Reflected entropy in AdS/CFT

Souvik Dutta

University of Illinois Urbana-Champaign

Based on: SD & Thomas Faulkner 1905.00577

Quantum Information & String Theory, 2019

Yukawa Institute of Theoretical Physics, 京都大学

June 8th, 2019 Kyoto, Japan

Motivation

- Field theory encoded as (local geometric objects in?) dual gravity theory

$$S(A) = -\,\mathrm{Tr}(\rho_A \ln \rho_A) = \frac{\mathrm{Area}(m_A)}{4G_N} + \mathcal{O}(G_N^0)$$

[Ryu, Takayanagi `08]
[Faulkner, Lewkowycz, Maldacena `13]
[Lewkowycz, Maldacena, `13]

- HRT surface (time dep.) = Entanglement entropy [Hubeny, Rangamani, Takayanagi `07]

  Cross-section of Wheeler-De Witt patch = Complexity? [Stanford, Susskind`14]

  Max. vol time slice = Information metric [Miyaji, Numasawa, Shiba, Takayanagi, Watanabe`15]

  Entanglement wedge = density matrix [Czech, Karczmarek, Nogueira, van Raamsdonk `12]

  Minimal cross-section of entanglement wedge = ? Masamichi, Koji's talks

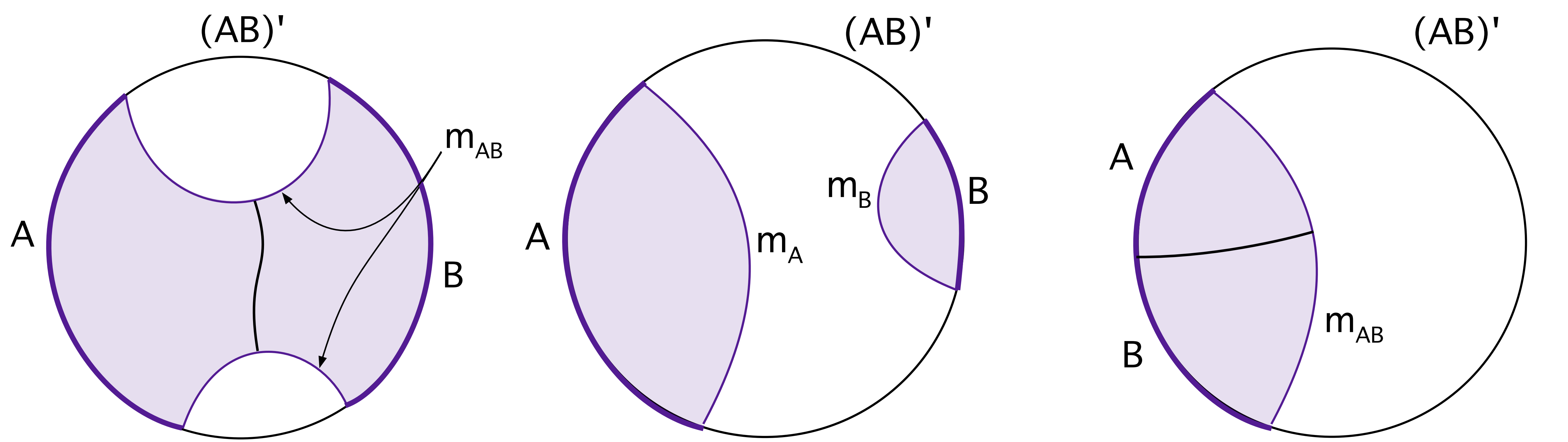
  - Entropy of purification: $E_W(A:B) = E_P(A:B)$ [Takayanagi, Umemoto, `17]
  - Reflected entropy: $E_W(A:B) = \frac{1}{2} S_R(A:B)$ [SD, Faulkner, `19]

Slicing the entanglement wedge

- Set of bulk points spacelike separated from m_{AB} and on the side of AB.

[Czech, Karczmarek, Nogueira, van Raamsdonk`12]

[Headrick, Hubeny, Lawrence, Rangamani `14]



$E_W(A:B) \neq 0$ | $E_W(A:B) = 0$ | $E_W(A:B) \neq 0$; divergent

A, B disjoint

$\partial A \cap \partial B \neq \phi$

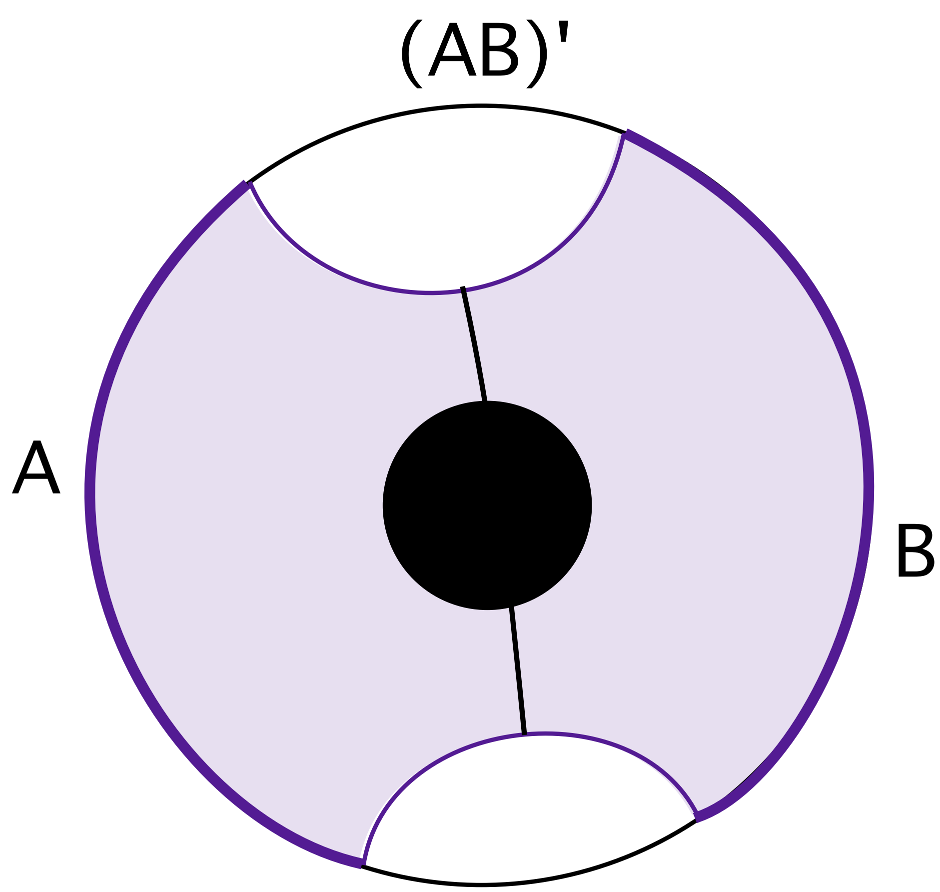
[Takayanagi, Umemoto, `17]

Slicing the entanglement wedge

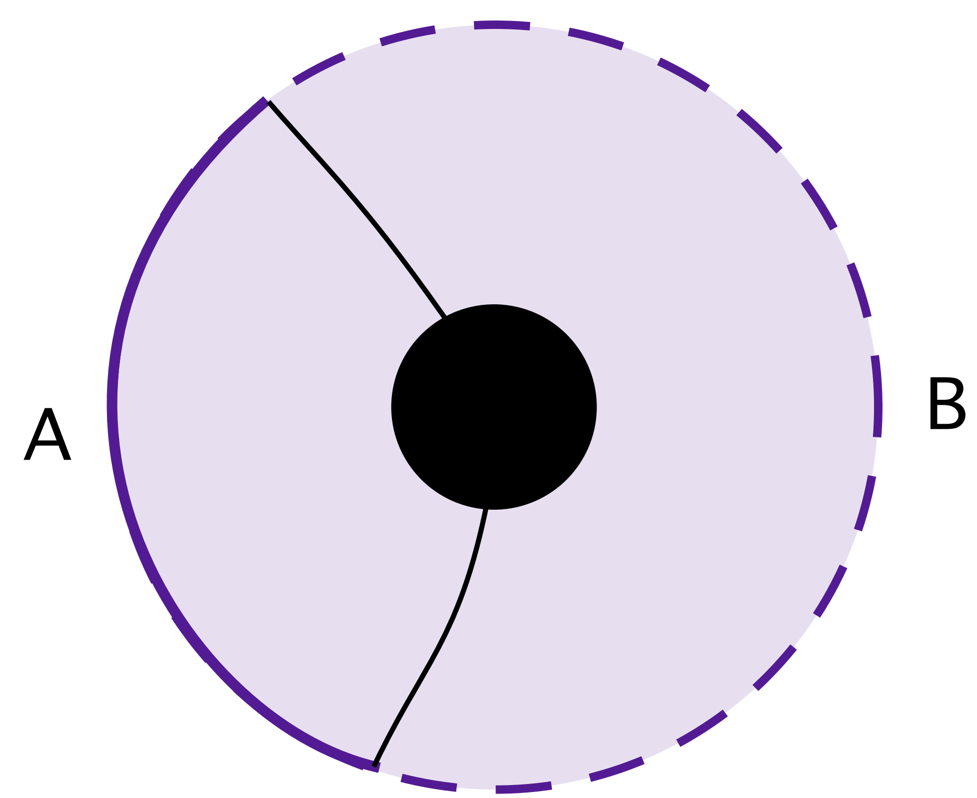
- Set of bulk points spacelike separated from m_{AB} and on the side of AB.

[Czech, Karczmarek, Nogueira, van Raamsdonk`12]

[Headrick, Hubeny, Lawrence, Rangamani `14]



$(AB)' \neq 0$; $E_W(A : B)$ finite



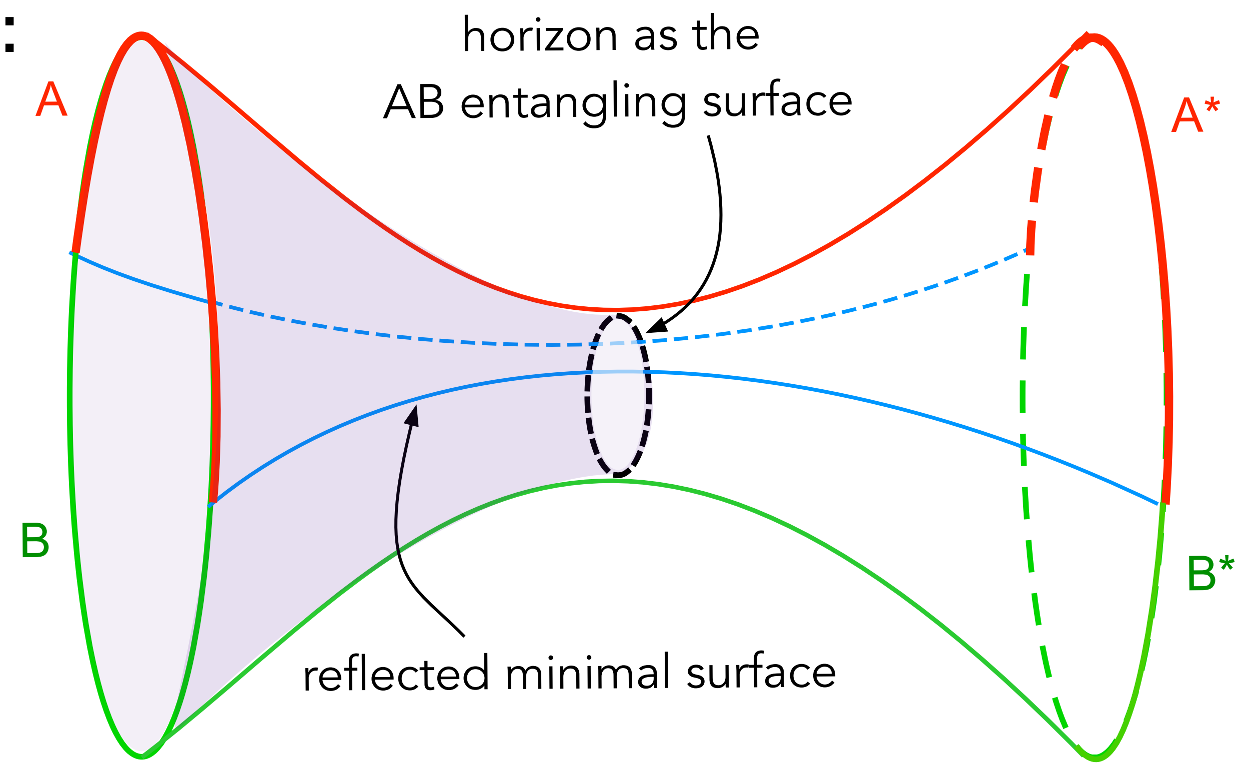
$(AB)' = 0$; $E_W(A : B)$ divergent

Thermal state of a CFT; bulk dual is an AdS black hole.

Warm-up example

$$E_W(A:B) = \frac{1}{2} S(AA^\star)_{TFD} \equiv \frac{1}{2} S_R(A:B)$$

Example:



$$\rho_{AB} = e^{-\beta H}$$

$$|TFD\rangle = \sum_i e^{-\beta E_i/2} |E_i\rangle_L \otimes |E_i\rangle_R$$

[Maldacena `01]

Mixed state is a thermal state in AB; bulk dual to $|\mathrm{TFD}\rangle$ is eternal black hole.

The duality proposal

- For ANY mixed state ρ_{AB},

$$E_W(A:B) = \frac{1}{2} S(AA^\star)_{\sqrt{\rho_{AB}}} \equiv \frac{1}{2} S_R(A:B)$$

$\sqrt{\rho_{AB}}$: canonical purification

- $\rho_{AB}^{1/2} \in \mathscr{H}_A \otimes \mathscr{H}_B \longleftrightarrow |\sqrt{\rho_{AB}}\rangle \in \mathrm{End}(\mathscr{H}_A) \otimes \mathrm{End}(\mathscr{H}_B)$
  - space of linear maps/matrices acting on $\mathscr{H}_A$
  - forms a Hilbert space, $\langle \sigma_A | \sigma'_A \rangle = \mathrm{Tr}_A \sigma_A^\dagger \sigma'_A$
  - $\mathrm{End}(\mathscr{H}_A) \cong \mathscr{H}_A \otimes \mathscr{H}_A^\star$
- $\implies |\sqrt{\rho_{AB}}\rangle \in \left(\mathscr{H}_A \otimes \mathscr{H}_A^\star\right) \otimes \left(\mathscr{H}_B \otimes \mathscr{H}_B^\star\right)$

  such that $\mathrm{Tr}_{A^\star B^\star} |\sqrt{\rho_{AB}}\rangle\langle\sqrt{\rho_{AB}}| = \rho_{AB}$

$$\rho_{AB} = e^{-\beta H}$$

$$\downarrow$$

$$\sum_i e^{-\beta E_i/2} |E_i\rangle_L \otimes |E_i\rangle_R$$

Reflected entropy: properties

$$S_R(A:B) = S(AA^\star)_{\sqrt{\rho_{AB}}}$$

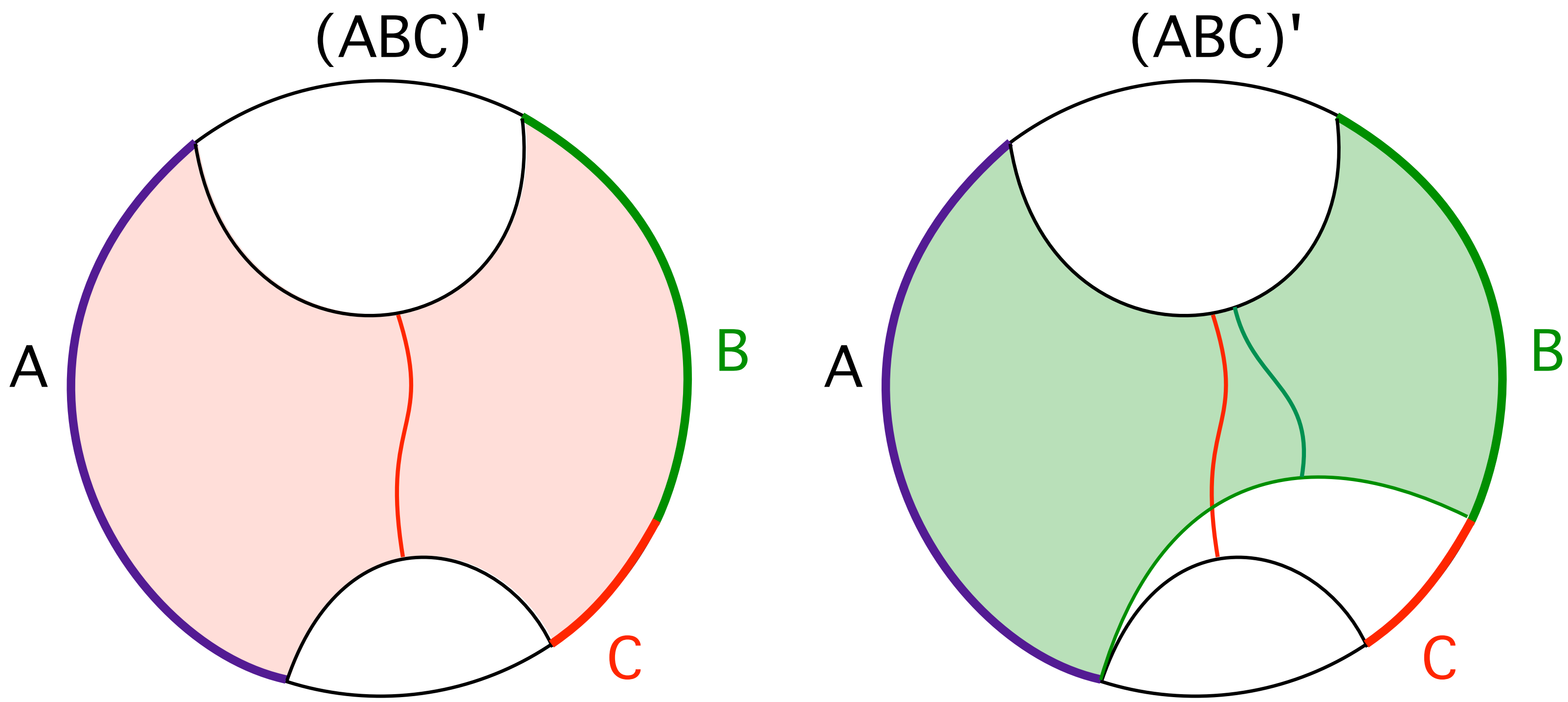
- If ρ_{AB} is pure: $S_R(A:B) = S(A) + S(A^\star) = 2S(A)$
- If ρ_{AB} is factorized: $\sqrt{\rho_{AB}} = \sqrt{\rho_A} \otimes \sqrt{\rho_B} \implies S_R(A:B) = 0$
- Measures entanglement and classical correlations.

$$0 \leq I(A:B) \leq S_R(A:B) \leq 2 \min\,[S(A), S(B)]$$

- E wedge inequality: $S_R(A:BC) \geq I(A:B) + I(A:C)$ [Hayden, Headrick, Maloney `11]
- Monotonicity (?): $S_R(A:BC) \geq S_R(A:B)$

Reflected entropy: properties

$$S_R(A:B) = S(AA^\star)_{\sqrt{\rho_{AB}}}$$



Expected from

- Ent wedge nesting
- $E_W(A:BC) \geq E_W(A:B)$

[Chen, Dong, Lewkowycz, Qi `18]

[Faulkner, Li, Wang`17]

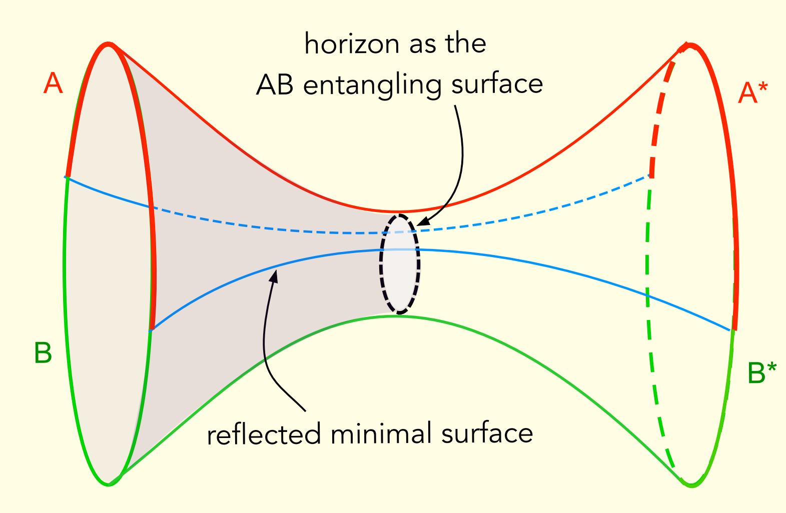
- Monotonicity (?): $S_R(A:BC) \geq S_R(A:B)$

Rènyi version ✓

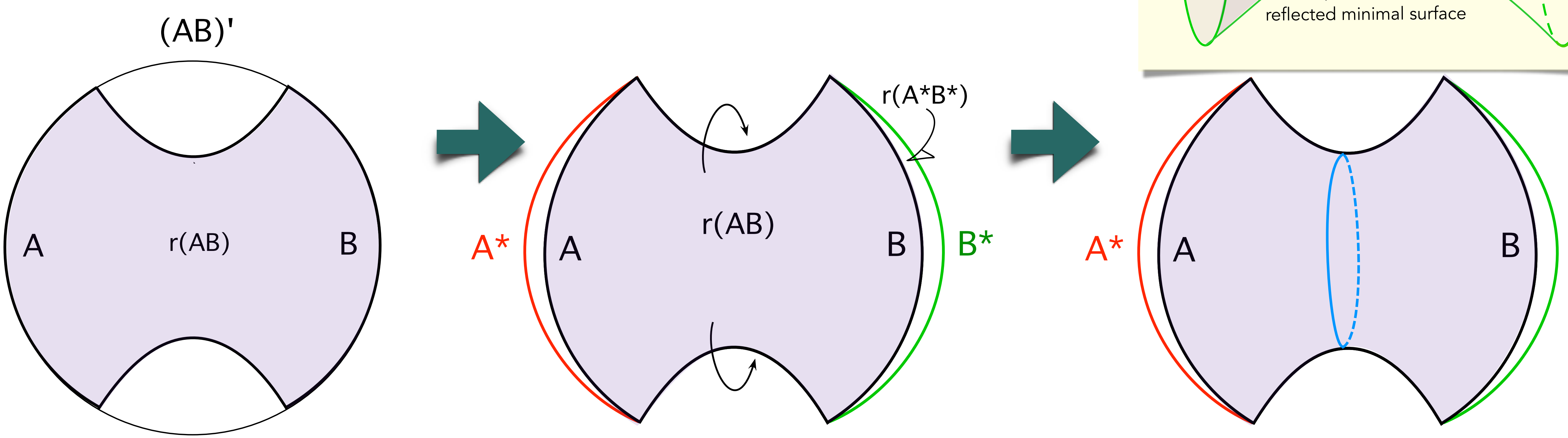
$n \to 1$?

Reflected minimal surfaces

$$S_R(A:B) = \frac{\mathrm{Ar}(m_{AA^\star})}{4G_N} + \dots$$



$m_{AA^\star}$: Bulk surfaces that compute the reflected entropy



$\partial A \cap \partial B = \phi$

Hilbert space split as A, B and (AB)'

$r(AB) \cup r(A^\star B^\star)$

(glued along m_{AB})

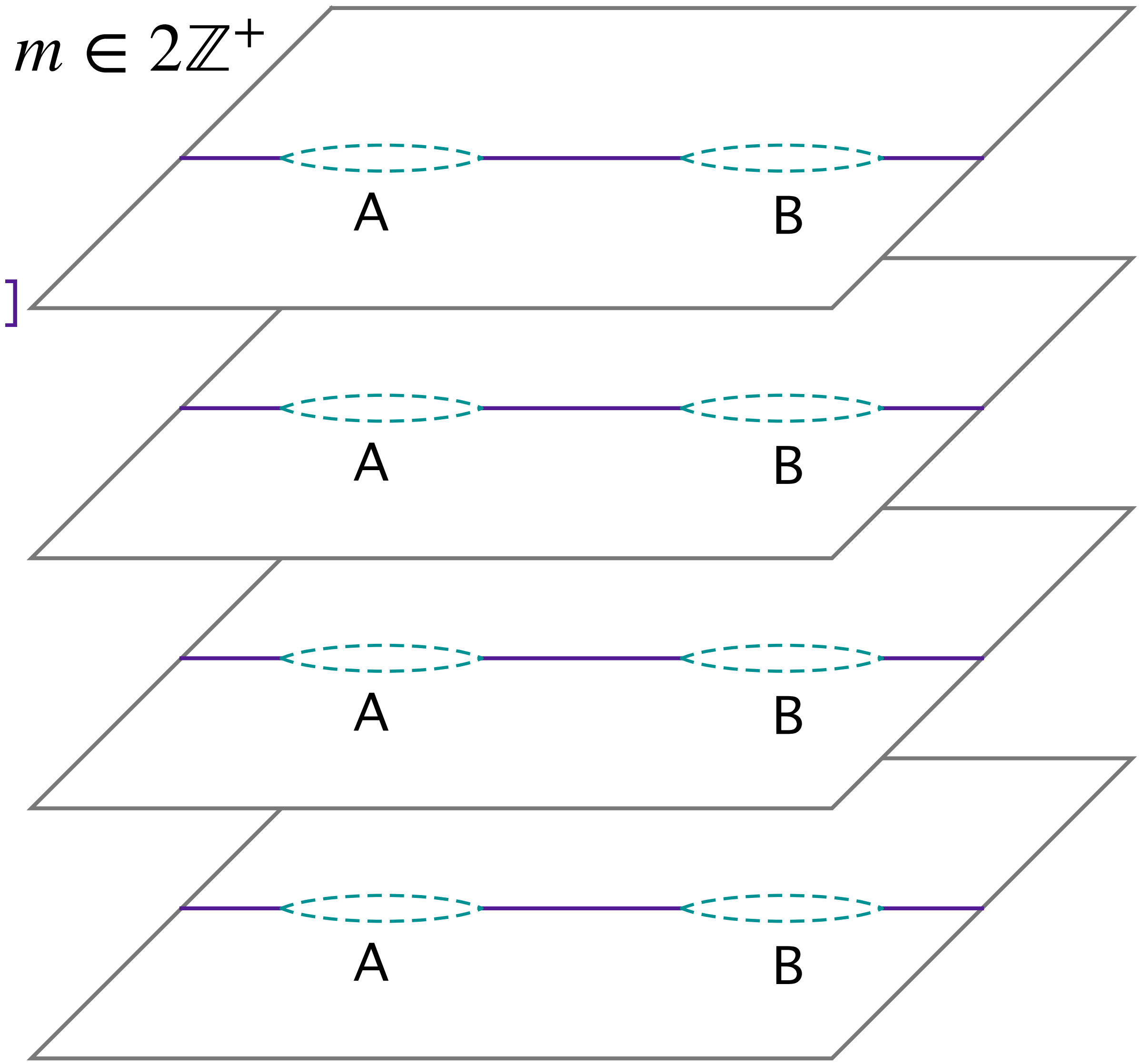
dual to $\sqrt{\rho_{AB}}$

$m_{AA^\star}$ is non-contractible cycle in glued spacetime

Replica trick for reflected entropy

Construct canonical purification $|\rho_{AB}^{1/2}\rangle$, and the corresponding bulk dual.

- Path integral description for $|\rho_{AB}^{m/2}\rangle$, $m \in 2\mathbb{Z}^+$
- by "splitting" usual replica trick for calculating $\mathrm{Tr}\,\rho_{AB}^{m}$ [Calabrese, Cardy, `05]

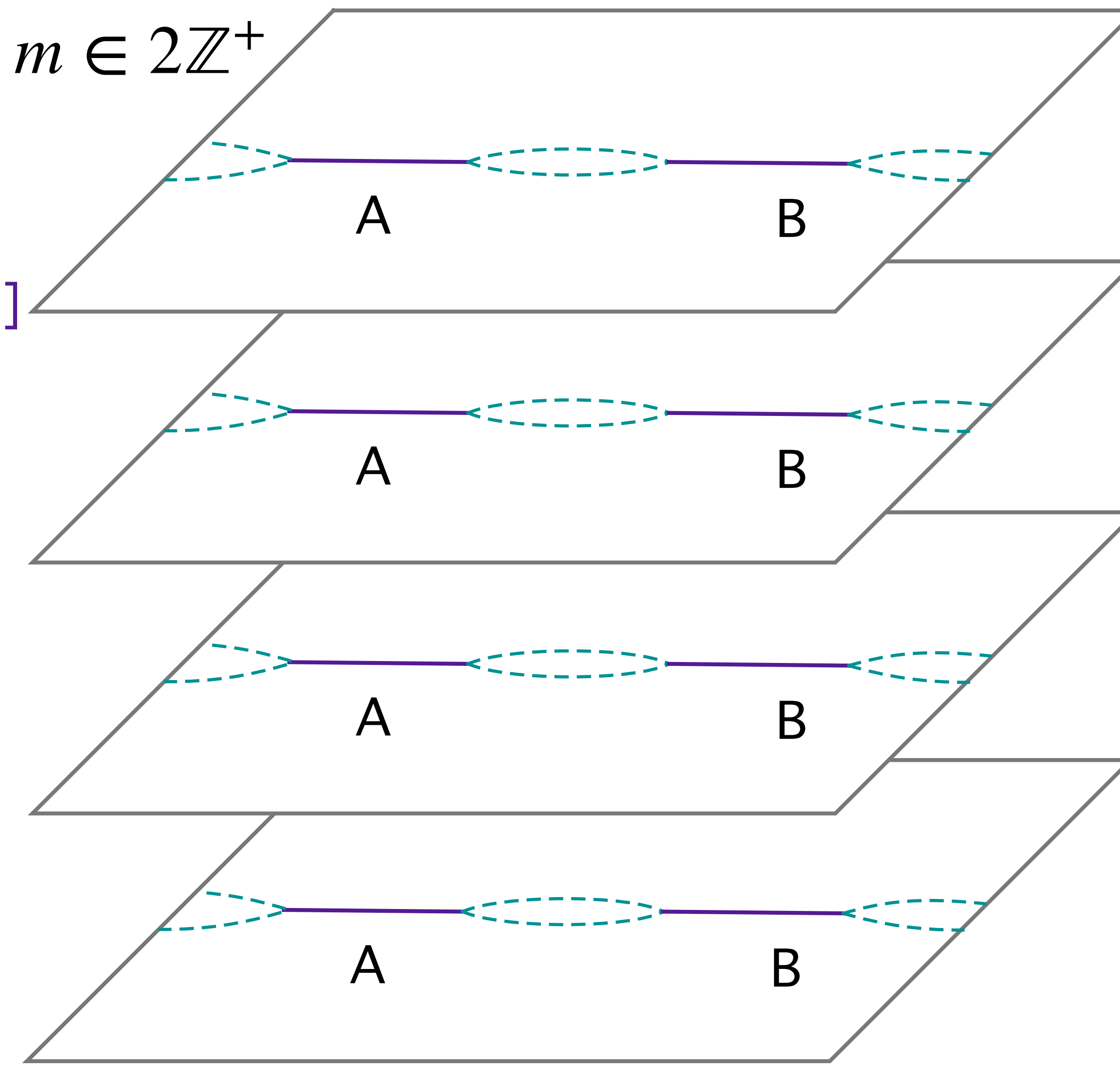


$m = 4$

Replica trick for reflected entropy

Construct canonical purification $|\rho_{AB}^{1/2}\rangle$, and the corresponding bulk dual.

- Path integral description for $|\rho_{AB}^{m/2}\rangle$, $m \in 2\mathbb{Z}^+$
- by "splitting" usual replica trick for calculating $\mathrm{Tr}\,\rho_{AB}^m$ [Calabrese, Cardy, `05]
- $\because\ \mathrm{Tr}\,\rho_{AB}^m = \mathrm{Tr}\,\rho_{(AB)'}^m$

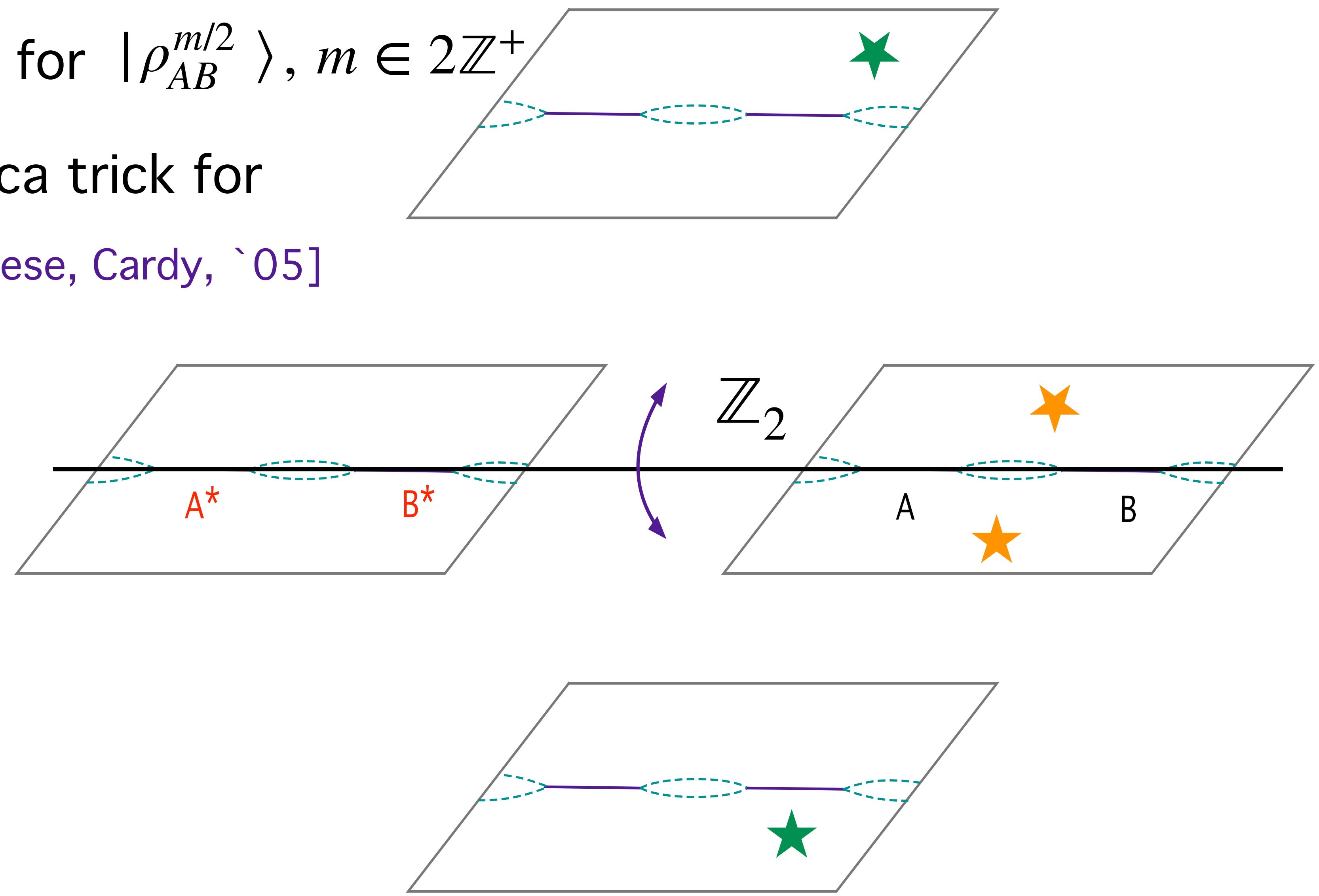


$m = 4$

Replica trick for reflected entropy

Construct canonical purification $|\rho_{AB}^{1/2}\rangle$, and the corresponding bulk dual.

- Path integral description for $|\rho_{AB}^{m/2}\rangle,\ m \in 2\mathbb{Z}^+$
- by "splitting" usual replica trick for calculating $\mathrm{Tr}\,\rho_{AB}^{m}$ [Calabrese, Cardy, `05]
- $\because\ \mathrm{Tr}\,\rho_{AB}^{m} = \mathrm{Tr}\,\rho_{(AB)'}^{m}$
- Global $\mathbb{Z}_2$



$m = 4$

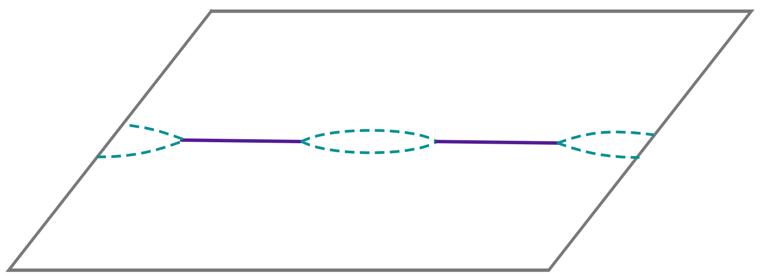
Replica trick for reflected entropy

Construct canonical purification $|\rho_{AB}^{1/2}\rangle$, and the corresponding bulk dual.

- Path integral description for $|\rho_{AB}^{m/2}\rangle,\ m \in 2\mathbb{Z}^+$
- by "splitting" usual replica trick for calculating $\mathrm{Tr}\,\rho_{AB}^m$ [Calabrese, Cardy, `05]
- $\because\ \mathrm{Tr}\,\rho_{AB}^m = \mathrm{Tr}\,\rho_{(AB)'}^m$
- Global $\mathbb{Z}_2$

  Can treat the Euclidean P.I. on lower half as $|\rho_{AB}^{m/2}\rangle$

$$\rho_{AA^\star BB^\star}^{(m)} = \frac{1}{\mathrm{Tr}\rho_{AB}^m}|\rho_{AB}^{m/2}\rangle\langle\rho_{AB}^{m/2}|$$



$m = 4$

Rènyi reflected entropy

- To find the EE of AA* in the state $\rho^{(m)}_{AA^\star BB^\star}$, we first trace over BB* and then compute the Rènyi entropy for replica index $n \in \mathbb{Z}^+$

$$S_n(AA^\star)_{\rho_{AB}^{m/2}} = \frac{1}{n-1} \ln \mathrm{Tr} \left(\rho^{(m)}_{AA^\star}\right)^n$$

- In 2 dimensions, 4-point function of twist operators

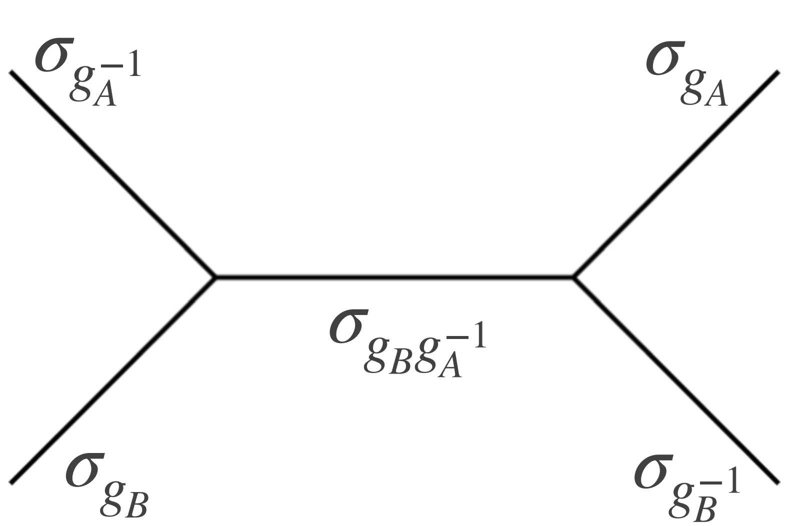
$$S_n(AA^\star)_{\rho_{AB}^{m/2}} \propto \ln \left\langle \sigma_{g_A}(a_1)\sigma_{g_A^{-1}}(a_2)\sigma_{g_B}(b_1)\sigma_{g_B^{-1}}(b_2)\right\rangle$$

$$h_{g_A} = h_{g_B} = \frac{nc}{24}\left(m - \frac{1}{m}\right)$$

- For large-c CFTs with sparse spectrum

$$\lim_{\{m,n\}\to 1} S_n(AA^\star) \approx \frac{2c}{3} \log \left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right)$$

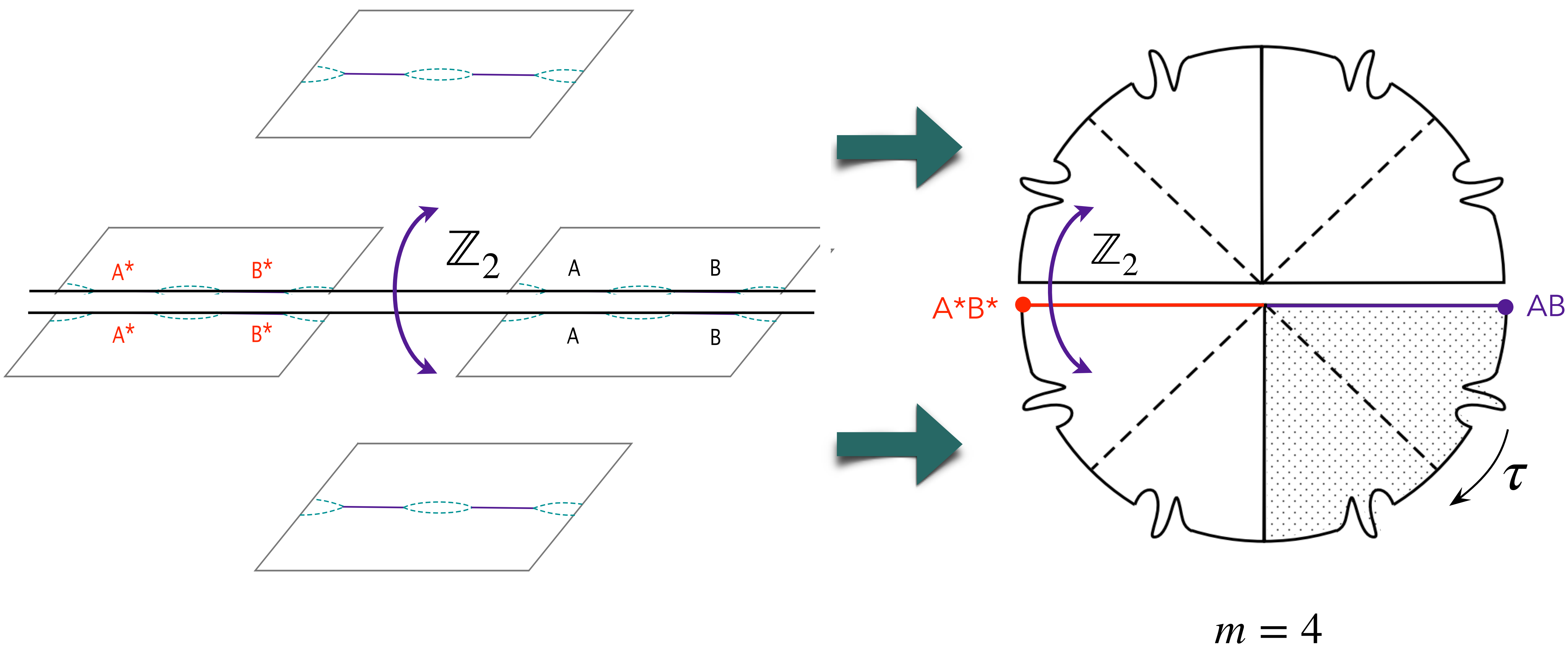
✔Cross-section using AdS_3



[Caputa, Miyaji, Takayanagi, Umemoto, `19]

Establishing $2E_W(A:B) = S_R(A:B)$

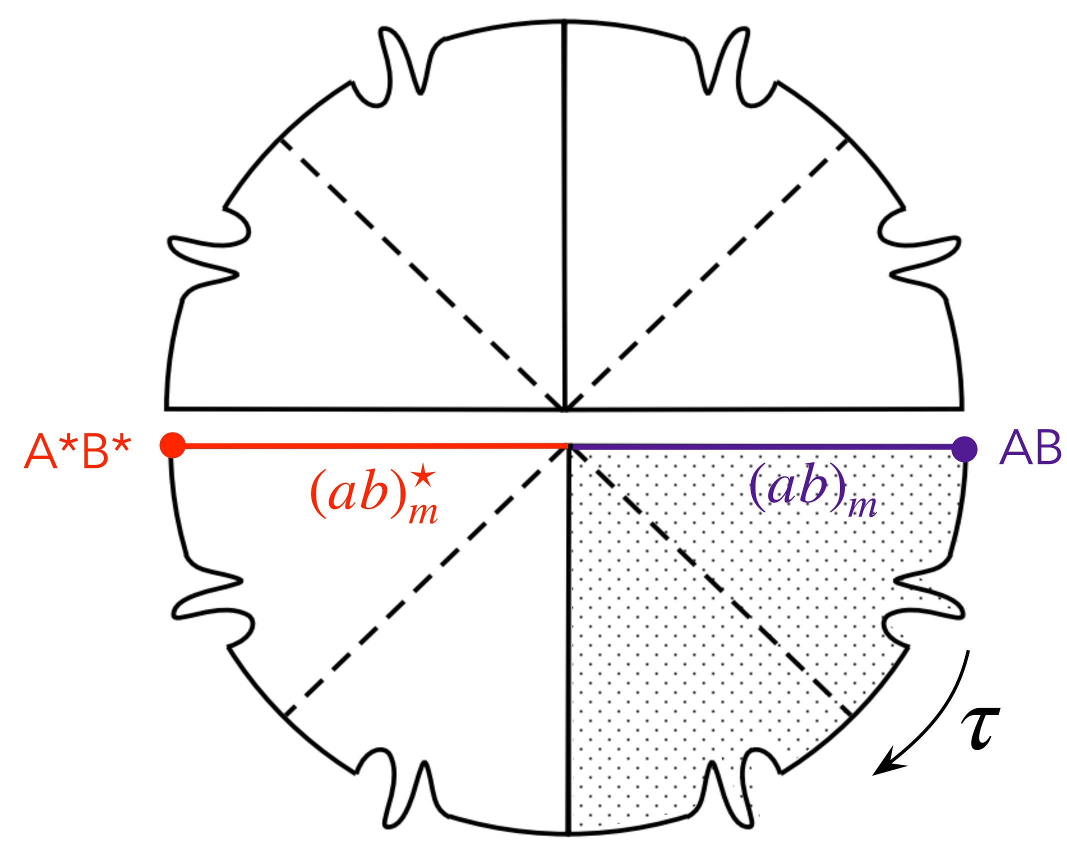
- Fill in bulk à la [Lewkowycz, Maldacena`13]



Establishing $2E_W(A:B) = S_R(A:B)$

- Fill in bulk à la [Lewkowycz, Maldacena`13] Gabriel's talk

$(ab)_m$ and $(ab)^\star_m$ are entanglement wedges for AB, $A^\star B^\star$



$m = 4$

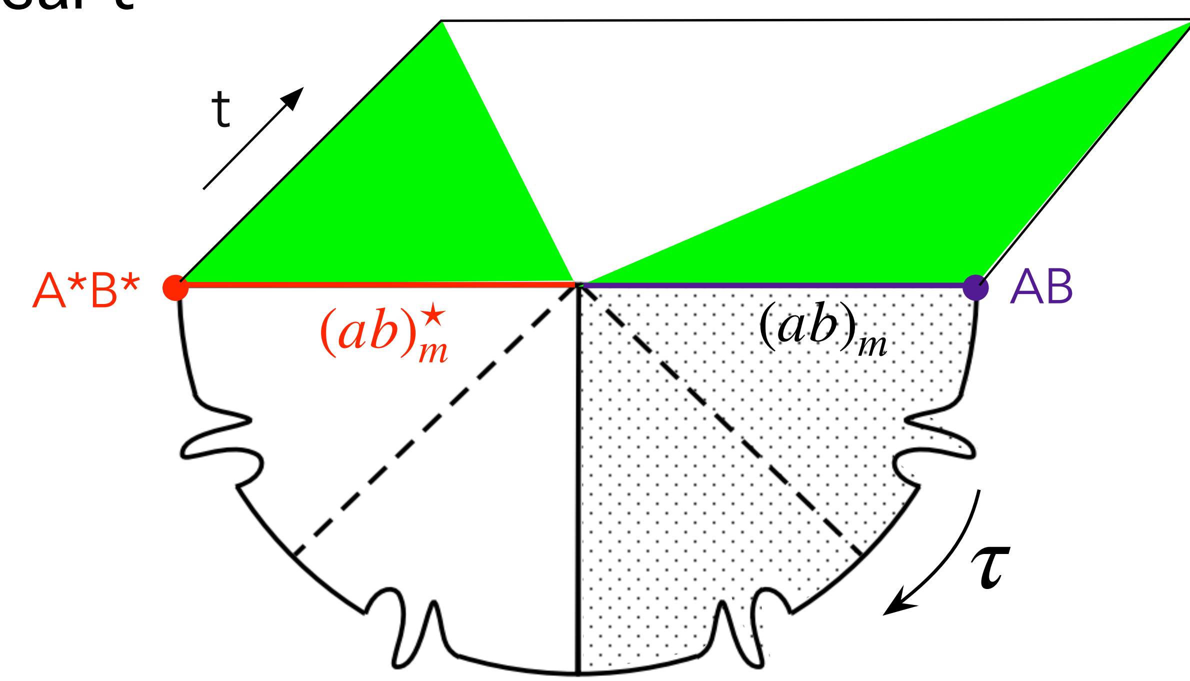
Establishing $2E_W(A:B) = S_R(A:B)$

- Fill in bulk à la [Lewkowycz, Maldacena`13] Gabriel's talk

  $(ab)_m$ and $(ab)_m^\star$ are entanglement wedges for AB, $A^\star B^\star$

- Bulk dual to purification $|\rho_{AB}^{m/2}\rangle$:

  $(ab)_m \cup (ab)_m^\star$ + analytic cont. to real t



$m = 4$

Establishing $2E_W(A:B) = S_R(A:B)$

- Fill in bulk à la [Lewkowycz, Maldacena`13]

  $(ab)_m$ and $(ab)_m^\star$ are entanglement wedges for AB, $A^\star B^\star$

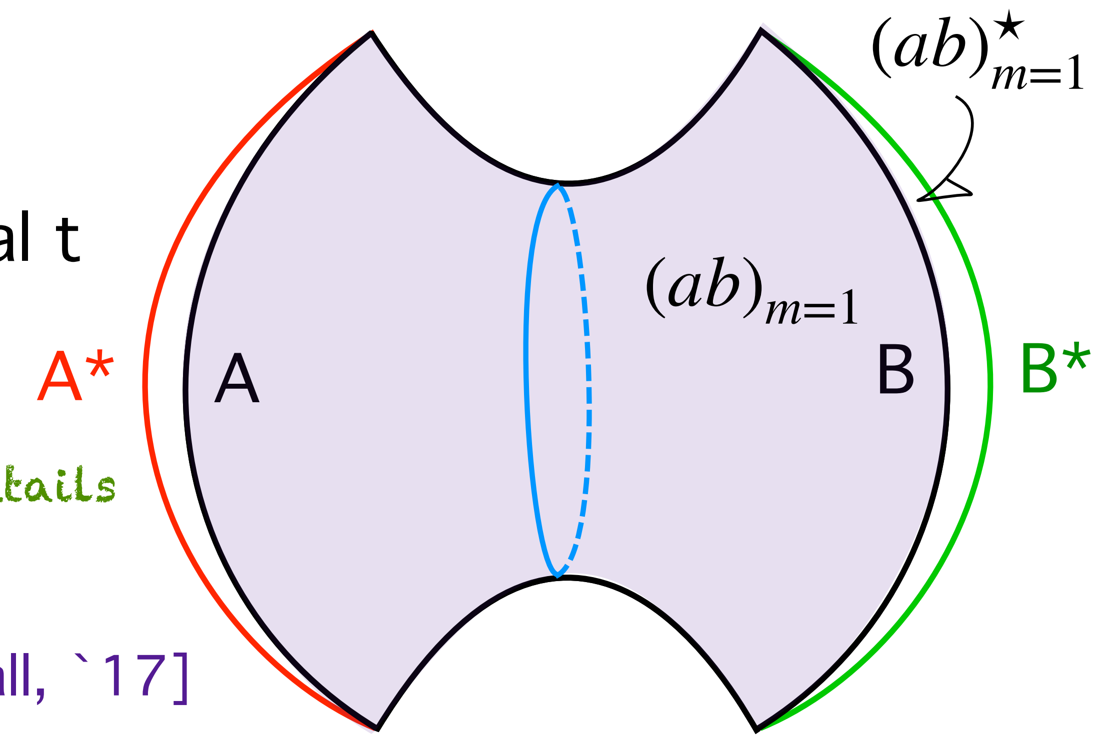
- Bulk dual to purification $|\rho_{AB}^{m/2}\rangle$:

  $(ab)_m \cup (ab)_m^\star$ + analytic cont. to real t

- In the limit $m \to 1$:

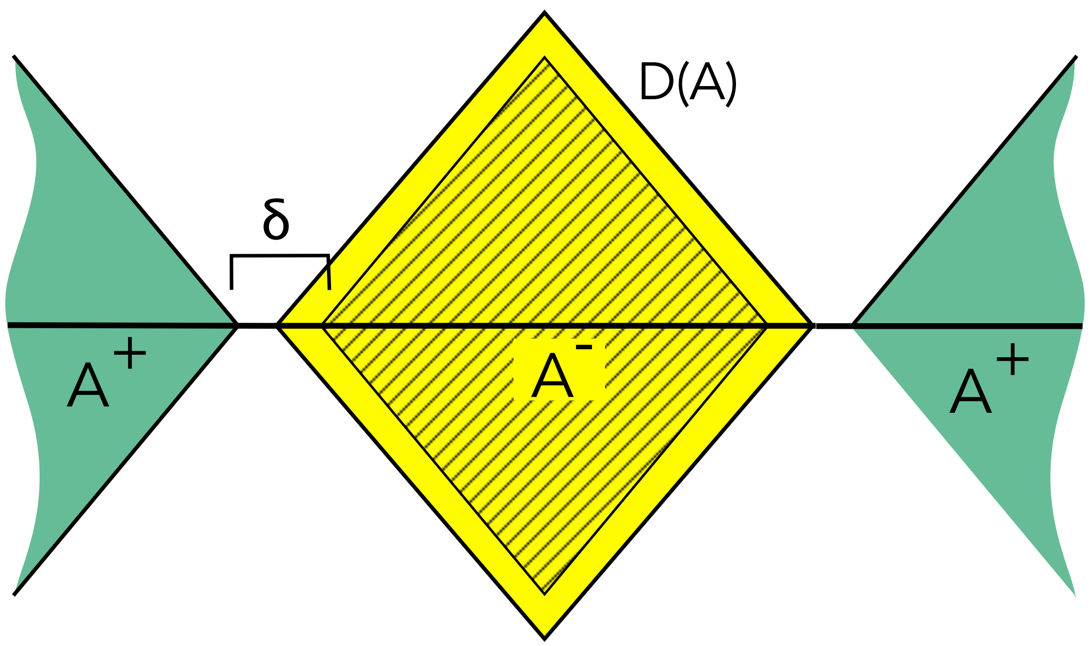
  See Arvin's talk for details

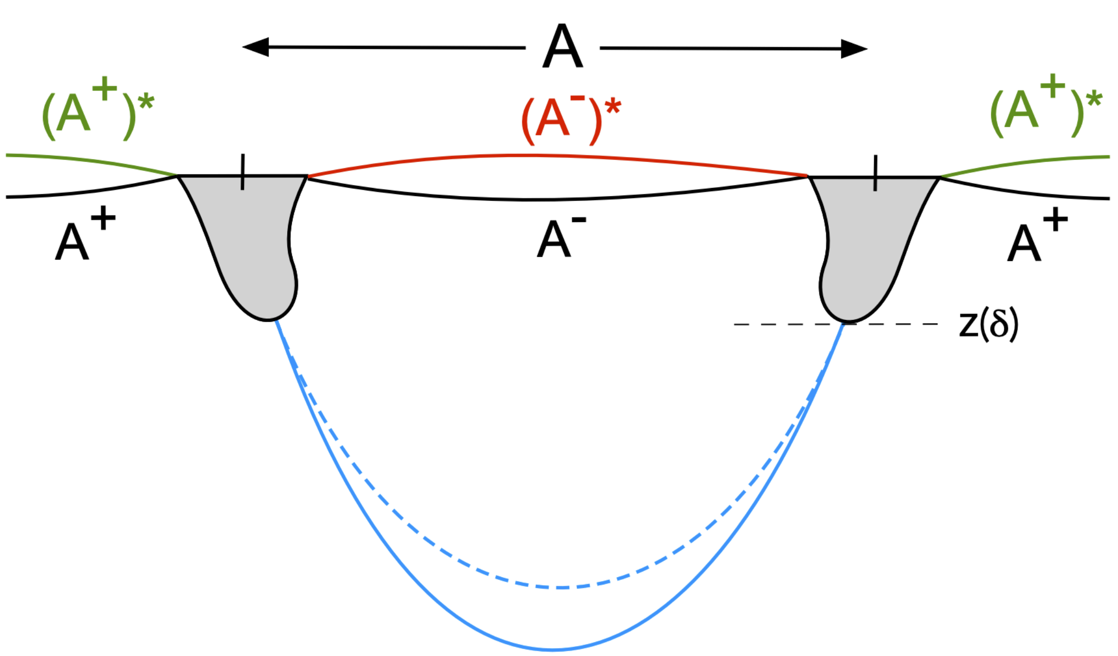
  - Spacetime gluing prescription across extremal surface [Engelhardt, Wall, `17]
  - $\mathbb{Z}_m$ fixed point becomes RT surface for AB
  - By construction, $(ab)_{m=1} = r(ab)$, $(ab)^\star_{m=1} = r(a^\star b^\star)$



$S_R(A:B)$ as geometric regulator

- Reflected entropy as natural regulator for EE in QFTs
- Similar to the Mutual Information regulator [Casini, Huerta, Myers, Yale, `15]





Boundary: $\mathscr{D}(A^-) \subset \mathscr{D}(A) \subset \mathscr{D}((A^+)^c)$

$$\frac{1}{2}I(A^+:A^-) \leq \frac{1}{2}S_R(A^-:A^+)$$

Bulk: Entanglement wedge for $A^- \cup A^+$

$$z(\delta) \propto \delta \implies \frac{1}{2}S_R(A^-:A^+) \approx S_{EE}^{(\delta)}(A)_\psi$$

[Calabrese, Cardy, `05]

ありがとうございます