Holographic Entanglement of Purification from Conformal Field Theories

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Collaboration with
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Motivation

- Entanglement entropy is not convenient to capture quantum entanglement of mixed states.

- Entanglement entropy of mixed state contains classical correlations, so in particular is nonzero for unentangled mixed states.

- One idea to do better job, is to consider entanglement entropy of purifications, and minimize E.E over certain set C of purifications.

    - Consider a mixed state ρ_{AB} on $H_A \otimes H_B$.

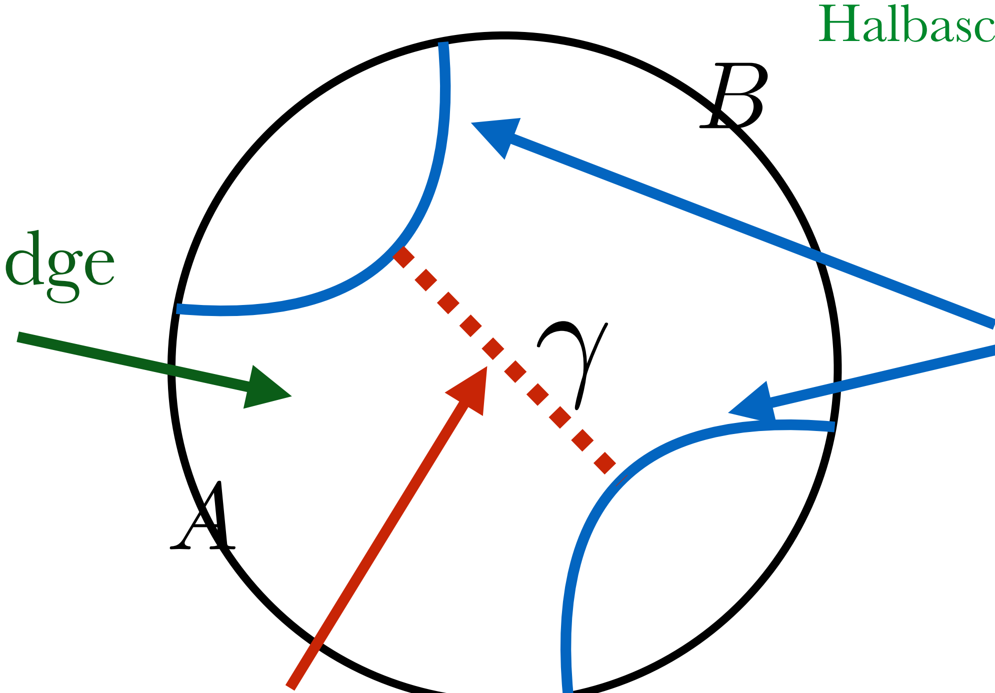
$$E_{\mathcal{C}}[\rho_{AB}] := \underset{\substack{\mathrm{Tr}_{H_{A'} \otimes H_{B'}} |\Psi\rangle\langle\Psi| = \rho_{AB} \\ |\Psi\rangle \in H_A \otimes H_{A'} \otimes H_B \otimes H_{B'} \\ |\Psi\rangle \in \mathcal{C}}}{\mathrm{Min}} S_{|\Psi\rangle}(AA')$$

Motivation

$$E_{\mathcal{C}}[\rho_{AB}] := \underset{\substack{\mathrm{Tr}_{H_{A'} \otimes H_{B'}} |\Psi\rangle\langle\Psi| = \rho_{AB} \\ |\Psi\rangle \in H_A \otimes H_{A'} \otimes H_B \otimes H_{B'} \\ |\Psi\rangle \in \mathcal{C}}}{\mathrm{Min}} S_{|\Psi\rangle}(AA')$$

- When C contains all possible purifications, this quantity is called Entanglement of Purification.
- Proposal for gravity dual of EoP: Entanglement wedge cross section.

[Takayanagi, Umemoto][Nguyen, Devakul, Halbasch, Zaletel, Swingle]



$$E_P[\rho_{AB}] = \frac{\mathrm{Area}[\gamma]}{4G_N}$$

Motivation & Main Results

- Computing EoP is extremely hard, therefore proving this equality is also difficult.
- In this work, we will consider a particular preparation of purification of the given mixed state, using continuous tensor network.

$$|\Psi\rangle \in H_A \otimes H_B \otimes H_E$$

$$\rho_{AB} = \mathrm{Tr}_{H_E} |\Psi\rangle\langle\Psi|$$

- We can associate entanglement entropy $S_{AA'}(|\Psi\rangle)$ for each factorization of auxiliary Hilbert space H_E.

$$H_E = H_{A'} \otimes H_{B'}$$

Motivation & Main Results

- We consider all such (allowed) factorizations, and minimize the associated entanglement entropy, defining E_C.

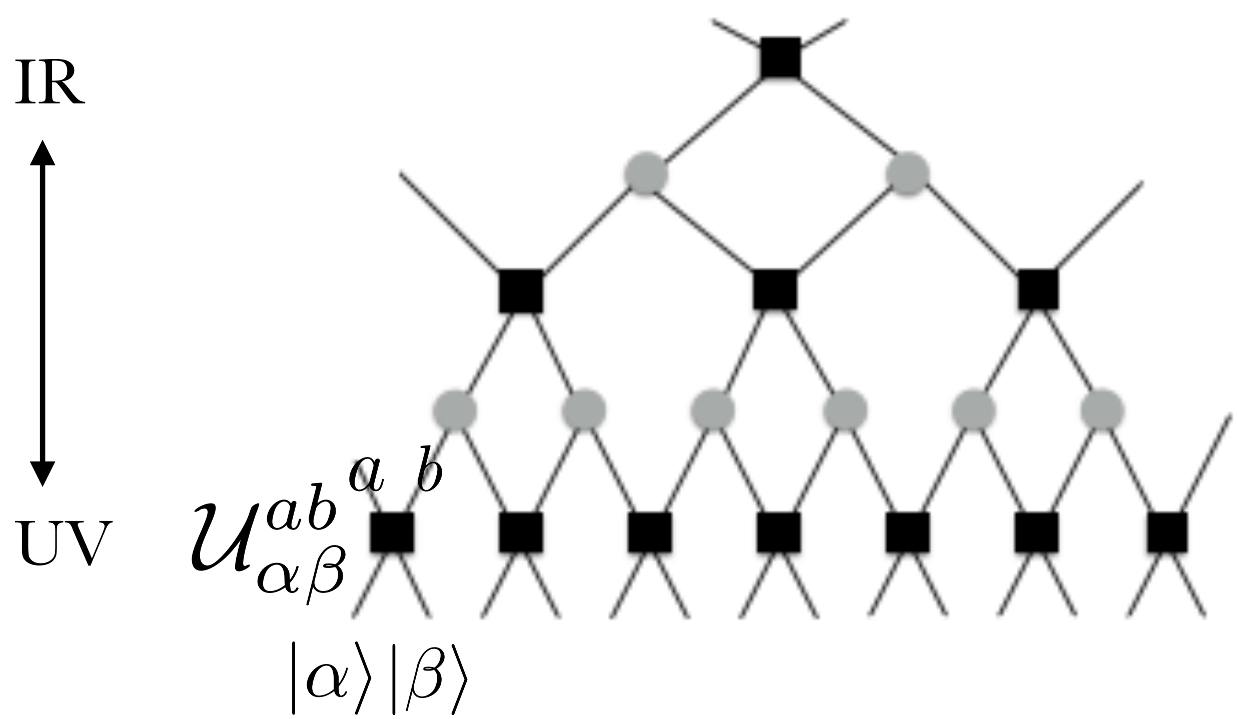
$$E_{\mathcal{C}}[\rho_{AB}] := \underset{\substack{\mathrm{Tr}_{H_{A'}\otimes H_{B'}} |\Psi\rangle\langle\Psi| = \rho_{AB} \\ |\Psi\rangle \in H_A \otimes H_{A'} \otimes H_B \otimes H_{B'} \\ |\Psi\rangle \in \mathcal{C}}}{\mathrm{Min}} S_{|\Psi\rangle}(AA')$$

- It turns out that, for 2d CFT ground state, when A & B are adjacent intervals, E_C gives Entanglement wedge cross section.

$$E_{\mathcal{C}}[\rho_{AB}] = \frac{\mathrm{Area}[\gamma_{EWCS}]}{4G_N}$$

Tensor Network

- Tensor Network: Efficient representation of ground state wave function. [White][Vidal]



$$|0\rangle \otimes |0\rangle$$

$$\Uparrow$$

$$\sum_{a,b} \mathcal{U}^{00}_{ab} |a\rangle \otimes |b\rangle$$

$$\Uparrow$$

$$\sum_{c,d,e,f} \mathcal{U}^{cd}_{0a} \mathcal{U}^{b0}_{ef} \sum_{ab} \mathcal{U}^{00}_{ab} |c\rangle \otimes |d\rangle \otimes |e\rangle \otimes |f\rangle$$

- Entanglement structure of CFT state is encoded in the geometry, in a very similar manner as that of AdS/CFT, or Ryu-Takayanagi formula.

➡ • Tensor network = Timeslice of bulk spacetime in AdS/CFT [Swingle]

Continuous Tensor Network from Weyl Transformation

[M.M, Takayanagi, Watanabe] [Caputa, Kundu, M.M, Takayanagi, Watanabe]

- We apply Weyl transformation to CFT ground state wave function.

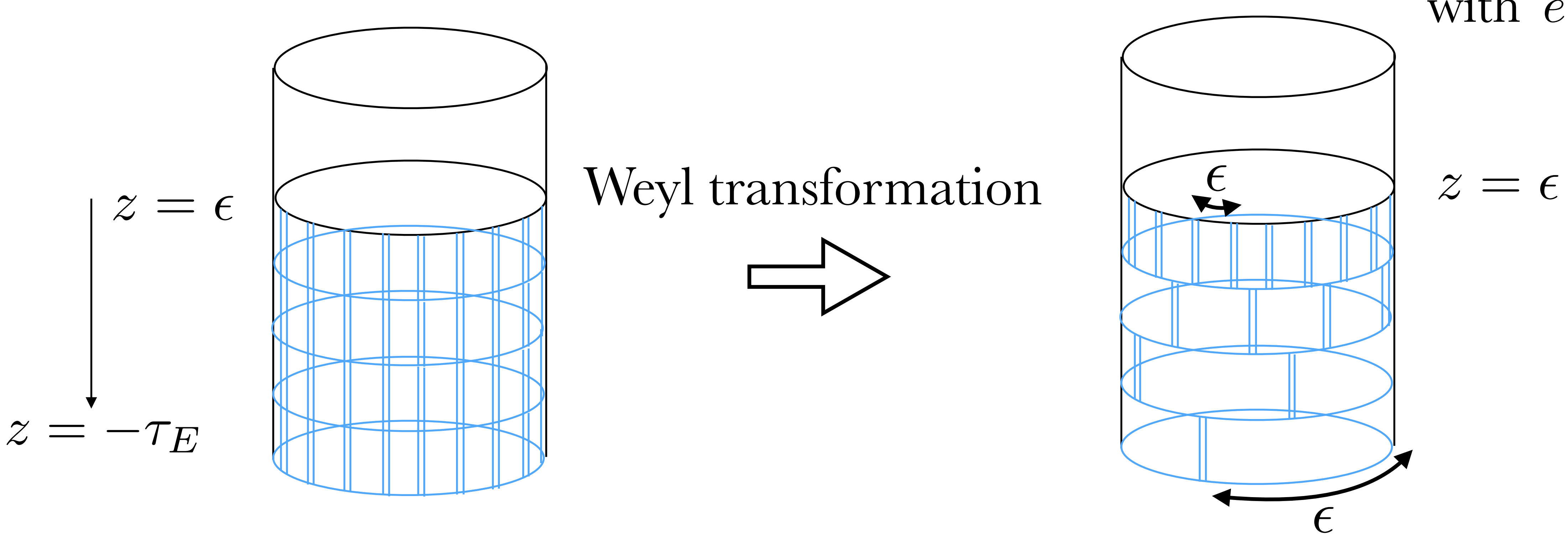
$$\langle \phi_0 | \Omega_{vac} \rangle \;\propto \int_{\substack{\tau < -\epsilon \\ \phi(x,-\epsilon)=\phi_0(x)}} \mathcal{D}\phi(x,\tau)\; e^{-\int_{-\infty}^{-\epsilon} d\tau \int dx \mathcal{L}_E[\phi(x,\tau)]}$$

- Such Weyl transformation makes the effective lattice spacing position dependent, introducing tensor network structure in the path integral.

$$ds^2 = \frac{dz^2 + dx^2}{\epsilon^2}$$

$$ds^2 = e^{2w(z,x)}(dz^2 + dx^2)$$

with $e^{w(\epsilon,x)} = 1/\epsilon$



Continuous Tensor Network from Weyl Transformation

- Because of the conformal invariance, the wave function remains unchanged, up to overall constant; Liouville action.

$$\langle \phi_0 | \Omega_{vac} \rangle \propto \int_{\substack{\tau < -\epsilon \\ \phi(x,-\epsilon)=\phi_0(x)}} \mathcal{D}\phi(x,\tau)\; e^{-\int_{-\infty}^{-\epsilon} d\tau \int dx \mathcal{L}_E[\phi(x,\tau)]}$$

$$= \boxed{e^{-S_L[w]}} \int_{\substack{z > \epsilon \\ \phi(x,z=\epsilon)=\phi_0(x)}} \mathcal{D}\phi(x,z)\; e^{-\int_{\epsilon}^{\infty} dz \int dx \mathcal{L}_E[\phi(x,z),w(x,z)]}$$

Liouville action: $S_L[w] = \frac{c}{24}\int_{\epsilon}^{\infty} dz \int dx \left[(\partial w)^2 + e^{2w} - \frac{1}{\epsilon^2} \right]$ [Polyakov]

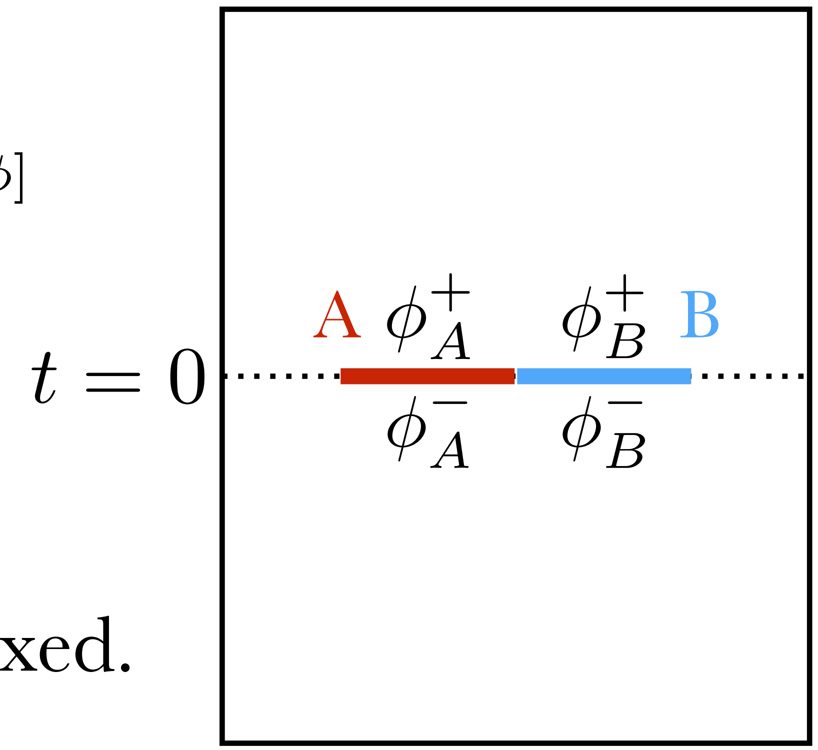
- Such Liouville action corresponds to number of tensors of tensor network[Czech], which is to be minimized.

Weyl factor satisfies Liouville equation (+ sources), when the path-integral is "optimized".

Continuous Tensor Network for Mixed State

- We consider wave function reduced density matrix of CFT ground state.

$$[\rho_{AB}]_{(\phi_A^- \phi_B^-),(\phi_A^+ \phi_B^+)} \propto \int_{\phi(\pm 0, x_{A,B}) = \phi^{\pm}_{A,B}} \mathcal{D}\phi \; e^{-S[\phi]}$$



- We apply Weyl transformation to this wave function, keeping metric on subregions A & B fixed.

$$ds^2 = dy d\tilde{y} \quad \Rightarrow \quad ds^2 = \frac{\epsilon^2}{\left(\mathrm{Im}\sqrt{\frac{y-a}{b-y}}\right)^2} \cdot \frac{(b-a)^2}{4|b-y|^3|y-a|} dy d\tilde{y}$$

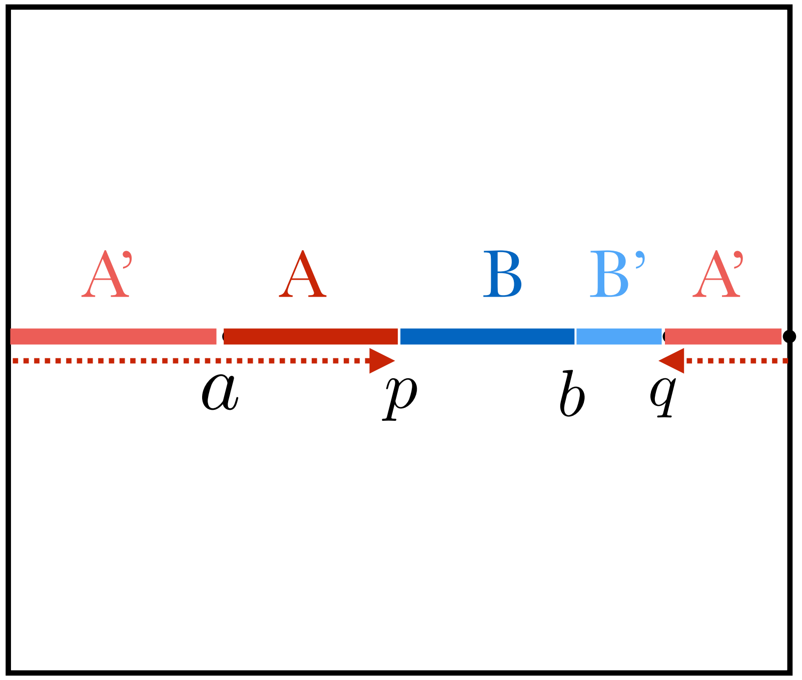
- The state at $t = 0$ gives a purification of ρ_{AB}.

Entanglement Entropy of Purification

[Caputa, M.M, Takayanagi, Umemoto]

- We consider arbitrary partition of auxiliary system, and consider entanglement entropy of AA'.

- We consider 2d CFT and A and B are adjacent intervals.



Original subsystems

$$A = [a,\ p] \quad B = [p,\ b]$$

Auxiliary subsystems

$$A' = [-\infty,\ a] \cup [q,\ \infty] \quad B' = [b,\ q]$$

$$ds^2 = \frac{\epsilon^2}{(\mathrm{Im}\sqrt{\frac{y-a}{b-y}})^2} \cdot \frac{(b-a)^2}{4|b-y|^3|y-a|} dy d\tilde{y}$$

- Using twist field, entanglement entropy of AA' is

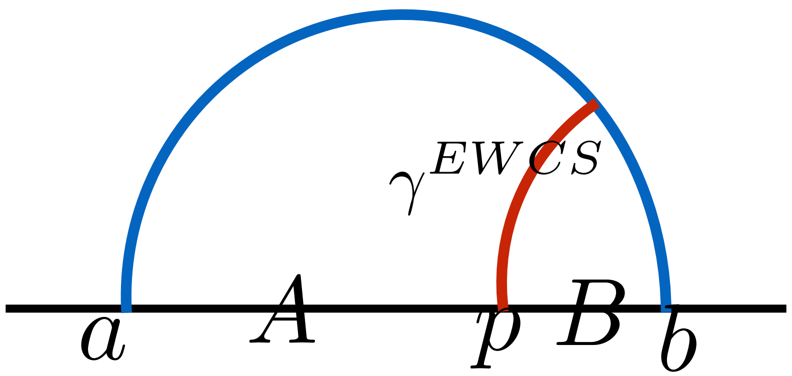
$$S_{AA'}(q) = \frac{c}{6}\log\Big[\frac{(b-a)(q-p)^2}{2\epsilon(q-a)(a-b)}\Big]$$

Entanglement Entropy of Purification

[Caputa, M.M, Takayanagi, Umemoto]

- Minimizing the entanglement entropy over all partitions, resulting entanglement entropy gives entanglement wedge cross section!

$$S_{A\tilde{A}} = \frac{c}{6}\log\left[\frac{2(p-a)(b-p)}{\epsilon(b-a)}\right] = \frac{\text{Area}(\gamma^{EWCS})}{4G_N}$$



- This implies,
  - A field theory calculation of EWCS.
  - Optimized path-integral geometry corresponds to the entanglement wedge.

Discussions

- The result so far is general, independent from large c and field theory content.
- This is O.K as long as we are considering adjacent intervals, but how about non-adjacent intervals?
- Can we identify the optimized metric as timeslice of entanglement wedge?
- Relation to canonical purification? [Dutta, Faulkner]
- Higher dimensions etc.