Boundary, Anomaly, and Entanglement

Masataka Watanabe (AEC, ITP, U. Bern)

29. May, 2019

YITP, Kyoto U.

Based on work in progress with

Simeon Hellerman and Domenico Orlando

Cutting to the chase...

- Since I only have twenty minutes, let me just go straight to the point already.
- Point 1: The concept of entanglement entropy is ill-defined for two-dimensional QFTs with a gravitational anomaly.
- Point 2: For non-anomolous theories, even when the concept of entanglement entropy exists, modular Hamiltonians are generically non-unique. It is also often not possible to choose the canonical one out of them choices.
- Audience: Excuse me? I have not the faintest idea....
- Audience: Have you gone mad?
- Okay fine. Let me give you a five-minute version of the proof of the first no-go theorem for you to understand.

No-go theorem on tensor factorization

- I am going to prove that there are no notion of tensor factorization of Hilbert space commonly assumed in quantum information theory.
- Tensor factoirzation is the Hilbert space structure

$$\mathcal{H}_{\Sigma} = \mathcal{H}_A \otimes \mathcal{H}_B,$$

  where the spatial slice Σ is a disjoint union of two spatial regions A and B.
- Side comment: I allow for any ultraviolet modifications of the theory attempting to make the factorization happen, like adding a heavy scalar field to the pure QED so that Wilson lines factorize [Casini, Huerta, Rosanbal 2013, Harlow 2015].

Step 1: Boundary Hamiltonian

- Let us first assume there exists a tensor product structure of Hilbert space mentioned earlier.
- One can therefore define an operation of taking a partial trace.
- Also, I will denote the (fully renormalised) Hamiltonian of the system as H. We demand the Hamiltonian be local, in the sense that H is a sum of local operators $H(x)$.
- In other words,

$$[H, \mathcal{O}(x)] = \dot{\mathcal{O}}(x),$$

where $\dot{\mathcal{O}}$ denotes the time derivative of $\mathcal{O}$.

Step 1: Boundary Hamiltonian

- Now we define the following quantity,

$$H_A^{[\epsilon]} \equiv -\frac{1}{\epsilon} \log \left(\mathrm{tr}_B \left[e^{-\epsilon H} \right] \right).$$

This is called the modular Hamiltonian at temperature ϵ^{-1}.

- Equivalently, one can work also with the following physically intuitive definition,

$$H_A^{[\epsilon]} \equiv \frac{\mathrm{tr}_B \left[H e^{-\epsilon H} \right]}{\mathrm{tr}_B \left[e^{-\epsilon H} \right]} = -\frac{d}{d\epsilon} \log \left(\mathrm{tr}_B \left[e^{-\epsilon H} \right] \right).$$

where it is easy to see that ϵ^{-1} serves as the UV cutoff.

- Up to multiplicative constants at each order in ϵ, these two definitions are the same. Especially check for yourself they match for $\epsilon \to 0$.

Step 1: Boundary Hamiltonian

- By taking the partial trace, because such a procedure is local, the Euclidean time evolution $e^{-\epsilon H}$ is never modified at more than ϵ distance away from the boundary $x = 0$ of A and B.
- Remark: We are heavily using the physical requirement that this partial trace does not introduce any correlations that weren't there in the original Hamiltonian. In other words, we only consider local tensor factorizations.
- Therefore we have $H_A^{[\epsilon]}(x) = H(x)$ for $x > \epsilon$.

Step 1: Boundary Hamiltonian

- Taking $\epsilon \to 0$, we recover

$$H_A(x) \equiv \lim_{\epsilon \to 0} H_A^{[\epsilon]}(x) = H(x)$$

  for $x > 0$.
- On the boundary, in order to properly take the limit $\epsilon \to 0$, one also needs to consider boundary relevant operators, whose coefficients scaling as $\epsilon^{\Delta - 1}$. For example, boundary cosmological constant will scale as ϵ^{-1}.
- The boundary renormalisation procedure ensures that the limit can be taken properly, and H_A now becomes a conformal invariant Hamiltonian defined on a space with boundary having finite boundary entropy [Affleck, Ludwig 1991, Friedan, Konechny 2003].

Step 2: CFT on a space with a boundary

- We now have a CFT on spatial slice $\mathbb{R}^+$ with Hamiltonian density $H(x)$.
- The conformal Ward identity on a space with boundary now becomes the following

$$T(z) - \bar{T}(\bar{z}) = \partial_0 \mathcal{O}_{\text{boundary}},$$

which dictates that the flow of energy never gets dissipated outside the spatial slice.

Step 2: CFT on a space with a boundary

- It is a standard procedure to change boundary conditions into boundary states $|B\rangle$, by exchanging the role of space and time.
- The conformal Ward identity are then translated into the following equation

$$\left(L_n - \bar{L}_{-n}\right)|B\rangle = 0$$

- Playing with the Virasoro algebra, we get

$$c_L = c_R,$$

where $c_{L,R}$ are the left- and right- central charges. In other words, BCFTs always have a vanishing gravitational anomaly, $c_L - c_R$.

Step 3: Taking the contrapositive

- You can now take the contrapositive of the whole process.

$$\boxed{\text{existence of tensor factorization}}$$
$$\Downarrow$$
$$\boxed{\text{consistent boundary condition}}$$
$$\Downarrow$$
$$\boxed{\text{vanishing gravitational anomaly}}$$

- A theory with a non-vanishing gravitational anomaly cannot have a unitary boundary condition, and hence is not compatible with the existence of the partial trace.

Generalised Nielsen-Ninomiya theorem

- One can prove the generalised Nielsen-Ninomiya theorem as a corollary.
- Because any theory with a lattice realisation can be trivially tensor factorized (by taking a free boundary condition), this means that a theory with a non-vanishing gravitational anomaly cannot be put onto a lattice.
- This includes the original two-dimensional Nielsen-Ninomiya theorem that one chiral fermion cannot be realised on a lattice, because now we have $c_L = 1/2$ while $c_R = 0$

FAQs

- I expect lots of question marks inside your head now.
- This is partly your fault partly mine.
- Your fault is that you are too contaminated by a lattice intuition.
- While mine is that I didn't give an explicit definition of tensor factorization without using such an intuition.

FAQs

- Question 1: The Hilbert space of gauge theories are in general non-factorizable in the literal mathematical sense, but this can be amended by modifying the theory in the UV. Doesn't the same thing happen here?
- Answer: No you can't. The anomaly is an RG invariant object. You cannot remove an anomaly by modifying the theory in the UV, as long as you wish that theory to flow to the same fixed point as the original theory.

FAQs

- Question 2: What exactly do you mean by tensor factorization and partial traces in continuum theories? There shouldn't exist tensor factorization at all in continuum field theories.
- Answer: It is a common misconception that tensor factorization doesn't make sense at all in continuum theories. It is neither true that AQFT helps make the definition more rigorous.
- There certainly are theories with tensor factorizations in suitably regulated sense of the word.

FAQs

- What *is* correct, is that there just are multiple ways in which to relate the total Hilbert space $\mathcal{H}$ with the tensor factorized Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The number of such choices are roughly the number of boundary universality classes of the theory [Ohmori, Tachikawa 2014].
- So rather than an equality like $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, we should be thinking about a map $\mathcal{M}$

$$\mathcal{M} : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B$$

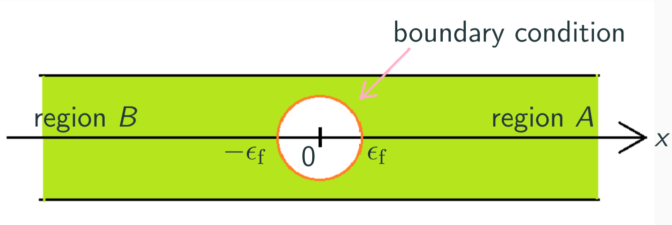
- This has a physical effect on the computation of modular Hamiltonians!

FAQs

- This means when you are computing $\rho_A \equiv \text{tr}_B(\rho)$, what you actually mean is the following
$$\rho_A \equiv \text{tr}_B\left(\mathcal{M}\rho\mathcal{M}^\dagger\right) = \text{tr}_B\left(\mathcal{M}^\dagger\mathcal{M}\rho\right).$$
So in other words the $\mathcal{F}[\mathcal{M}] \equiv \mathcal{M}^\dagger\mathcal{M} : \mathcal{H} \to \mathcal{H}$ modifies the density matrix itself!
- The effect of $\mathcal{F}[\mathcal{M}]$ should be as local as possible. In order to capture the physics of the original density matrix using *e.g.*, the entanglement entropy, we do not want to introduce correlations about the size of spatial slice itself.
- This rules out the non-local definition of the partial trace used in [Holzhey, Larsen, Wilczek 1994]. One can compute the entanglement entropy even for gravitational anomalous theories using it, but it won't reflect the correct physical correlation of the original theory.

FAQs

- They better not be exactly local though. In order for the tracing operation to not diverge, we want to smear it a bit using a small scale ϵ_{f}. And this is exactly why you have to introduce a scale in computing the entanglement entropy.
- I will not tell you the most general way to define such (almost)-local maps, as it can be not very intuitive.
- One construction of such a map $\mathcal{F}[\mathcal{M}]$ is to use the path integral on a strip with a hole of size ϵ_{f} cut out with a certain boundary condition [Ohmori, Tachikawa 2014].



FAQs

- Question 3: Why can't we just use the Cardy-Calabrese formalism? Wouldn't that give a canonical way to define the entanglement spectrum?
- Surprisingly we can't. The CC formalism doesn't define a consistent entanglement spectrum which can be seen at large Renyi indices.
- It never ever corresponds to a consistent spectrum of unitary matrices while that of reduced density matrices should. The CC operator should be supplemented by infinite numbers of bulk/boundary irrelevant operators [Cardy, Calabrese 2010].
- Boundary relevant operators also contribute, which even leads the theory into different boundary universal classes [Ohmori, Tachikawa 2014].

FAQs

- Question 4: Can you add to the theory a spectator sector to cancel the gravitational anomaly that decouples in the IR?
- It for sure works for lattice factorizations, but not for the entanglement entropy.
- Even though such degrees of freedom are decoupled away from the boundary, they are not on the boundary. They are always coupled to the original degrees of freedom by an $O(1)$ amount at the entangling surface.

FAQs

- Question 5: Can you use AQFT to define the (relative) modular Hamiltonian even for the anomalous case?
- No. AQFT requires that the Hilbert space be approximated by a finite dimensional one.
- This assumption is tantamount to assuming a lattice realisation of the theory.
- Also, AQFT makes it look as if there is one and the only modular Hamiltonian for each state, but as I have explained, that is missing out an important point.
- My take on such a construction is that they only define the bare modular Hamiltonian, in which they do not care about the behaviour of the boundary degrees of freedom.

Take-home messages

- One cannot have a tensor factorization of the Hilbert space in two-dimensional QFTs with a gravitational anomaly.
- This cannot be cured by any procedures which do not modify the infrared physics nor correlations of the original theory.
- Higher-dimensional generalizations are possible.
- It is also related to the lattice realisability of continuum theories.
- Even when modular Hamiltonians exist, one should also be careful about their non-uniqueness coming from what boundary universality classes are realised.

Take-home messages

- The generalisation of Nielsen-Ninomiya theorem towards three dimensions (or in general odd dimensions) could be interesting regarding various discrete anomalies.
- Holographic implications are still not clear. There are indeed explicit constructions of holographic CFTs with non-vanishing gravitational anomaly. What does (quantum corrections to) the Ryu-Takayanagi formula compute in such cases?
- At least it is plausible that any holographic two-dimensional CFTs must have $|c_L - c_R| / (c_L + c_R) \ll 1$ to make sense of the Ryu-Takayanagi formula at a classical level. How do we prove this?