Weyl Anomaly and Magnetization Current for M5 Branes in Background 3-form Flux

YIPQS long-term and Nishinomiya-Yukawa memorial workshop
Quantum Information and String Theory 2019

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1701.04275 with Miao and Guo,
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Outline

1. M5-branes

- The decoupling limit of N coincident M5-branes is given by an interacting (2,0) superconformal theory in 6 dimensions.
- Low energy effective description (easier): e.g. non-abelian equation of motion for self dual tensor field (Chu 2011; +Kao 2012; +Vanichchapongjaroen 12, 13; +Isono 13; Niarchos, Siampos 13)
- Fundamental formulation (hard):
  1. DLCQ instanton quantum mechanics (Aharony, Berkooz,Kachru, Seiberg, Silverstein 1997)
  2. Deconstruction (Arkani-Hamed, Cohen, Kaplan, Karch, Motl 2001)
  3. 5d SYM (Douglas; Lambert,Papageorgakis,Schmidt-Sommerfeld 2011)

  Yet so far there is no convincing check of these proposals. A satisfactory description for a system of multiple M5-branes is still missing.

M5-branes wish list

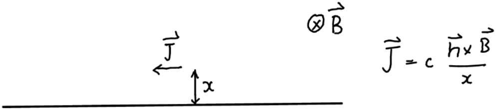
- The fundamental theory, no matter how it is defined, should:
  1. describe non-trivial interaction of (2,0) superconformal multiplets (Witten)
  2. contain BPS states of self-dual strings (Strominger)
  3. explain the S duality of the $\mathcal{N} = 4$ SYM (Montone, Olive)
  4. make apparent the N^3 entropy behaviour (Klebanov, Tseytlin)
- Motivation 1: Add new items to this M5 wish list!
  We argue for the presence of a Weyl anomaly of the form in any 6d CFT:

$$\mathcal{A} := \partial_\varphi W[e^{2\varphi} g_{\mu\nu}]\big|_{\mathrm{const.}\,\varphi=0} = \int_{M_6} b H_{\mu\nu\lambda}^2$$

2. Novel boundary effects

Another motivation is to study novel effect of boundary CFT.
Casimir effect arises from energetic response of the vacuum to the presence of boundary.

- In this talk, I want to talk about a new kind of response of the vacuum to the boundary:



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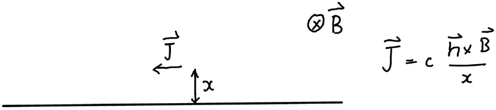
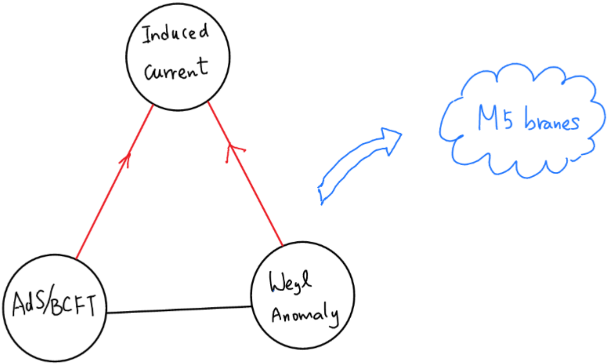


- This can be derived from exact analysis of BCFT (section 1) or AdS/BCFT (section 2).

- In section 3, we will apply it to boundary M5-branes system and learn something about the Weyl anomaly in 6d.



Outline

1.1.1. Casmir effect in BCFT

In general, for a d-dimensional BQFT, vev of renormalized stress tensor has the asymptotic behaviour near boundary:

$$\langle T_{ij} \rangle = x^{-d} T_{ij}^{(d)} ... + x^{-1} T_{ij}^{(1)} + \cdots , \quad x \sim 0,$$

x is proper distance from boundary, and

$$T_{ij}^{(d)} = \alpha_0 h_{ij}, \quad T_{ij}^{(d-1)} = 2\alpha_1 \bar{k}_{ij},$$
$$T_{ij}^{(d-2)} = \frac{-4\alpha_1}{d-1} n_{(i} h_{j)}^l \nabla_l k - \frac{4\alpha_1}{d-2} n_{(i} h_{j)}^l n^p R_{lp} + \cdots + t_{ij}$$
$$t_{ij} := \lceil \beta_1 C_{ikjl} n^k n^l + \beta_2 \mathcal{R}_{ij} + \beta_3 k k_{ij} + \beta_4 k_i^l k_{lj} \rceil,$$

where n_i, h_{ij} and $\bar{k}_{ij}$ are respectively the normal vector, induced metric and the traceless part of extrinsic curvature of the boundary P.

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where n_i, h_{ij} and $\bar{k}_{ij}$ are respectively the normal vector, induced metric and the traceless part of extrinsic curvature of the boundary P.

- The Casimir coefficient $(\alpha_0, \alpha_1, \beta_i)$ fixes the shape dependence of the leading Casimir effects of BCFT.
- For BCFT, conformal symmetry requires $\alpha_0 = 0$.

1.1.2. Casimir effects from Weyl Anomaly

Boundary Weyl anomaly

Weyl anomaly

$$\mathcal{A} := \partial_\sigma W[e^{2\sigma} g_{ij}]|_{\sigma=0} = \int_M \langle T_i^i \rangle,$$

where

$$\left\langle T_i^i \right\rangle = \left\langle T_i^i \right\rangle_M + \delta(x_\perp) \left\langle T_a^a \right\rangle_P .$$

- Weyl anomaly are classified in terms of curvature invariants.
  3d: $\left\langle T_i^i \right\rangle = \delta(x)[b_1 \mathcal{R} + b_2 \mathrm{Tr}\bar{k}^2]$
  4d:
  $\left\langle T_i^i \right\rangle = \frac{c}{8} \mathrm{Tr} C^2 - \frac{a}{16\pi^2} E_4 + \delta(x)[\frac{a}{16\pi^2} E_4^{\mathrm{bdy}} + b_3 \mathrm{Tr}\bar{k}^3 + b_4 C^{ac}{}_{bc} \bar{k}^b{}_a]$
- Bulk central charges c do not depend on BC. Boundary central charges b_i depend on BC in general.

Theorem: Consider BQFT, the variation of the Weyl anomaly under an arbitrary variaion of the metric can be measured by the 1-point function of the renormalized stress tensor (Chu, Miao 17, 18)
"Integrability condition":

$$\boxed{(\delta\mathcal{A})_{\partial M} = \left(\frac{1}{2}\int_{x\geq\epsilon}\sqrt{g}\,T^{ij}\delta g_{ij}\right)_{\log(1/\epsilon)}}$$

where $(\delta\mathcal{A})_{\partial M}$ is the boundary terms in the variations of Weyl anomaly and T^{ij} is the renormalized bulk stress tensor.

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Note that the right hand side must give an exact variation, this imposes strong constraints on the possible form of the stress tensor near the boundary.

Corollary: The energy momentum tensor $T_{\mu\nu}$ of BCFT has universal behaviour near the boundary. (Chu, Miao 17, 18)

"universal" means the Casimir coefficients does not need to be computed case by case. They are indeed specified entirely in terms of the boundary central charges of the theory.

E.g. 3d BCFT, Weyl anomaly has only boundary contributions

$$\mathcal{A} = \int_{\partial M} \sqrt{h}(b_1 \mathcal{R} + b_2 \mathrm{Tr}\bar{k}^2),$$

$$LHS = (\delta \mathcal{A})_{\partial M} = b_2 \int_{\partial M} \sqrt{h}\Big[(\frac{\mathrm{Tr}\bar{k}^2}{2} h^{ab} - 2\bar{k}^a_c k^{cb})\delta h_{ab} + 2\bar{k}^{ab}\delta k_{ab}\Big].$$

$$RHS = \left(\frac{1}{2}\int_{x\geq\epsilon} \sqrt{g}\, T^{ij}\delta g_{ij}\right)_{\log(1/\epsilon)}$$

$$= -\alpha_1 \int_P \sqrt{h}[(\frac{\mathrm{Tr}\bar{k}^2}{2} h^{ab} - 2\bar{k}^a_c k^{cb})\delta h_{ab} + 2\bar{k}^{ab}\delta k_{ab}]$$

$$+ \int_P \sqrt{h}[(\frac{\beta_3}{2} - \alpha_1) k \bar{k}^{ab}\delta h_{ab} + \frac{\beta_4}{2}\lceil k^a_c k^{cb}\rceil \delta h_{ab}].$$

We obtain $\boxed{\alpha_1 = -b_2, \quad \beta_3 = -2b_2, \quad \beta_4 = 0.}$

4d BCFT. Similar analysis and results for 4d.

- It is remarkable that the Casimir coefficients are completely determined by the boundary central charges.
- The relations between them are universal and independent of BC and theory.

1.2.1. Chiral Anomaly and Transport

Two famous Anomaly induced transport of charges:

- CME: chiral magnetic effect (Vilenkin 80; Giovannini Shaposhnikov 98; Froehlich etal 98)

$$\mathbf{J}_V = \sigma_{(\mathcal{B})V}\mathbf{B} \quad \mathbf{J}_A = \sigma_{(\mathcal{B})A}\mathbf{B},$$

  where the chiral magnetic conductivities are

$$\sigma_{(\mathcal{B})V} = \frac{e\mu_A}{2\pi^2}, \quad \sigma_{(\mathcal{B})A} = \frac{e\mu_V}{2\pi^2}$$

- CVE: chiral vortical effect (Kharzeev, Zhitnitsky 07; Erfmemger etal 09; Son etal 09; Landsteiner etal 11)

$$\mathbf{J}_V = \sigma_{(\mathcal{V})V}\omega, \quad \mathbf{J}_A = \sigma_{(\mathcal{V})A}\omega,$$

  where the chiral vortical conductivities are

$$\sigma_{(\mathcal{V})V} = \frac{\mu_V\mu_A}{\pi^2}, \quad \sigma_{(\mathcal{V})A} = \frac{\mu_V^2+\mu_A^2}{2\pi^2} + \frac{T^2}{6}$$

- Note that these induced current occurs only in a material system where the chemical potentials are non-vanishing.

Q. In the presence of boundary, can the phenomena of induced current occur in vacuum like Casimir effect? if so, how is it possible?

1.2.2. Weyl Anomaly and induced current

- Just as energy momentum tensor, the renormalized current also admits an asymptotic expansion near the boundary:

$$\langle J_i \rangle = x^{-3} J_i^{(3)} + x^{-2} J_i^{(2)} + x^{-1} J_i^{(1)}, \quad x \sim 0,$$

where x is the proper distance from the boundary and $J_i^{(n)}$ depend only on the background geometry and the background vector field strength.

- Imposing current conservation

$$\nabla_i \langle J^i \rangle = 0,$$

we obtain the gauge invariant solutions

$$J_\mu^{(3)} = 0, \qquad J_\mu^{(2)} = 0,$$
$$J_\mu^{(1)} = \alpha_1 F_{\mu\nu} n^\nu + \alpha_2 \mathcal{D}_\mu k + \alpha_3 \mathcal{D}_\nu k_\mu^\nu + \alpha_4 \star F_{\mu\nu}\, n^\nu$$

Under an arbitrary of A_μ, one can similarly establish the integrability condition

$$\boxed{(\delta\mathcal{A})_{\partial M} = \left(\int_M \sqrt{g} J^\mu \delta A_\mu\right)_{\log\frac{1}{\epsilon}}}$$

Using it, the current coefficients are determined completely in terms of central charges.

- Consider background $U(1)$ gauge field, e.g. QED

$$\mathcal{A} = \int_M \sqrt{g}[b_1 F_{\mu\nu} F^{\mu\nu} + \text{metric part}], \quad b_1 = -\beta(e)/(2e^3), \text{ centeral charge.}$$

This implies that $(\delta\mathcal{A})_{\partial M} = -4b_1 \int_{\partial M} \sqrt{h} F^b{}_n \delta a_b$. Matching it with the RHS of the integrability condition, we obtain

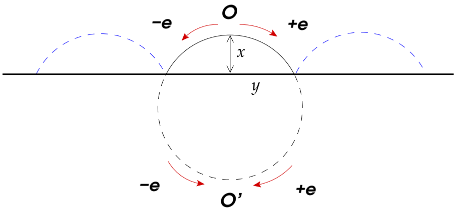
$$\alpha_1 = 4b_1, \quad \alpha_2 = \alpha_3 = \alpha_4 = 0.$$

- Explicitly, we obtain the induced current near the boundary of a BQFT:

$$\boxed{\mathbf{J} = \frac{e^2 c}{\hbar} \frac{4 b_1 \mathbf{n} \times \mathbf{B}}{x}, \quad x \sim 0}$$

Q. What is the physics of this current?

- The current is a result of charge separation due to vacuum fluctuation in the presence of external B-field.



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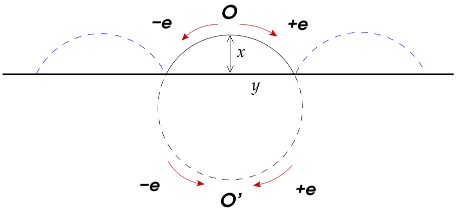
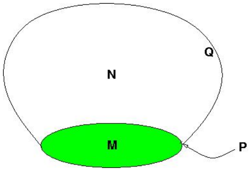


- Equivalently, it can also be seen as arising from the magnetization of the vacuum due to presence of boundary: $\mathbf{J} = \nabla \times \mathbf{M}$.

Outline

2.1. Statement of AdS/BCFT

- Consider d dimensional CFT defined on $R^{1,d-1}$. There is a $SO(2, d)$ conformal symmetry. This is realized in holography as isometries of the bulk of AdS$_{d+1}$ space
- When a boundary is introduced, the full conformal symmetry is reduced at the boundary. Takayanagi proposed to extend that the gravity dual is given by gravity in a $d+1$ dimensional manifold N whose boundary is given by M and Q. (Takayanagi 11)



The bulk gravity action is given by

$$I = \int_N \sqrt{G}(R - 2\Lambda) + 2\int_M \sqrt{g}K + 2\int_Q \sqrt{h}(K - T) + 2\int_P \sqrt{\sigma}\theta,$$

T measures the boundary degrees of freedom (g-function)

(Takayanagi 2011)

- The central issue is the determination of the location of Q in the bulk. Takayanagi proposed to impose Neumann boundary condition on Q to fix its position:

$$\text{EOM of } Q: \quad K_{\alpha\beta} - (K - T)h_{\alpha\beta} = 0, \qquad (*1)$$

This gives a set of second order DE.

- However since Q is of co-dimension one,the location of Q is determined by a single embedding function:

$$z = z(x^i), \quad \text{here } x^i = \text{coordinates of } M$$

The tensorial embedding equation (*1) generally imposes too many constraints and it does not has solution for general shape P of BCFT.

- Alternatively, it is possible to impose on Q a mixed BC and this leads to a scalar embedding equation:

$$\boxed{(1-d)K + dT = 0} \qquad (*2)$$

(Chu, Guo, Miao 17; Chu, Miao 17)

- As there is only one embedding function for Q, it is natural that there is a single condition for it.
- The scalar embedding equation give rises to a consistent proposal of BCFT, e.g. giving rises to the correct form of boundary Weyl anomaly, the correct form of near boundary expansion of the stress tensor, the correct relation between the Casimir coefficients and the boundary central charge etc.
- It turns out that the tensorial embedding equation is also consistent provided that one previously ignored important ingredient is incooperated.

Non FG (Fefferman-Graham) expansion

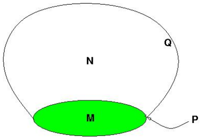
- In the standard AdS/CFT, FG expansion of the bulk metric is assumed:

$$ds^2 = \frac{dz^2 + g_{ij} dx^i dx^j}{z^2}, \quad g_{ij} = g^{(0)}_{ij} + z^2 g^{(1)}_{ij} + \cdots .$$

$g^{(0)}_{ij}$ is the metric of BCFT on M. $g^{(1)}_{ij}$ is fixed by Einstein equation:

$$g^{(1)}_{ij} = -\frac{1}{d-2}\left(R^{(0)}_{ij} - \frac{R^{(0)}}{2(d-1)} g^{(0)}_{ij}\right).$$

- However in the case of AdS/BCFT, the dual manifold N has discontinuity at the corner P where Q and M meet.



- We have to give up the assumption that the bulk manifold has a metric that can be FG expanded in small z near M!
- However, as we lose the perturbative FG expansion, solving Einstein equation becomes impossible unless another expansion is found! We can use an expansion about the boundary P!
- Convenient to use the Gauss normal coordinates. The metric $g^{(0)}_{ij}$ of the BCFT reads (P is located at $x = 0$):

$$ds_0^2 = dx^2 + (\sigma_{ab} + 2xk_{ab} + x^2 q_{ab} + \cdots)dy^a dy^b.$$

- We found a new systematic construction of non-FG expanded metric by employing k_{ab}, q_{ab} etc as expansion parameter, but keeping both the z and x dependence as exact.

  In this way, we are able to construct a perturbative solution to the bulk Einstein equation: (Chu, Miao 2017)

$$ds^2 = \frac{dz^2 + dx^2 + \left(\delta_{ab} - 2x\bar{k}_{ab} f\left(\frac{z}{x}\right)\right) dy^a dy^b}{z^2} + \cdots .$$

- At the order $O(k)$, the Einstein equation has the solution

$$f(s) = 1 - \lambda_1 \frac{s^d \, {}_2F_1\left(\frac{d-1}{2}, \frac{d}{2}; \frac{d+2}{2}; -s^2\right)}{d},$$

where λ_1 is free parameter.

- $\lambda_1 = 0$ gives FG. More free constants appear in higher order terms of the solution.
- The existence of new free parameters in the bulk background allows the tensorial embedding equation be solved also consistently..

2.2 Application: Induced current near boundary

- To investigate the renormalized current in holographic models of BCFT, we add a Maxwell action:

$$I = \int_N \sqrt{G}[R - 2\Lambda - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}] + 2\int_Q \sqrt{\gamma}[K - T]$$

- Consider 4d and planar boundary, one can solve for the bulk Maxwell equation and BC, and obtain

$$\mathcal{A}_a = F_{xa}\sqrt{x^2 + z^2}, \tag{2}$$

where F_{xa} is the field strength at the boundary.

- The holographic current is

$$\langle J^a \rangle = \lim_{z\to 0} \frac{\delta I}{\delta A_a} = \lim_{z\to 0} \sqrt{G}\mathcal{F}^{za} = -\frac{F_{ax}}{x} + \cdots$$

Same as field theory analysis!

Outline

3.1 Boundary string current from holography

- Consider a charged particle moving on the worldline C: $x^\mu = x^\mu(\tau)$. The motion gives a current and a coupling to gauge field:

$$J^\mu(x) = \delta^{(d-1)}(x - x(\tau))\frac{dx^\mu(\tau)}{d\tau}.$$

$$\int_M J_\mu A^\mu = \int_C A_\mu dx^\mu$$

- Similarly, movement of strings gives the higher 2-form current and a coupling to the 2-form potential $B_{\mu\nu}$:

$$J_{\mu\nu} = \delta^{(d-2)}(x - x(\sigma,\tau))\epsilon^{\alpha\beta}\frac{\partial x^\mu}{\partial\sigma^\alpha}\frac{\partial x^\nu}{\partial\sigma^\beta}.$$

$$\int_M J_{\mu\nu}B^{\mu\nu} = \int_\Sigma B_{\mu\nu}dx^\mu dx^n.$$

Q. Any implication of knowing the existence of such a coupling?

Consider a BCFT in 6d and denote the Weyl anomaly as $\mathcal{A}$. The gravitational part is well understood. (Deser, Schwimmer)

$$\langle T^\mu_\mu \rangle = \frac{1}{(4\pi)^{d/2}} \left(\sum_j c_{dj} I^{(d)}_j - (-1)^{\frac{d}{2}} a_d E_d \right).$$

- Q. What about the contribution from background gauge field?
- Claim: One can similarly establish the relation

$$\boxed{(\delta \mathcal{A})_{\partial M} = \Big(\int_{M_\epsilon} J_{\mu\nu} \delta B^{\mu\nu} \Big)_{\log 1/\epsilon}.}$$

Thus, knowing the current, or vice versa, would allow us to learn something about the anomaly structure of the 6d CFT.

- Consider a 6d BCFT dual to the the bulk action with an H-field

$$I = \int d^7x\sqrt{G}(R - 2\Lambda - \frac{1}{12}H^2_{\mu\nu\lambda})$$

Using the BCFT holography, one finds a string current parallel to the boundary when a H-field strength is turned on

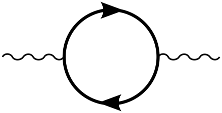
$$J_{ab} = b_1 \frac{H_{abx}}{x}.$$

3.2. Prediction for the Weyl anomaly in 6d

- The relation $(\delta\mathcal{A})_{\partial M} = \left(\int_{M_\epsilon} J_{\mu\nu}\delta B^{\mu\nu}\right)_{\log 1/\epsilon}$ predicts the Weyl anomaly in the 6d CFT:

$$\boxed{\mathcal{A} = \int_M \frac{b_1}{12} H^2}$$

- It is interesting to understand how matter fields would couple to the $B_{\mu\nu}$ field (covariant derivatives?) and give rises to the Weyl anomaly.



M5-branes system

- For a system of N M5-branes, the gravity dual is given by $AdS_7 \times S^4$.
- Restoring the units, b_1 is given by

$$b_1 = \frac{R^5}{16\pi G_7} = \frac{N^3}{3\pi^3},$$

where $G^{(7)} = G^{(11)}/R_S^4$, $R_S = l_P(\pi N)^{1/3}$ is the 4-sphere radius and $R = 2R_S$ is the AdS_7 radius.
- Therefore for a system of N M5-branes with boundary, one finds for the singlet current

$$\boxed{J_{ab} = \frac{N^3}{3\pi^3}\frac{H_{abx}}{x}.}$$

This results suggest that there is N^3 degrees of freedom in the interacting (2,0) theory.

Outline

1. Induced Current as Exact Universal Behaviour of BCFT

2. Induced Current from AdS/BCFT

3. Application to 6d: Weyl anomaly from Induced String Current.

Summary and Discussions

Conclusions and Discussions

1. We found new universal relations for near boundary behaviour of stress tensor and electric current.
2. In the presence of a magnetic field, we predict a magnetization current in the vicinity of the boundary of QED vacuum. This is a kind of magnetic Casimir effect.
3. For BCFT, we have constructed a new class of bulk metric that is based on an expansion (non-FG) of the exterior curvature (and its derivative) of the field theory boundary. This is a crucial ingredient for a consistent proposal of AdS/BCFT.
4. We predict a Weyl anomaly for the M5-branes system.

Open questions:

- What is the origin of the Weyl anomaly of M5-brane?
- What is the implicition on the partition function?
- Can one understand the magnetization of the vacuum may be understood in terms of NCG?

$$[x^i, x^j] = i\theta^{ij}.$$

- What about the M5-brane worldvolume?

$$[x^i, x^j, x^k] = i\theta^{ijk}?$$

Thank you!

String loop covariant derivatives

- As $B_{\mu\nu}$ is coupled to a worldsheet, the natural way to construct a covariant derivative is by considering a string functional $\Psi(C)$.
- Define loop derivative

$$\partial_{\mu\nu}\Psi := \frac{\delta\Psi(C)}{\delta\sigma^{\mu\nu}} := \lim_{\Delta\sigma^{\mu\nu}\to 0} \frac{\Psi(C+\delta C)-\Psi(C)}{\Delta\sigma^{\mu\nu}}$$

- A general unitary transformation that depends on the loop takes the form

$$\Psi(C) \to \Psi(C) \exp(i \int_C \alpha),$$

where $\alpha = \alpha_\mu dx^\mu$ is a 1-form.

- The derivative

$$\mathcal{D}_{\mu\nu}\Psi := (\partial_{\mu\nu} - iB_{\mu\nu})\Psi$$

is covariant: $\mathcal{D}_{\mu\nu}(e^{i\int\alpha}\Psi) = e^{i\int\alpha}\,\mathcal{D}_{\mu\nu}\Psi$ since $B_{\mu\nu}$ transforms as

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu$$

It is more natural to consider a theory of self-dual string and think of the Weyl anomaly as a property of the effective action of the string partition function.