Towards a unified picture of complexity for quantum fields

Michal P. Heller

aei.mpg.de/GQFI

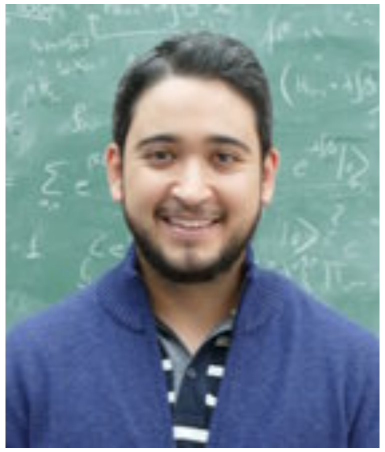
Alexander von Humboldt
Stiftung/Foundation

Path integral optimization as circuit complexity

Michal P. Heller

aei.mpg.de/GQFI

based on **1904.02713** with



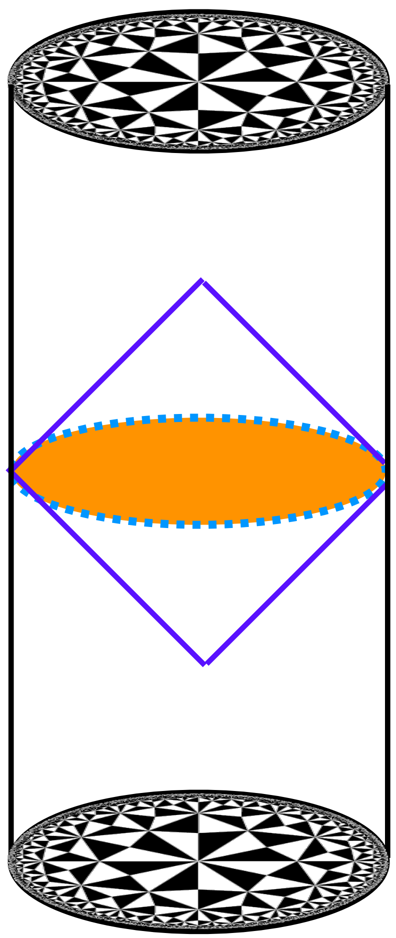
H. Camargo

R. Jefferson

J. Knaute

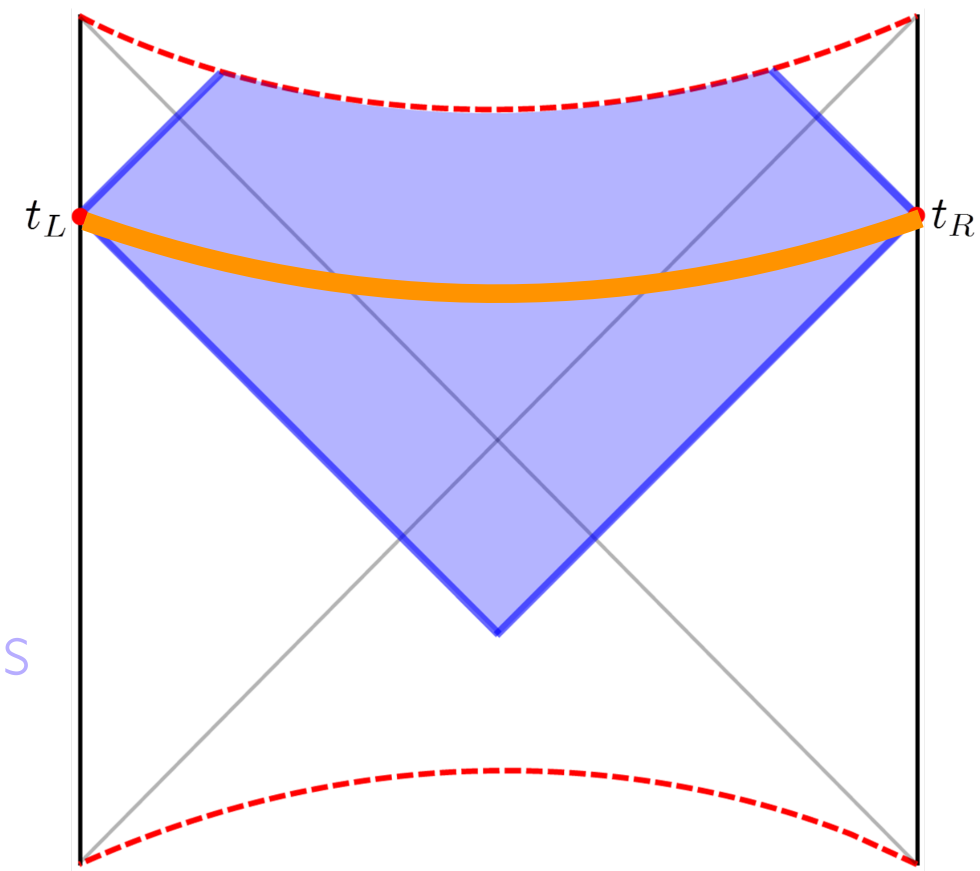
Motivation 1: holographic complexity proposals

1402.5674 by Susskind, **1509.07876** by Brown et al., …



$\mathcal{C}_V \sim$ Volume of codim-1 max volume bulk slice

$\mathcal{C}_A \sim$ Action in codim-0 bulk region with null bdries



What do holography complexity proposals stand for in hQFT?

Motivation 2: complexity in free QFTs

To my taste, $\mathcal{C}_V$ and $\mathcal{C}_A$ looked a lot like calculating HRT surfaces before first works on entanglement entropy in QFT (pioneers: 1980s, explosion > 2004)

Geometric approach to complexity (of operators):

quant-ph/0502070 by Nielsen, ...

$$|T\rangle = \mathcal{P}e^{-\int_{\kappa_1}^{\kappa_2} d\kappa \sum_I \mathcal{O}_I Y^I(\kappa)}|R\rangle \longrightarrow \mathcal{C}_{L_1} \sim \min\left[\int_{\kappa_1}^{\kappa_2} d\kappa \sum_I \eta_I |Y^I(\kappa)|\right]$$

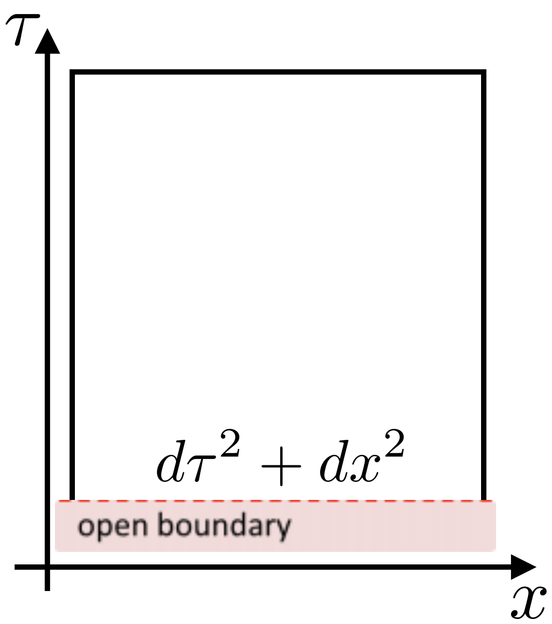
First applications to free QFTs on a lattice / cMERA regularization:

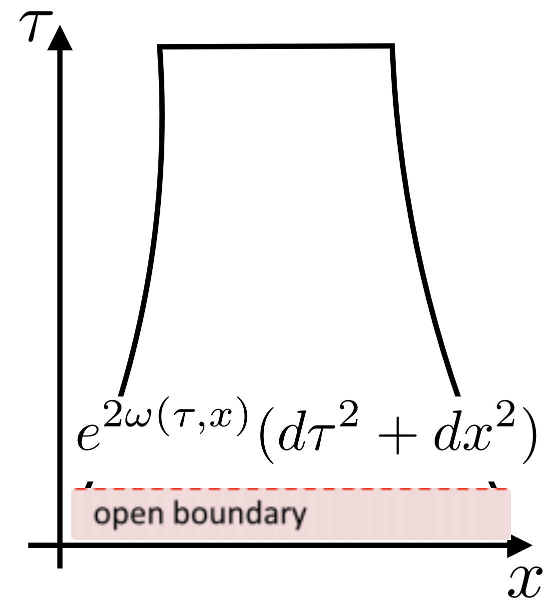
1707.08582, 1707.08570 by Jefferson & Myers, 1807.07075, 1810.05151, ...

- unitary gates (for bosons): $\mathcal{O}_I \sim i\,\phi_j\phi_l, \quad i\,\pi_j\,\pi_l \quad \text{and} \quad i\,\phi_{(j}\pi_{l)}$
- for very fine-tuned cost function, $\mathcal{C}_{\text{certain}\,L_2}$, one can get exact results
- circuits then also use very non-local gates, e.g. $\mathcal{O}_I = i\,\phi_{\text{here}}\,\phi_{\text{other galaxy}}$
- the good: similar divergence structures to holography (universality?)
- the bad (but expected): very different time dependence than holography

Motivation 3: path integral optimization in CFT_2

1703.00456 by the Kyoto group, …





$$\tilde{\Psi}[\phi_0(x)] = \exp\left\{\boxed{\frac{c}{24\pi}\int_{0^+}^{\infty} d\tau \int_{-\infty}^{\infty} dx \left(\frac{\mu_0}{\epsilon^2}e^{2\omega} + \dot{\omega}^2 + \omega'^2\right)}\right\} \times \Psi[\phi_0(x)]$$

The Liouville action (covariant)

ϵ : lattice spacing

minimization

optimal path integral defined on H_2 (relation to $\mathcal{C}_V$ unclear, but tempting) path integral optimization

$$\stackrel{?}{=}$$

circuit depth minimization

see the original papers and **1706.00965** by Czech for a qualitative TN interpretation

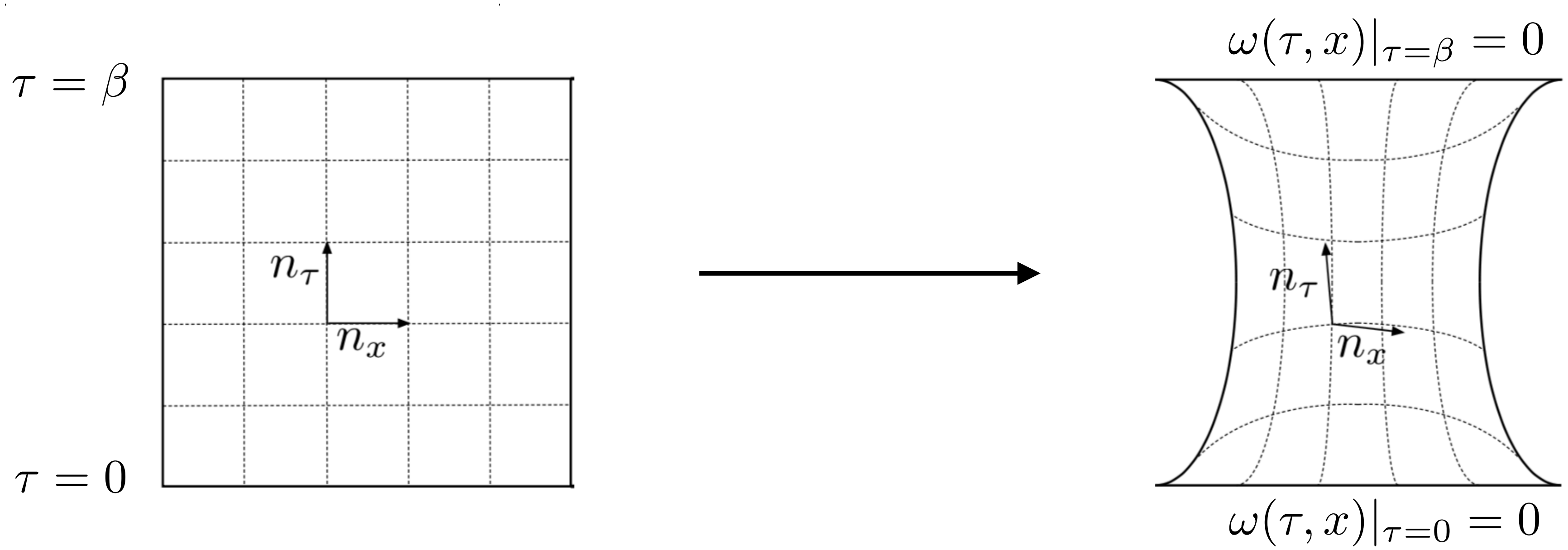
Setup

From now on we focus entirely on CFT_2 on a line (with no assumptions on c)

The object of interest will be $\rho_\beta = e^{-\beta H}$ and we will <u>ignore normalization</u>

Matrix elements of ρ_β are computed by Euclidean path integral on $[0,\beta]\times\mathbb{R}$

We preserve the <u>operator</u> when deforming the metric $e^{2\,\omega(\tau,x)}(d\tau^2+dx^2)$:



$$\mathcal{P}e^{-\int_0^\beta d\tau\int_{-\infty}^{\infty}dx\,T_{tt}(x)} \longrightarrow \text{???}$$

From Euclidean path integrals to circuits in CFT_2

1807.02501 by Milsted & Vidal

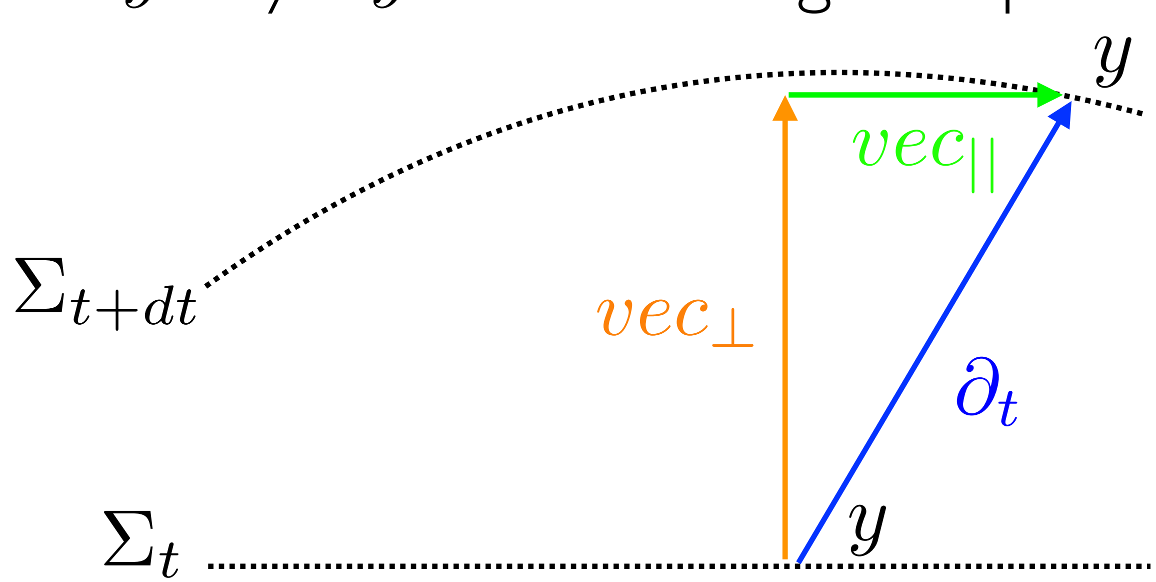
There is a very simple prescription from path integrals on

$$(a^2+b^2)\,dt^2+2b\,dt\,dy+dy^2$$

to circuits involving components $T_{t_M t_M}$ and $T_{t_M y}$ on $-dt_M^2+dy^2$:

$$\mathcal{P}\exp\left[-\int_{t_i}^{t_f}dt\int_{-\infty}^{\infty}dy\,\{a(t,y)\,T_{t_M t_M}(y)+i\,b(t,y)T_{t_M y}(y)\}\right]$$

The basic idea is that we want to generate repeatedly a trafo from a slice at t to a slice $t+dt$ in a y - by - y fashion using flat space generators:



$|vec_\perp| = a$

$|vec_{||}| = \pm b$

The basic idea

If we then take as gates Hermitian exps of $T_{t_M t_M}$ and unitary exps of $i\,T_{t_M y}$, then the thing to do is to consider a coordinate trafo

$$e^{2\omega}(d\tau^2 + dx^2) \quad\longrightarrow\quad (a^2+b^2)\,dt^2 + 2\,b\,dt\,dy + dy^2$$

This simultaneously does two things

- it expresses $\mathcal{P}\exp\left[-\int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dy\,\{a(t,y)\,T_{t_M t_M}(y) + i\,b(t,y)T_{t_M y}(y)\}\right]$

  in terms of ω

- it guarantees that the operatorial expression still reproduces ρ_β

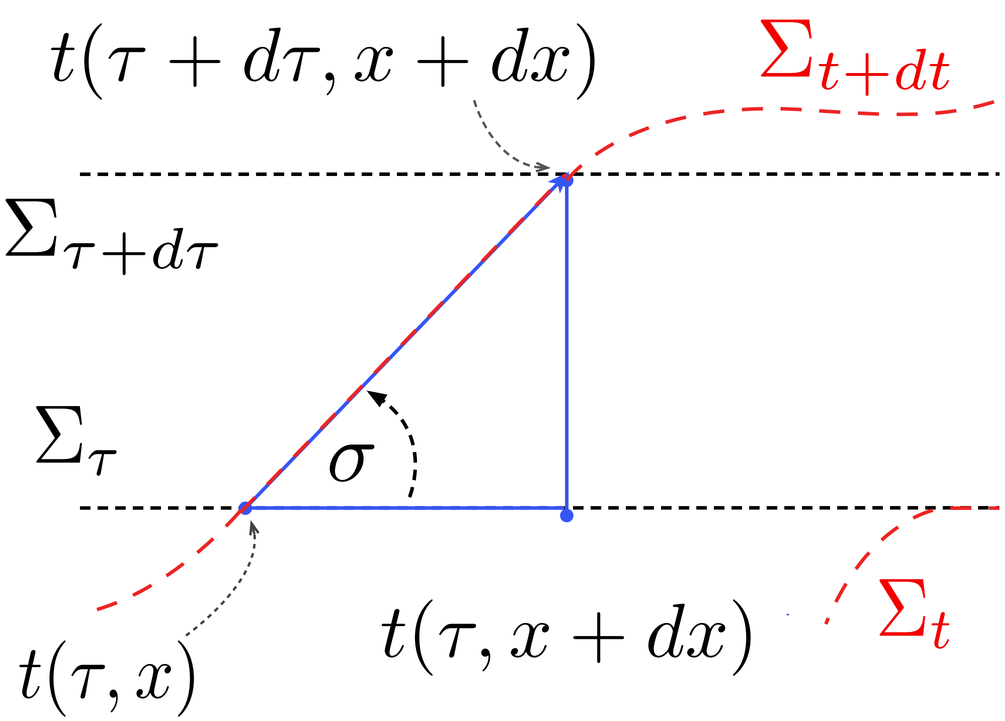
One such trafo is $t=\tau$ and $y = \displaystyle\int_0^x e^{\omega(\tau,\xi)}d\xi$, but there are more

Time slicing freedom

From this point of view, it is more convenient to look the other way around

$$(a^2+b^2)\,dt^2 + 2\,b\,dt\,dy + dy^2 \xrightarrow{\begin{cases} t = t(\tau,x) \\ y = y(\tau,x) \end{cases}} e^{2\omega}(d\tau^2 + dx^2)$$

This gives us an additional function on top of ω; we can* take it to be σ:



Cost function — an example

We want to view $\mathcal{P}\exp\left[-\int_{t_i}^{t_f} dt \int_{-\infty}^{\infty} dy \,\{a(t,y)\, T_{t_M t_M}(y) + i\, b(t,y) T_{t_M y}(y)\}\right]$

as $\mathcal{P}e^{-\int_{\kappa_1}^{\kappa_2} d\kappa \sum_I \mathcal{O}_I Y^I(\kappa)}$ and count gates. Perhaps the simplest thing to do is:

$$\text{cost}_{L_1} \sim \int \mathrm{d}t\,\mathrm{d}y\, \frac{1}{\epsilon^2}\, \left(|a| + \epsilon\, \eta_{\partial a} |\partial_y a| + \epsilon\, \eta_{\partial b} |\partial_y b| + \ldots\right)$$

$$\downarrow$$

$$\begin{aligned}&\int_0^\beta \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{e^\omega}{\epsilon^2} \Big\{ e^\omega \\ &+ \epsilon\, \eta_{\partial a} \big|(\dot{\omega} - \sigma')\, \sin\sigma + (\omega' + \dot{\sigma})\, \cos\sigma\big| \\ &+ \epsilon\, \eta_{\partial a} \big|(\omega' + \dot{\sigma})\, \sin\sigma - (\dot{\omega} - \sigma')\, \cos\sigma\big|\end{aligned}$$

σ-dependence makes it non-covariant. For $\omega(\tau) \longrightarrow \simeq \int_0^\beta \mathrm{d}\tau\, \frac{1}{\epsilon}\, \{e^{2\,\omega} + \epsilon\, \eta_{\partial B} |e^\omega\, \dot{\omega}|\}$

How to get Liouville as a cost function?

We tried very hard and kept failing for various reasons. However, it turns out that at least the following DBI-ish cost function does the job:

$$\int \mathrm{d}t\, \mathrm{d}y\, \frac{1}{\epsilon^2}\sqrt{a^2 + \epsilon^2 \eta_{(\partial a)^2}(\partial_y a)^2 + \epsilon^2\, \eta_{(\partial b)^2}(\partial_y b)^2 + \dots}$$

and only if the penalty factors are the same and if we expand in ϵ:

$$\simeq \boxed{\int \mathrm{d}\tau\, \mathrm{d}x \left\{ \frac{e^{2\omega}}{\epsilon^2} + \frac{1}{2}\eta_{(\partial a)^2}\left(\dot{\omega}^2 + \omega'^2\right)\right.} \longleftarrow \text{Liouville action!!!}$$
$$\left. + \frac{1}{2}\eta_{(\partial a)^2}\left(\dot{\sigma}^2 + \sigma'^2\right) + \underbrace{\boxed{\eta_{(\partial a)^2}(\omega'\dot{\sigma} - \dot{\omega}\sigma')}}_{\text{total derivative}} + \dots \right\}$$

σ decouples to NLO in ϵ, which we interpret as restoration of covariance path integral optimization

=

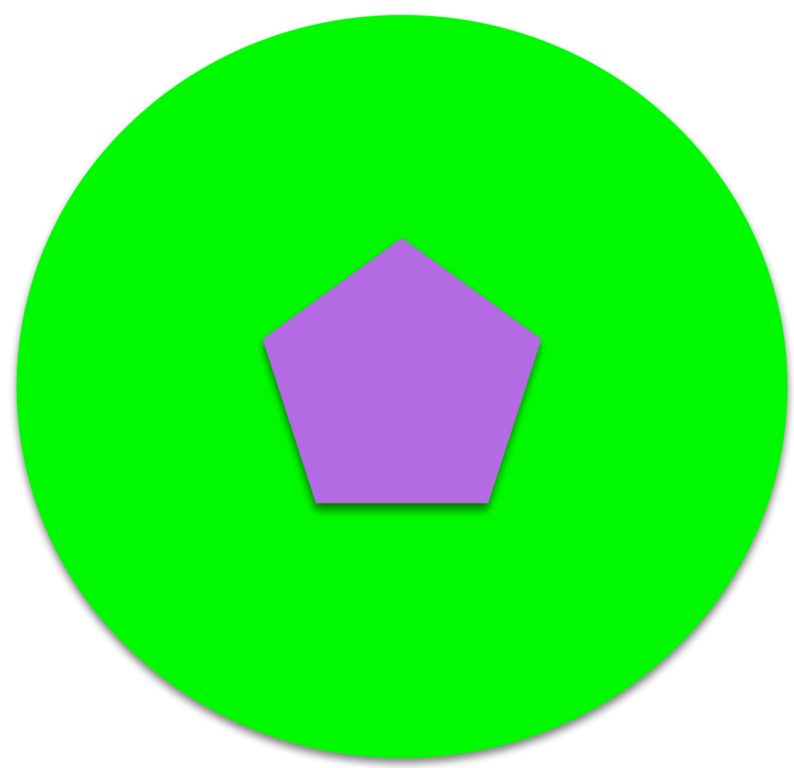
circuit depth minimization

Summary 1904.02713 with Camargo, Jefferson and Knaute

To my taste, $\mathcal{C}_V$ and $\mathcal{C}_A$ looked a lot like calculating HRT surfaces before first works on entanglement entropy in QFT (pioneers: 1980s, booming > 2004)

This led to first works on defining complexity in QFTs with two approaches developed: geometric gate counting and path-integral optimization

What I presented today is that



path-integral optimization

$\subset$

geometric gate counting

Outlook 1904.02713 with Camargo, Jefferson and Knaute

In our derivation, Liouville appears only as a NLO approximation in UV cut-off

However, optimization has been done by putting both terms in the expansion to be of the same order; therefore, the expansion is likely to break down

This provides a strong incentive for considering higher order terms; we believe imposing covariance very much constrains choices of cost functions

Does cost function has sth to do in the end with the path-integration measure?

Setting aside Liouville, it seems to be the time to start thinking about cost functions as functionals of sources. Is it the path to prove $\mathcal{C}_{V/A}$?

a bit in vain of **1806.10144** by Belin et al., …