Holographic Complexity in the Jackiw-Teitelboim Gravity

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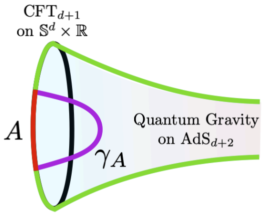
Based on "Holographic Complexity Equals Which Action?" JHEP02(2019)160, arXiv:1901.00014

Work with Hugo Marrochio, Robert C. Myers, Leonel Queimada, Beni Yoshida (Perimeter)

See also: poster presentation by Hugo on 19th June

Entanglement Probes the Bulk Spacetime

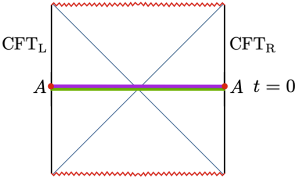


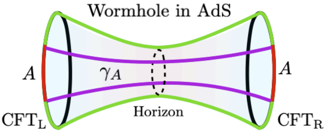
Holographic Entanglement Entropy: Ryu-Takayanagi formula

Entanglement entropy S_A for the region A in CFT
= Area of the minimal surface γ_A in AdS

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

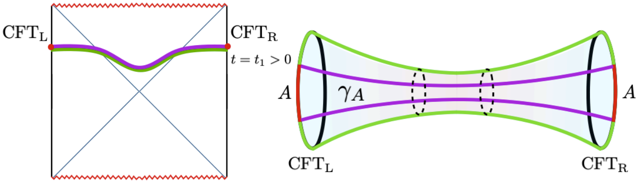
Can Entanglement Probe the Black Hole Interior?





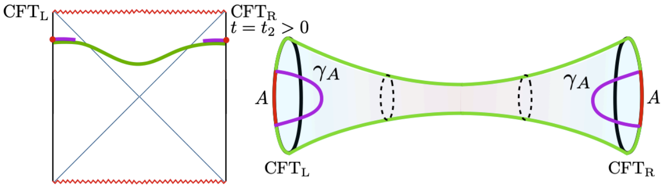
[Hartman-Maldacena '13]

Can Entanglement Probe the Black Hole Interior?



[Hartman-Maldacena ’13]

Can Entanglement Probe the Black Hole Interior?



[Hartman-Maldacena '13]

Entanglement

grows for a short time, stops growing after the system thermalizes

$\Updownarrow$ discrepancy

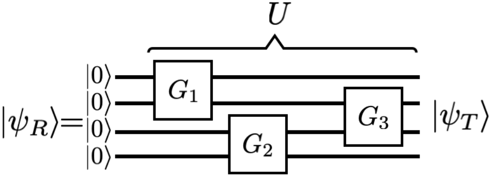
Wormhole

the growth lasts for a very long time

- Susskind '14
  "Entanglement is not enough to understand the rich geometric structures that exist behind the horizon"

Missing Link -Complexity?

- Quantity encoding that growth in the quantum state?
  → Susskind proposed: “complexity” of the q uantum state
- Complexity: min # of operations necessary to get a particular state



- Quantum circuit model:

$$|\psi_T\rangle = U|\psi_R\rangle$$

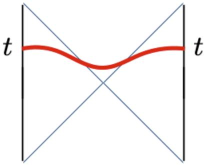
  $|\psi_T\rangle$: a target state $|\psi_T\rangle$; a simple reference state (eg. $|0\rangle|0\rangle \cdots |0\rangle$)
  U: unitary transformation built from a particular global set of gates
- Complexity = # of elementary gates in the optimal or shortest circuit
- Complexity is expected to grow linearly in time for a very long time in chaotic theories

Holographic Complexity

- Bulk quantity that probes the growth of the black hole interior?
  "Holographic complexity"
  [Susskind'14 Brown-Roberts-Susskind-Swingle-Zhao-Ying'16]

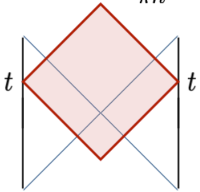
Complexity = Volume

$$\mathcal{C}_{\mathcal{V}}(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N \ell} \right]$$



Complexity = Action

$$\mathcal{C}_{\mathcal{A}}(\Sigma) = \frac{I_{\mathrm{WDW}}}{\pi\hbar}$$



WDW patch
=union of all the spatial slices
anchored at a given boundary time

Holographic Complexity is really complexity?

- At least for examples which have been tested, both CA and CV lead to linear growth at late times

$$\frac{dC}{dt} \sim S\,T$$

- Responses to insertions of operators (precursors) are well represented by the shockwave geometries

Both defs always reproduce the expected behavior of complexity?

- AdS_2/SYK duality is a good place to test!

SYK model: quantum mechanical model of fermions
→definition of complexity could be well understood

AdS_2: described by the Jackiw-Teitelboim gravity
→ simple enough to allow explicit computations both for CV and CA

$\Uparrow$
Today's focus!

Similar arguments done in [Brown-Gharibyan-Lin-Susskind-Thorlacius-Zhao '18]

Jackiw-Teitelboim Gravity

- JT model: $1+1$-dimensional dilaton gravity [Teitelboim '83 Jackiw '85]

$$
\begin{aligned}
I_{JT} &= \frac{\Phi_0}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\,\sqrt{-g}R + 2\int_{\partial\mathcal{M}} d\tau K\right] \\
&= \frac{1}{16\pi G_N}\left[\int_{\mathcal{M}} d^2x\,\sqrt{-g}\Phi(R+\frac{2}{L_2^2}) + 2\int_{\partial\mathcal{M}} d\tau\Phi(K-\frac{1}{L_2})\right]
\end{aligned}
$$

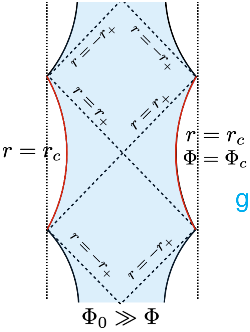
- 1st line: topological term with a const. dilaton $\Phi_0 \to$ Euler character
- 2nd line: terms depending on a dynamical dilaton $\Phi \to$ give EOM

$$
\begin{aligned}
0 &= R + \frac{2}{L_2^2}\,, \\
0 &= \nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi + g_{\mu\nu}\frac{1}{L_2^2}\Phi
\end{aligned}
$$

Nearly AdS_2 Solution

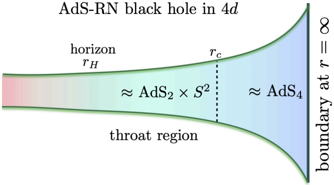
- AdS_2 solution

$$\Phi = \frac{\Phi_c}{r_c} r\,,\ ds^2 = -\frac{r^2 - r_+^2}{L_2^2} dt^2 + \frac{L_2^2}{r^2 - r_+^2} dr^2$$



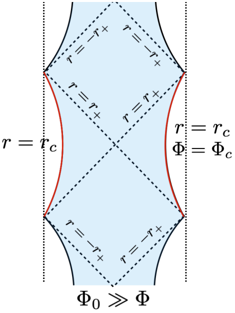
- Focus on the region $\Phi_0 \gg \Phi$
  $\Leftrightarrow$ spacetime cut-off at $r = r_c$ where $\Phi_0 \gg \Phi_c$
  [Maldacena-Stanford-Yang '16]

$\rightarrow$JT model: effective description of the throat region of near-extremal RN black hole in higher dim.



Φ_0: area of the extremal bh , Φ: deviation of the area from the extremality

Nearly AdS$_2$ Solution



- AdS$_2$ solution represents a black hole with

$$T_{JT} = \frac{r_+}{2\pi L_2^2}$$

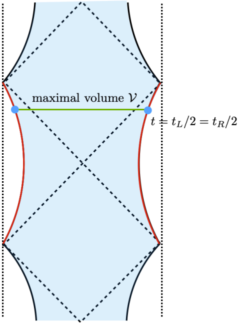
and

$$S_{JT} = \frac{\Phi_0 + \Phi(r_+)}{4G_N} = S_0 + \frac{\pi L_2^2}{2G_N}\frac{\Phi_c}{r_c}T_{JT}$$

$$M_{JT} = \frac{\Phi_c r_+^2}{16\pi G_N L_2^2 r_c} = \frac{\pi L_2^2}{4G_N}\frac{\Phi_c}{r_c}T_{JT}^2$$

- Extremal entropy S_0: associated to the extremal RN black hole in higher dimensions

Complexity=Volume in the JT Gravity



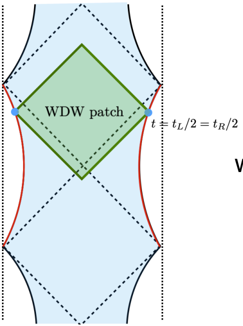
- Complexity in the CV proposal is computable analytically

$$\frac{dC_{\mathcal{V}}}{dt} \sim 8\pi S_0 T_{JT} \qquad \text{as } t \to \infty$$

- Complexity grows linearly in t as expected from the chaotic nature of the SYK

- $S_{JT} \sim S_0$: the number of dof
  T_{JT} : the scale for the rate at which new gates are introduced

Complexity=Action in the JT Gravity



- Complexity in the CA proposal

$$C_{\mathcal{A}} = \frac{I^{JT}_{WDW}}{\pi\hbar}$$

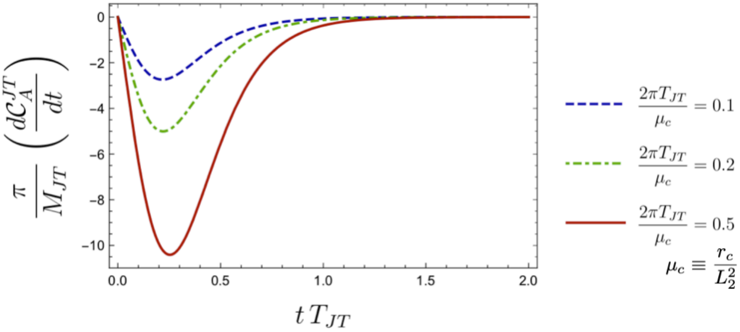
where

$$I^{JT}_{WDW} = I^{JT}_{bulk} + I^{JT}_{boundary}$$
$$I^{JT}_{boundary} = I^{JT}_{GHY} + I^{JT}_{joint} + I^{JT}_{bdry\ ct.}$$

- At late times, the contribution from $I^{JT}_{bulk} < 0$ and $I^{JT}_{boundary} > 0$ are exactly canceled out!

$$\frac{dC_{\mathcal{A}}}{dt} \sim 0 \qquad \text{as } t \to \infty$$

- C=A gives a different answer from C=V for the JT model!



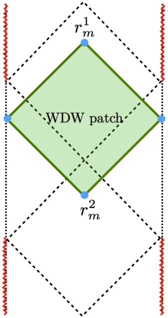
0
−2
−4
−6
−8
−10
0.0
0.5
1.0
1.5
2.0
$t\,T_{JT}$
$\frac{\pi}{M_{JT}}\left(\frac{d\mathcal{C}_A^{JT}}{dt}\right)$
$\frac{2\pi T_{JT}}{\mu_c} = 0.1$
$\frac{2\pi T_{JT}}{\mu_c} = 0.2$
$\frac{2\pi T_{JT}}{\mu_c} = 0.5$
$\mu_c \equiv \frac{r_c}{L_2^2}$

Complexity=Action for the RN black holes in $4d$

- JT model: derived from a dim reduction of the $4d$ Einstein-Maxwell theory → re-examine holographic complexity in $4d$

$$I_{EM} = \frac{1}{16\pi G}\int_{\mathcal{M}}(R + \frac{6}{L^2}) + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} K - \frac{1}{16\pi G_N}\int_{\mathcal{M}} F^2$$

- I_{EM} describes the electrically/ magnetically charged black holes



- Since $F^2 \sim B^2 - E^2$,

$$\frac{dI_{EM}}{dt} = \frac{1}{2G_N}\left[\frac{r^3}{L^2} \pm \frac{4\pi Q^2}{r}\right]_{r_M^2}^{r_m^1} \qquad \begin{cases} + : \text{electric} \\ - : \text{magnetic} \end{cases}$$

$$\frac{dC_{\mathcal{A}}}{dt} \sim \begin{cases} \frac{2\pi Q^2}{G_N}(1/r_- - 1/r_+) & : \text{electric} \\ 0 & : \text{magnetic} \end{cases}$$

- JT action: derived with an ansatz of magnetic solutions for the Maxwell field → consistent with $2d$!

Adding the Maxwell boundary term

- One can add the Maxwell bdy term to the original action I_{EM}

$$\tilde{I}_{EM}(\gamma) = I_{EM} + \frac{\gamma}{G_N}\int_{\partial\mathcal{M}} F^{\mu\nu}A_\mu n_\nu$$

n_ν: unit normal vector to the bdy

- It changes the behavior of the complexity

$$\frac{dC_{\mathcal{A}}(\gamma)}{dt} \sim \begin{cases} (1-\gamma)\frac{2\pi Q^2}{G_N}(1/r_- - 1/r_+) & : \text{electric} \\ \gamma\frac{2\pi Q^2}{G_N}(1/r_- - 1/r_+) & : \text{magnetic} \end{cases}$$

- When $\gamma = 1$, in contrast to the $\gamma = 0$ case

$$\frac{dC_{\mathcal{A}}(\gamma = 1)}{dt} \sim \begin{cases} 0 & : \text{electric} \\ \frac{2\pi Q^2}{G_N}(1/r_- - 1/r_+) & : \text{magnetic} \end{cases}$$

Role of the Maxwell boundary term?

- The Maxwell boundary term $I^{bdy}_{Max}(\gamma)$ for a physical boundary
  $\rightarrow$ changes the boundary condition of the Maxwell field A_μ
- In the Euclidean path-integral of quantum gravity,

different b.c. $\Leftrightarrow$ different thermodynamic ensemble

Specifically, (Q:charge, μ: "chemical potential" conjugate to charge Q)
[Hawking-Ross '95]

$$\text{Fixed-}Q\text{ ensemble}\begin{cases}\text{electric} & \text{with} \quad I^{bdy}_{Max}(\gamma=1) \\ \text{magnetic} & \text{with} \quad I^{bdy}_{Max}(\gamma=0)\end{cases} \quad \rightarrow \frac{dC_{\mathcal{A}}}{dt} \sim 0$$

$$\text{Fixed-}\mu\text{ ensemble}\begin{cases}\text{electric} & \text{with} \quad I^{bdy}_{Max}(\gamma=0) \\ \text{magnetic} & \text{with} \quad I^{bdy}_{Max}(\gamma=1)\end{cases} \quad \rightarrow \frac{dC_{\mathcal{A}}}{dt} \sim \frac{2\pi Q^2}{G_N}(1/r_- - 1/r_+)$$

Complexity=Action is sensitive to the thermodynamic ensemble?

Conclusion

- In the JT model, the $C_{\mathcal{A}}$ gives the different behavior from $C_{\mathcal{V}}$
  $\rightarrow$ the growth rate vanishes at late times!

- In $4d$, the similar behavior of $C_{\mathcal{A}}$ can be seen for the magnetic solutions described by I_{EM}

- In $4d$, introduction of the Maxwell bdy term changes the behavior of the complexity

- The complexity=action might be sensitive to the thermodynamic ensemble
  $\rightarrow$ Charge-confining b.c. : $\frac{dC_{\mathcal{A}}}{dt} \sim 0$
  Charge-permeable b.c. : $\frac{dC_{\mathcal{A}}}{dt} \sim \mathrm{const.}(\neq 0)$

- JT model corresponds to the charge-confining b.c.
  $\rightarrow$ vanishing growth of complexity

Thank you

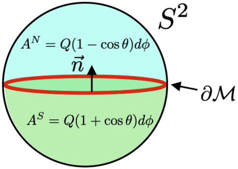
Maxwell boundary term for the magnetic solutions

Consider the contribution from the Maxwell bdy term

$$I^{bdy}_{Max} = \frac{1}{G_N} \int_{\partial \mathcal{M}} F^{\mu\nu} A_\mu n_\nu$$

n_ν: unit normal vector to the bdy

for the magnetic solutions $F_{\theta\phi} = \partial_\theta A_\phi = Q \sin\theta$



- Dirac string → different gauge fields for the northern/southerm hemi-sphere of S^2
- $\partial \mathcal{M}$ consists of the boundary of the northern/southerm hemi-sphere

- The dim reduction of the Maxwell bdy term for the magnetic case? → S^2 shrinks to a point: no $\partial \mathcal{M}$
- difficult to introduce the bdy term to the JT model to change the behavior of $C_{\mathcal{A}}$
- Alternatively, we can convert the bdy term into the bulk term by using the Stoke's theorem → different bulk action from the JT model