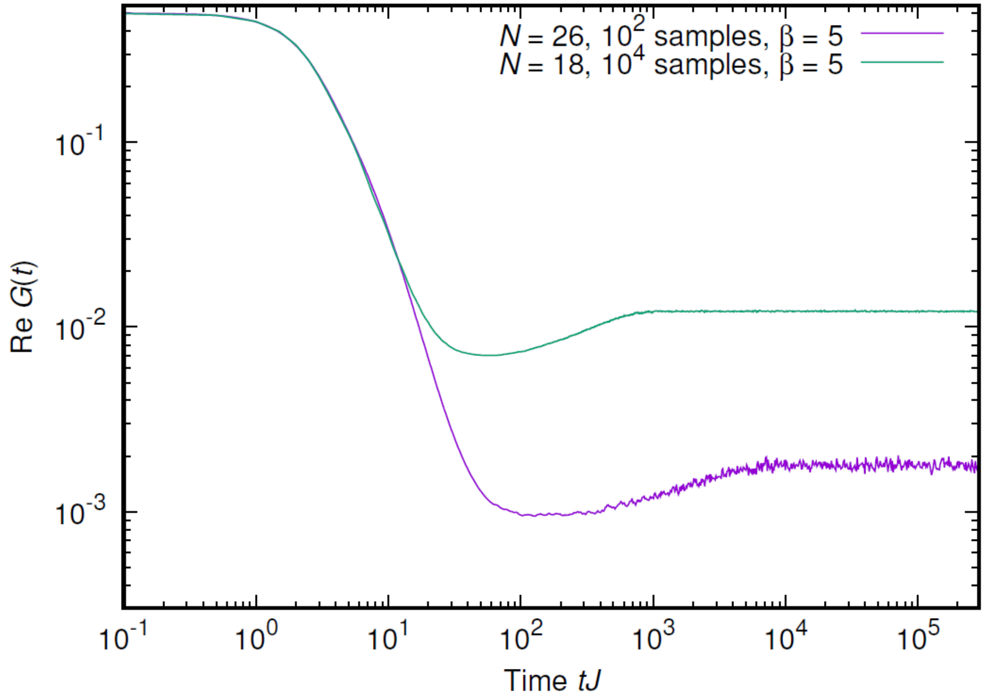
- Characterization of many-body quantum chaos
- The Sachdev-Ye-Kitaev model
- **The quantum Lyapunov spectrum**
- **The singular values of two-point correlators**
- Summary

[Cotler, MT et al., JHEP05(2017)118]

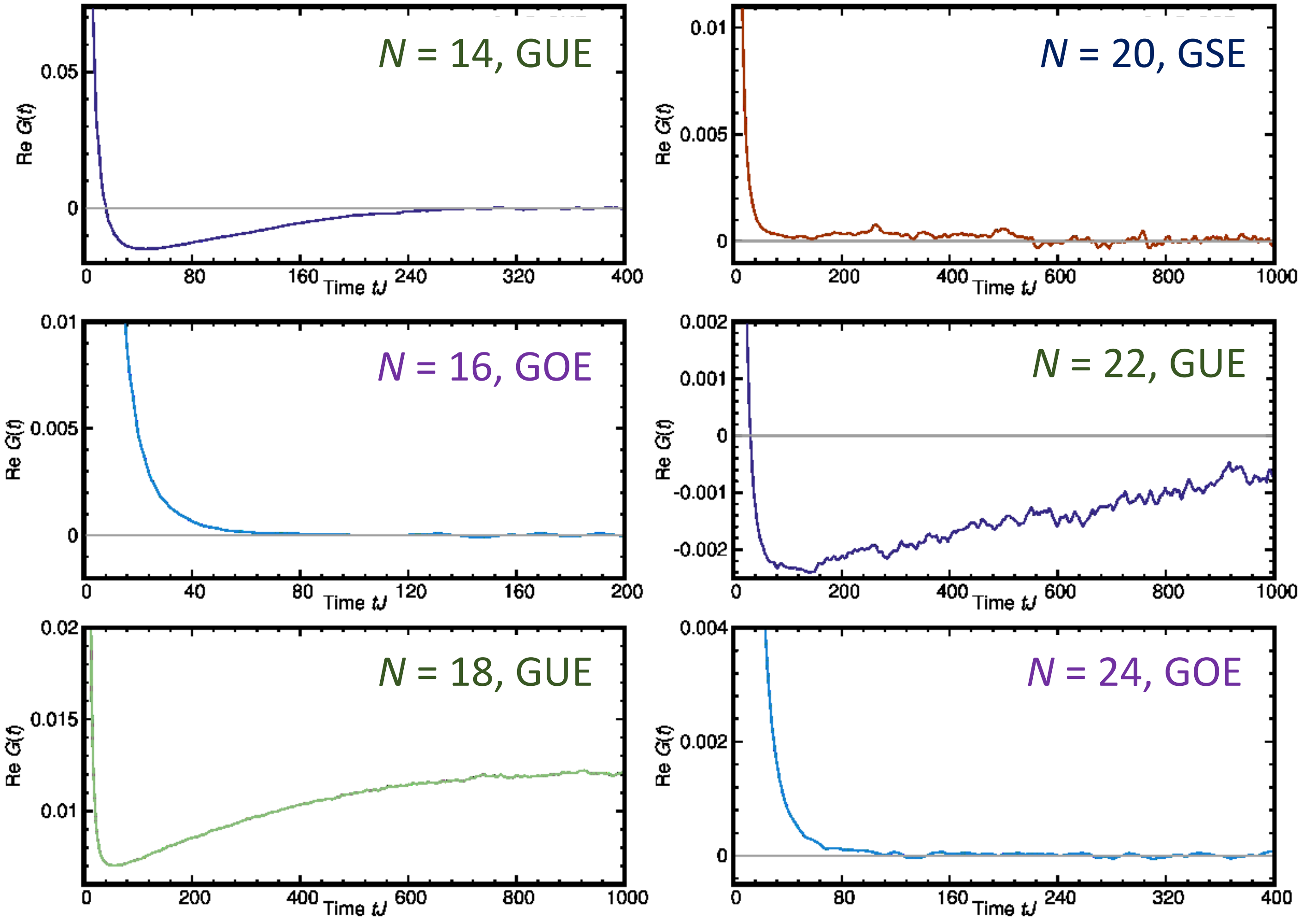
Correlation function

$$G(t) = \langle \hat{\chi}_a(t)\hat{\chi}_a(0)\rangle_\beta = \frac{1}{Z(\beta)}\sum_{m,n} e^{-\beta E_m}|\langle m|\hat{\chi}_a|n\rangle|^2 e^{i(E_m-E_n)t}$$

Dip-ramp-plateau structure for $N \equiv 2 \pmod 8$



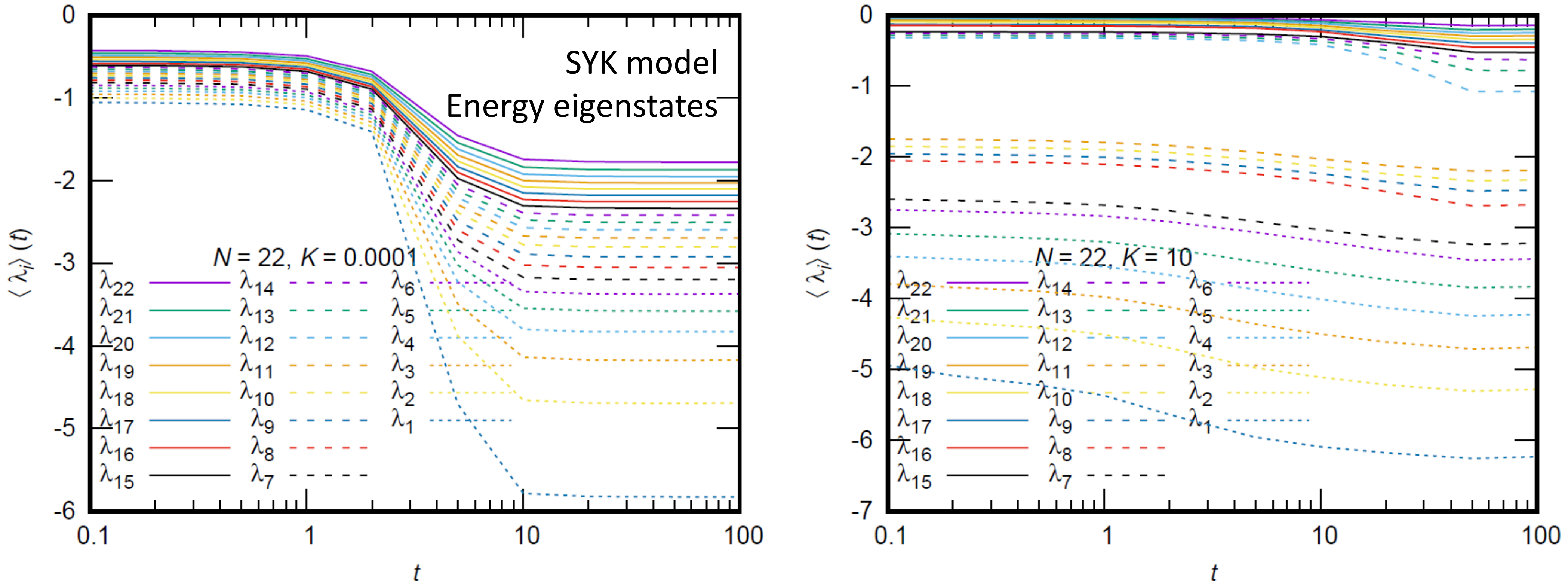
N mod 8	0	2	4	6
$\hat{X}$ maps H_E to	H_E	H_O	H_E	H_O
$\langle \text{even}|\chi|\text{odd}\rangle$		finite		0
Gaussian ensemble	GOE	GUE	GSE	GUE



Singular value statistics of two-point functions

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle \text{ for } \mathrm{SYK}_4 + K\, \mathrm{SYK}_2$$

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$



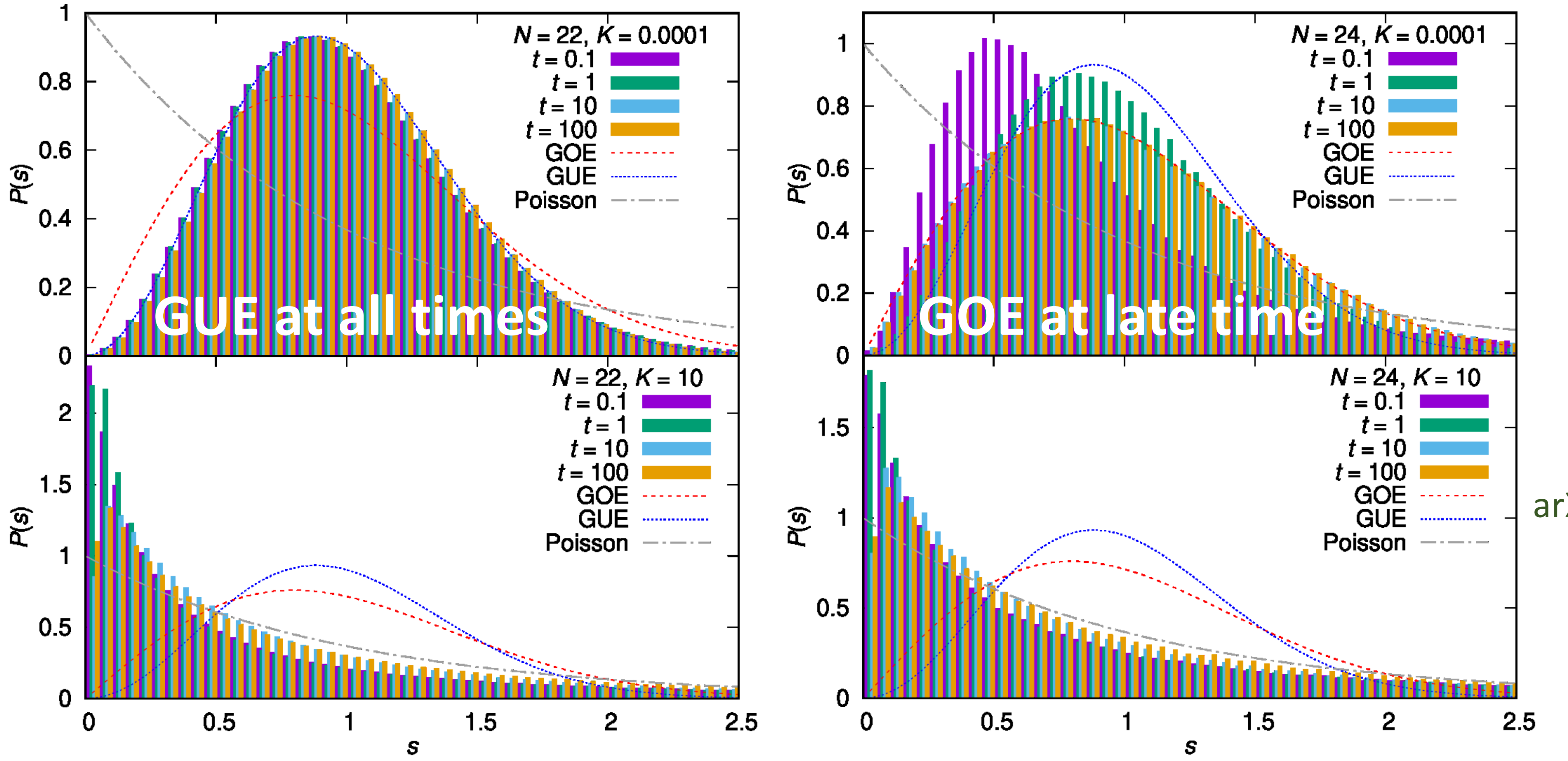
H. Gharibyan, M. Hanada, B. Swingle, and MT, arXiv:1902.11086

Singular value statistics of two-point functions

$$G_{ab}^{(\phi)} = \langle \phi | \hat{\chi}_a(t) \hat{\chi}_b(0) | \phi \rangle \text{ for } SYK_4 + K\, SYK_2$$

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$

larger $N/2$ exponents



arXiv:**1902.11086**

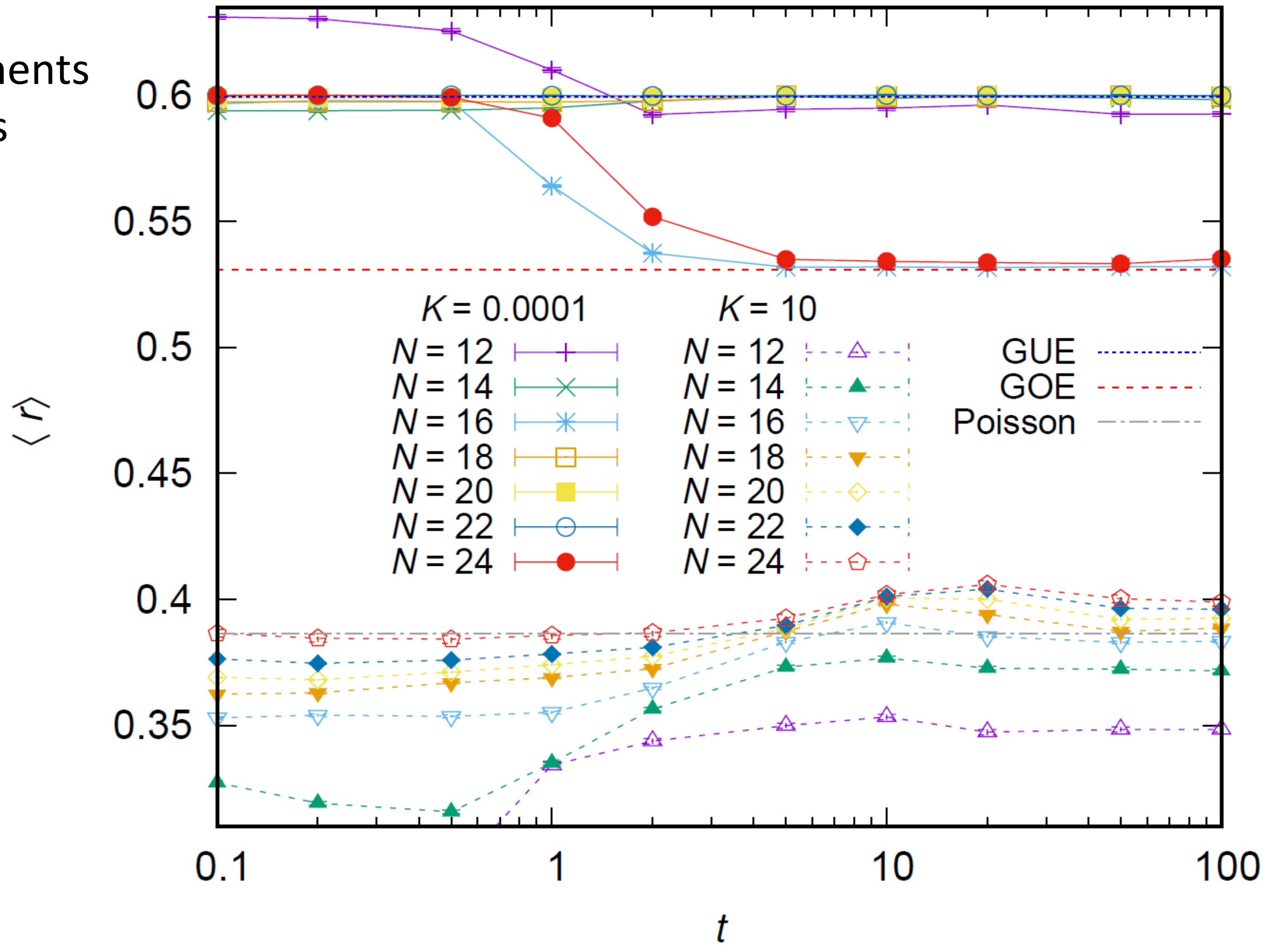
$\langle r \rangle$: average of the adjacent gap ratio $\frac{\min(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}{\max(\lambda_{i+1}-\lambda_i\,,\lambda_{i+2}-\lambda_{i+1})}$

Uncorrelated (Poisson): $2\log 2 - 1 \approx 0.386$

Correlated: larger (GOE: 0.5307, GUE: 0.5996 etc.) [Atas *et al.*, PRL 2013]

SYK, larger $N/2$ exponents

ϕ: energy eigenstates



N mod 8 = 2, 4, 6: GUE

N mod 8 = 0: GOE

(the matrix is symmetric)

$$G_{ab}^{(\phi)} = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle$$

$$\lambda_j = \log\left[\text{singular values of}\left(G_{ab}^{(\phi)}\right)\right]$$

fixed-i unfolded

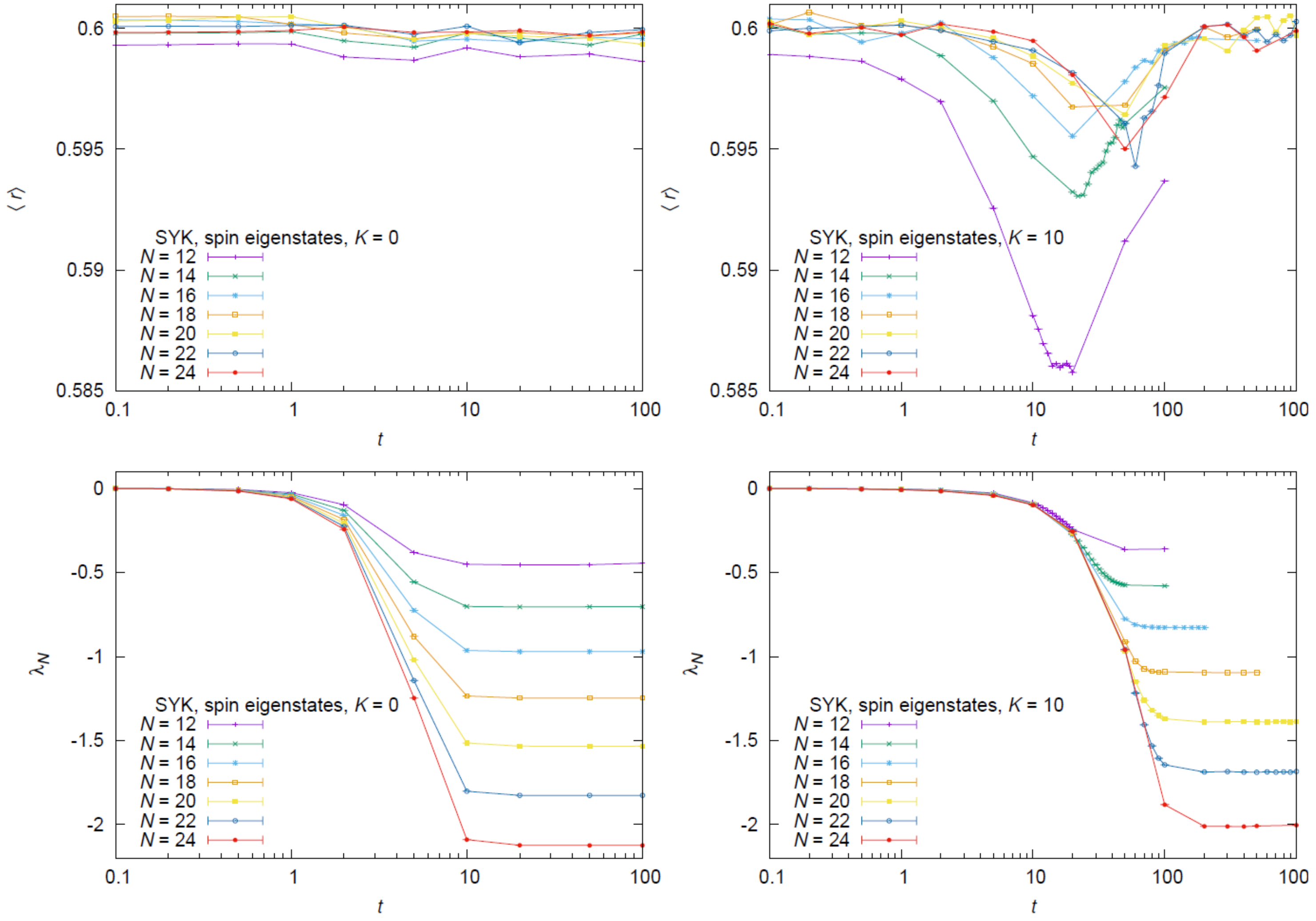
arXiv:**1902.11086**

At late time,

Random matrix behavior ⇔ chaotic

Random-matrix like for **complex fermion number eigenstates**, even for non-chaotic regime

Empty state in complex fermion description: state without long-range entanglement

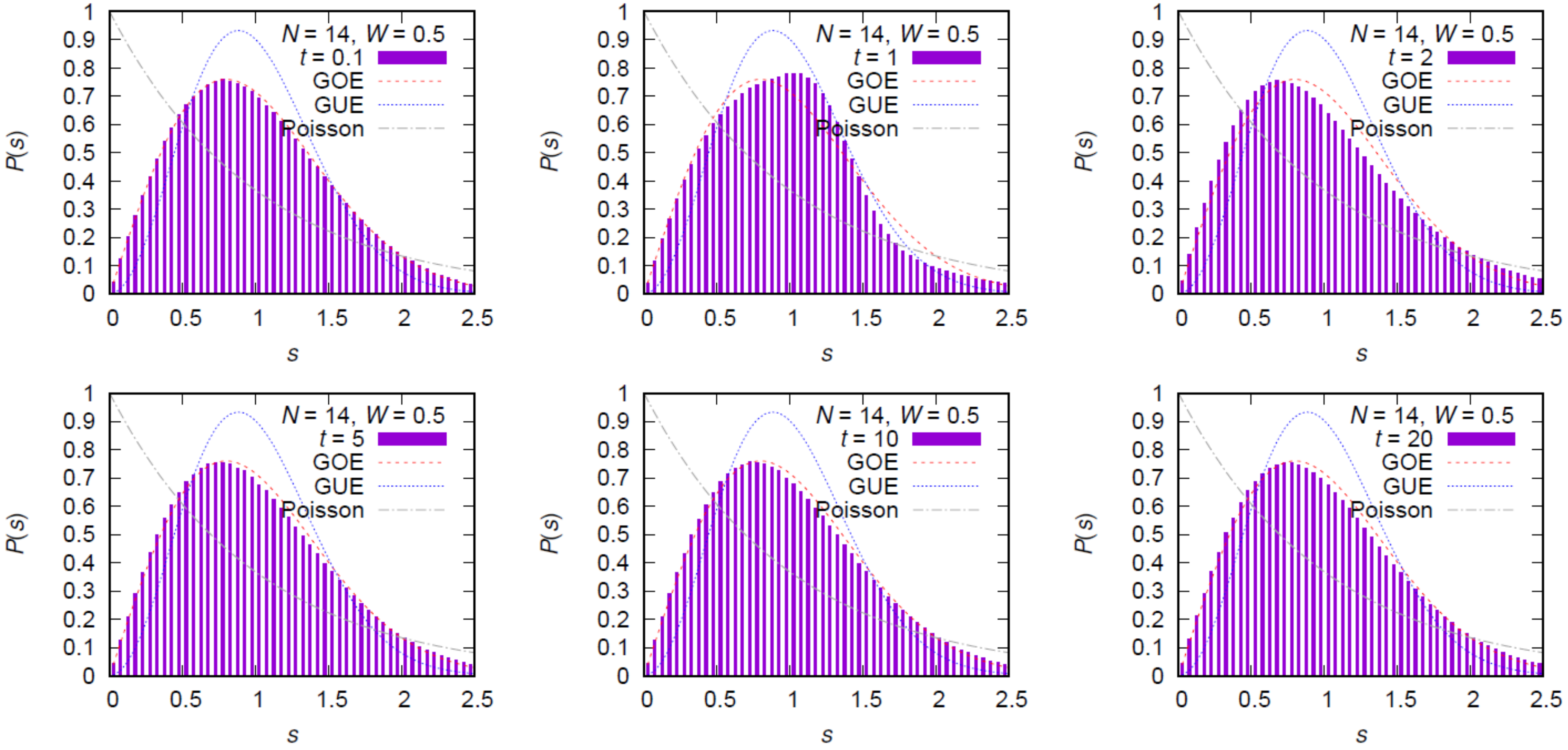


Two-point function for XXZ

$$\hat{H} = \sum_{i}^{N} \hat{S}_i \cdot \widehat{S_{i+1}} + \sum_{i}^{N} h_i \widehat{S_i^z} \qquad h_i \in [-W, W]$$

$$G_{ab}^{(\phi)} = \langle \phi | \widehat{\sigma^+}_a(t) \widehat{\sigma^-}_b(0) | \phi \rangle$$

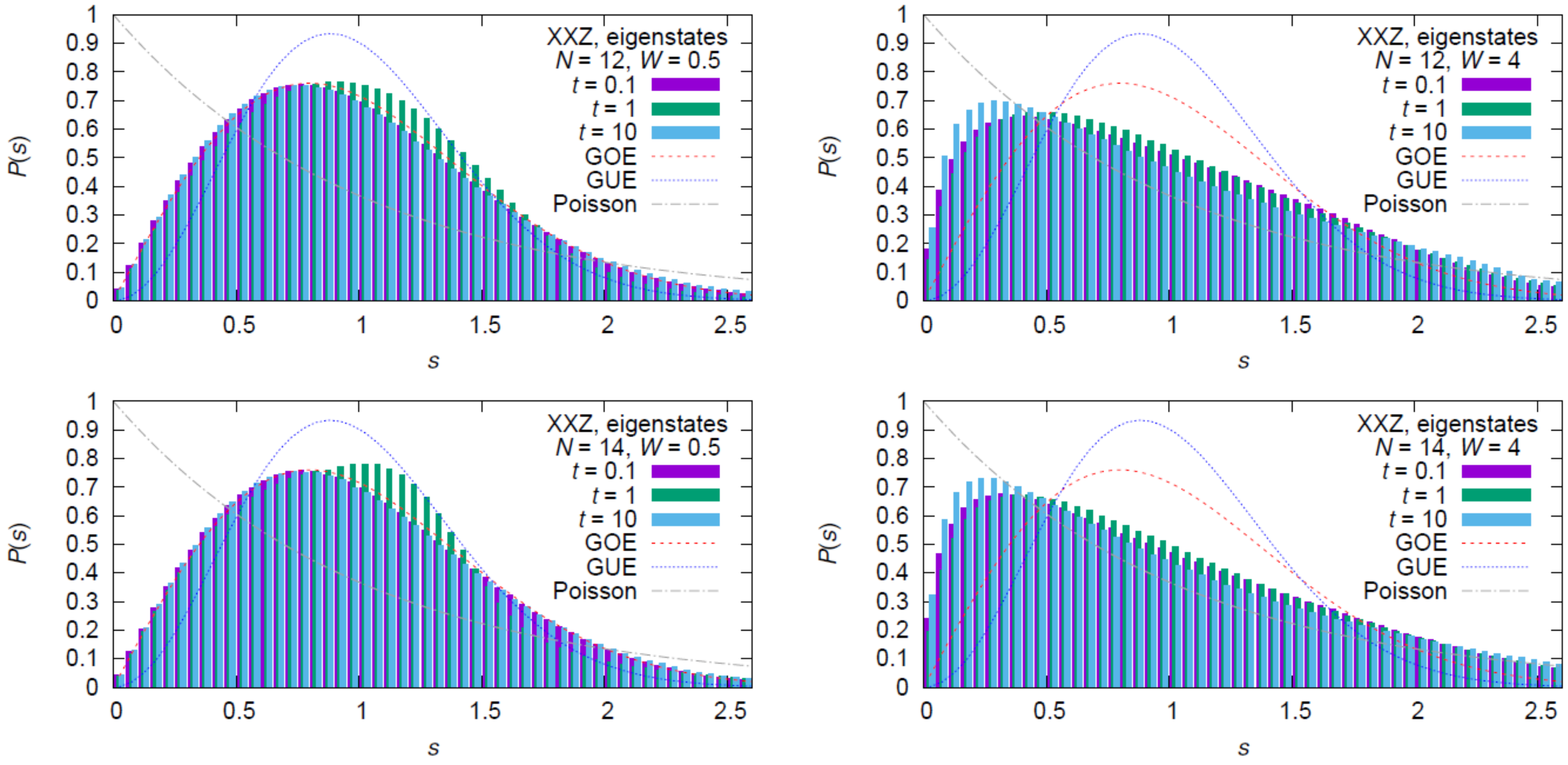
Energy eigenstates (not close to the spectral edges): GOE at short and long times for small *W*



Weak vs strong W

$$G_{ab}^{(\phi)} = \langle\phi|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)|\phi\rangle$$

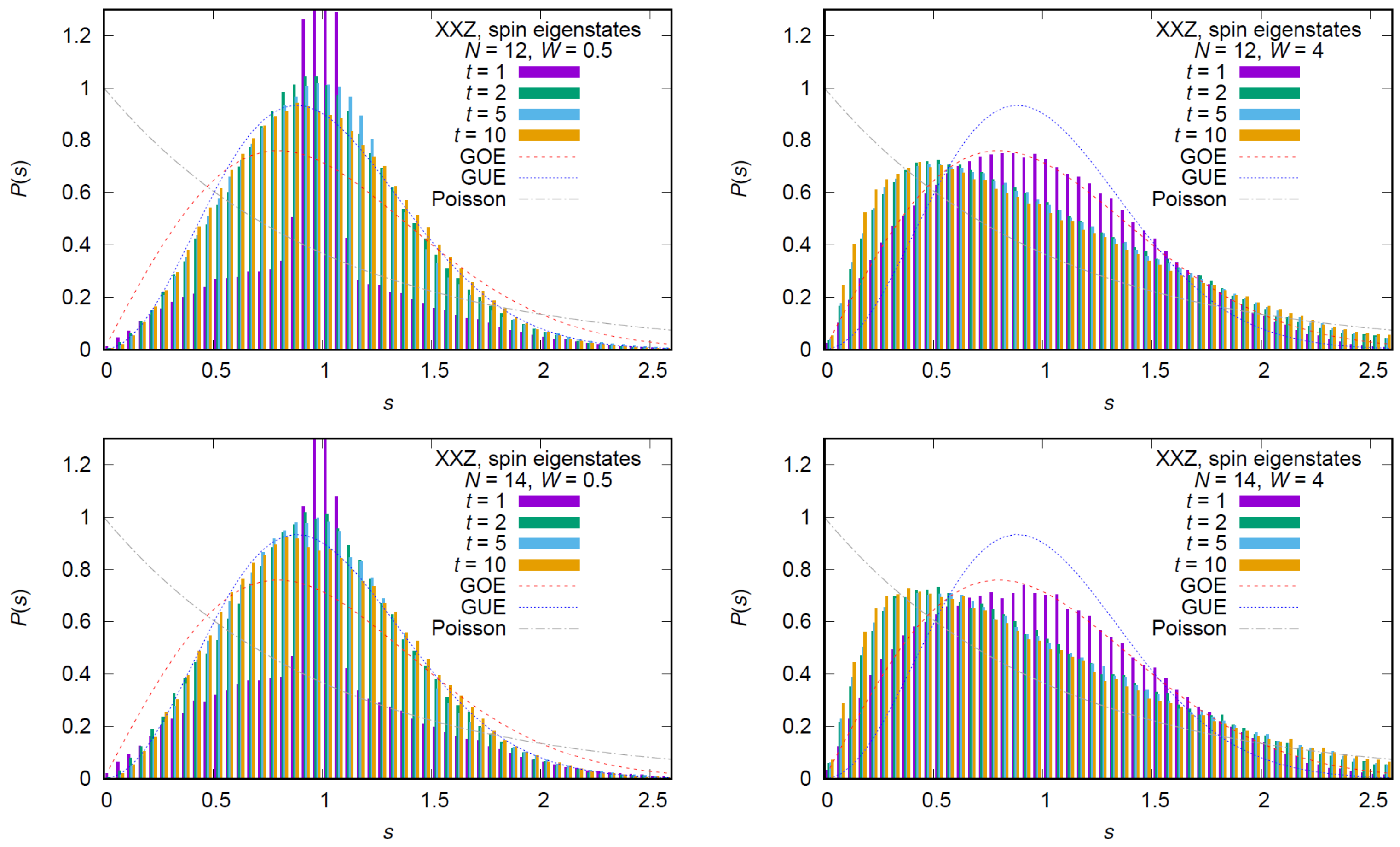
$$\hat{H} = \sum_i^N \widehat{S_i} \cdot \widehat{S_{i+1}} + \sum_i^N h_i \widehat{S_i^z} \qquad h_i \in [-W, W]$$



Energy eigenstates GOE at short and long times for small *W*, close to Poisson at any time for large *W*

XXZ model: Spin eigenstates → GUE

$$G_{ab}^{(\phi)} = \langle\phi|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)|\phi\rangle$$



Singular value statistics of two-point correlation function

Model	Chaotic (small K / small W)	Not chaotic (large K / large W)
SYK_4 + SYK_2	Energy eig. → GUE at late time except for $N \equiv 0 \pmod 8$: GOE Spin eig. → GUE at any time	Energy eig. → Poisson at any time Spin eig. → off from GUE at some time
XXZ + random field	Energy eig. → off from GOE at some time ($G_{ab}^{(\phi)} = \langle\phi\|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)\|\phi\rangle$ is symmetric) Spin eig. → converges to GUE ($G_{ab}^{(\phi)} = \langle\phi\|\widehat{\sigma^+}_a(t)\widehat{\sigma^-}_b(0)\|\phi\rangle$ is not symmetric)	Energy eig. → close to Poisson Spin eig. → approaches Poisson from RMT-like

H. Gharibyan, M. Hanada, B. Swingle, and MT, arXiv:1902.11086

Outlook / related recent works

- Euclidean time; two-point correlations in classical dynamics; experiments?
  - In progress
- Time scale?
  - cf. "Onset of Random Matrix Behavior in Scrambling Systems" (dip time determined by diffusion)
    H. Gharibyan, M. Hanada, S. H. Shenker, and MT, JHEP07(2018)124 (1803.08050)
- Many-body localization (MBL) in other systems?
  - cf. MBL in a finite-range SYK model
    A. M. García-García and MT, PRB **99**, 054202 (2019) (1801.03204)
- Relation between randomness and chaos?
  - cf. SYK_2 model: "Randomness and chaos in qubit models" (no need of chaos for slope-dip-ramp)
    Pak Hang Chris Lau, Chen-Te Ma, Jeff Murugan, and MT, Phys. Lett. B in press (1812.04770)
- Holographic interpretation?

Summary

- Many-body quantum chaos: characterizations
- The Sachdev-Ye-Kitaev model
- Quantum Lyapunov spectrum defined from local operators: characterizes quantum chaos [1809.01671]
  - Random matrix behavior in chaotic systems
  - Lyapunov growth
  - Fastest entropy production in the SYK model?

$\hat{L}_{ab}(t) = \sum_{j=1}^{N} \widehat{M}_{ja}(t)\widehat{M}_{jb}(t)$ for

$$\widehat{M}_{ab}(t) = \{\hat{\chi}_a(t), \hat{\chi}_b(0)\}$$

QLS: log(singular values of $\langle\phi|\hat{L}_{ab}(t)|\phi\rangle$)/(2t)

- Two-point correlation function: singular values exhibit random matrix behavior in chaotic cases [1902.11086]
  - Experiments should be possible with phase-sensitive measurements

$$G_{ab}^{(\phi)} = \langle\phi|\hat{\chi}_a(t)\hat{\chi}_b(0)|\phi\rangle$$

- Both characterizations of chaos demonstrated also for XXZ spin chain + random field