Exact Dualities and Entanglement Entropy

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How I learned to stop worrying and love EE

- **Entanglement entropy (EE)** is a basic measure of how much information about a **quantum state** is contained in a **subsystem**

- In QFT, the state is usually thermal ($\Rightarrow$ ground state at $T = 0$), and the subsystem is usually a spatial subregion

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- **Entanglement entropy (EE)** is a basic measure of how much information about a **quantum state** is contained in a **subsystem**
- In QFT, the state is usually thermal ($\Rightarrow$ ground state at $T = 0$), and the subsystem is usually a spatial subregion
- In this setup, EE probes the UV and diverges in the continuum limit. **This is OK!** For many purposes, the UV must be taken seriously
- One such purpose is to understand how EE maps under dualities. This question can be reliably answered for **exact dualities**, which hold in the far UV too. This will be explained in this talk

What are exact dualities?

- **"Duality"** has different meanings in different contexts, but it generally refers to a map between some set of operators or correlation functions in two different theories

  E.g. spins = fermions in 2D, particles = vortices in 3D, AdS = CFT

- A duality is **exact** if it holds at all energy scales, so that every operator/state/correlation function is mapped, all cutoffs are physical

- An exact duality is naturally presented as a **change of generating basis** of a single operator algebra. The dual generating sets mainly contain local operators (on potentially different geometric spaces)

Why study exact dualities?

- Most dualities in QFT are not exact:
  - Two theories may only be dual at low energies (e.g. Seiberg duality)
  - One (or both) of the dual theories need not even have a nonperturbative definition on its own (e.g. AdS/CFT)

  From this point of view, dualities may seem miraculous!

- Exact dualities are less miraculous. They can be rigorously proven, and sometimes they may explain more mysterious dualities

- Familiar examples of exact dualities: 2D bosonization, 3D particle-vortex, Abelian S-duality... and *maybe* level-rank

This talk

- Review and derive exact dualities in the following scope:
  - Hamiltonian framework (spatial lattice and continuous time)
  - Arbitrary lattices (even nontrivial homologies, Stiefel-Whitney classes)
  - Discrete group structure of the target space (Abelian d.o.f. only)

This talk

- Review and derive exact dualities in the following scope:
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- Many exact dualities fit into the following systematization:
  1. Start from **local dualities** that map local generators to each other
  2. **Twist** these dualities by coupling them to background gauge fields
  3. Make these background fields dynamical, get dualities involving the nonlocal **"disorder operators"**
  4. Note **obstructions** along the way: nontrivial SW classes, anomalies

- The behavior of EE under dualities will now become transparent: EE is invariant **if** boundary conditions are allowed to dualize

Local exact dualities

- **Local dualities** can be formulated after specifying the following:
  - Spatial lattice (d-dimensional, with branching structure)
  - Target space ($\mathbb{Z}_2$ for Ising spins and spinless fermions, $\mathbb{Z}_K$ for parafermions and clock models, U(1) for compact scalars)
  - D.o.f. locations (sites, links, plaquettes, etc)[1]
  - Statistics (a c-number element of the target space)

- It is **not** necessary to pick a Hamiltonian to define an exact duality! Dualities are not necessarily strong-weak (although that's when they're the most useful)

[1] A gauge constraint is assumed whenever d.o.f. live on chains of nonzero dimension.

Local exact dualities: details on the data needed

- A d-dimensional lattice $\mathbb{M}$ with a global ordering of vertices

- $\mathbb{Z}_K$ d.o.f. with local generators Φ, Π such that $\Phi^K = \Pi^K = \mathbb{1}$.
  These are canonical position/momentum operators if $\Pi\Phi = \mathrm{e}^{\frac{2\pi\mathrm{i}}{K}}\Phi\Pi$
  For $\mathbb{Z}_2$, $\Phi = X$ and $\Pi = Z$; for U(1), $\Phi = \mathrm{e}^{\mathrm{i}\phi}$ and $\Pi = \mathrm{e}^{-\mathrm{d}\phi\frac{\partial}{\partial\phi}}$

- D.o.f. on sites: matter models like Ising or compact scalar.
  D.o.f. on links: gauge theories, $\prod_u \Pi_{(v,u)} = \mathbb{1}$ on each site v

- Statistics of matter d.o.f. is captured by $\sigma \in \mathbb{Z}_K$ such that
  e.g. $\Phi_v\Phi_u = \sigma^{\theta(v,u)}\Phi_u\Phi_v$, where $\theta(v,u) = -\theta(u,v) = 1$ if $v > u$.
  For $K = 2$, $\sigma = -1$, let e.g. $\Phi \to \chi$, $\Pi \to \chi'$ (Majorana fermions)

Local exact dualities: $\mathbb{Z}_2$ bosonic matter

- This is the local version of Kramers-Wannier duality
- On one side, the theory has generators X_v, Z_v on all sites $v \in \mathbb{M}$
- The dual is a rank-$(d-1)$ $\mathbb{Z}_2$ gauge theory on the dual lattice $\mathbb{M}^\vee$. **Not all** generators map under the **local** duality:

$$X_v X_u = X^\vee_{(u,v)}, \quad Z_v = W^\vee_v = \prod_{s \subset v} Z^\vee_s$$

$v \in \mathbb{M}$ dualizes to a d-dim. cell in $\mathbb{M}^\vee$, and a link (v,u) dualizes to the $(d-1)$-dim. cell shared by the d-dim. cells u and v in $\mathbb{M}^\vee$

- Simplification in $d = 1$: $X_v X_{v+1} = Z^\vee_{v+\frac{1}{2}}, \quad Z_v = X^\vee_{v-\frac{1}{2}} X^\vee_{v+\frac{1}{2}}$

Local exact dualities: $\mathbb{Z}_2$ bosonic matter

$$X_v X_{v+1} = Z^\vee_{v+\frac{1}{2}}, \quad Z_v = X^\vee_{v-\frac{1}{2}} X^\vee_{v+\frac{1}{2}}$$

- Assume the $d=1$ lattice is periodic (a discretized circle)
- The local KW duality implies the global singlet constraints $\prod_v Z_v = \prod_v Z^\vee_{v+\frac{1}{2}} = \mathbb{1}$. Generally, local exact dualities are "singlet-singlet." In this case,

$$\frac{\text{Ising spins}}{\mathbb{Z}_2} = \frac{\text{Ising spins}^\vee}{\mathbb{Z}_2}$$

Global exact dualities: $\mathbb{Z}_2$ bosonic matter

- Background gauge fields change which sectors are mapped:

$$\eta_{v+\frac{1}{2}} X_v X_{v+1} = Z^\vee_{v+\frac{1}{2}}, \quad Z_v = X^\vee_{v-\frac{1}{2}} X^\vee_{v+\frac{1}{2}}, \quad \eta_{v+\frac{1}{2}} \in \mathbb{Z}_2$$

- **Twisted** constraints: $\prod_v Z_v = \mathbb{1}$, $\prod_v Z^\vee_{v+\frac{1}{2}} = \prod_v \eta_{v+\frac{1}{2}}$

- Make background gauge fields dynamical, add map $X_{v+\frac{1}{2}} = X^\vee_{v+\frac{1}{2}}$

$$\frac{\text{Ising spins}}{\mathbb{Z}_2} \times \mathbb{Z}_2 \text{ gauge theory} = \text{Ising spins}^\vee$$

Gauge-fixing $\Rightarrow X^\vee_{v+\frac{1}{2}}$ as a "disorder operator" (string of spin operators Z_v)

Other local/global dualities of bosons

- Larger target spaces, e.g. $\mathbb{Z}_K \xrightarrow{K\to\infty} \mathsf{U}(1)$ in $d=1$:

$$\frac{\text{compact scalar}}{\mathsf{U}(1)} = \frac{\text{compact scalar}^\vee}{\mathsf{U}(1)} \qquad \text{(a.k.a. T-duality)}$$

- Higher dimensions, e.g. in $d=2$

$$\frac{\text{Ising spins}}{\mathbb{Z}_2} = \frac{\mathbb{Z}_2 \text{ gauge theory}^\vee}{\mathbb{Z}_2 \text{ one-form}}$$

- All of them can be twisted, e.g. in $d=2$

$$\frac{\text{compact scalar}}{\mathsf{U}(1)} \times \mathsf{U}(1) \text{ topo. gauge theory} = \mathsf{U}(1) \text{ gauge theory}^\vee$$

Exact dualities with fermionic matter

- **In $d = 1$**, one local bosonization (Jordan-Wigner) duality is

$$Z_v = \mathrm{i}\chi'_v \chi_v, \quad X_v X_{v+1} = -\mathrm{i}\chi'_v \chi_{v+1}$$

- This duality can be twisted, e.g. via $\eta_{v+\frac{1}{2}} X_v X_{v+1} = -\mathrm{i}\chi'_v \chi_{v+1}$

- Making η dynamical in the usual way does **not** give a global duality. It is necessary to change the Gauss law:

$$X_{v-\frac{1}{2}} Z_v X_{v+\frac{1}{2}} = \mathbb{1} \longrightarrow X_{v-\frac{1}{2}} Z_v X_{v+\frac{1}{2}} = (-W)^{\delta_{1,v}}$$

- This is **1d flux attachment**. Gauge-fixing gives full Jordan-Wigner

$$Z_1 \cdots Z_{v-1} X_v = \chi_v, \quad Z_1 \cdots Z_{v-1} Y_v = \chi'_v$$

Exact dualities with fermionic matter

- **In** $d > 1$, local bosonization rules can be written on any lattice with a trivial second Stiefel-Whitney class [Chen, Kapustin, ĐR; Chen, Kapustin]

- Main lessons:
  - There is always flux attachment on the bosonic side
  - Depending on the lattice, a nontrivial $\mathbb{Z}_2$ background gauge field may need to be coupled to the fermions: the twists are "built in"
  - These background $\mathbb{Z}_2$ fields are **spin structures**
  - The higher-form symmetry is anomalous and can be twisted but not gauged. The zero-form $\mathbb{Z}_2$ symmetry can be gauged to give a global exact duality with dynamical spin structures, e.g. in $d = 2$:

$$\frac{\text{fermions}}{\mathbb{Z}_2} \times \mathbb{Z}_2 \text{ topo. gauge theory} = \text{flux-attached } \mathbb{Z}_2 \text{ gauge theory}$$

All this generalizes to bosonization of parafermions: their dual gauge theories have multiple units of $\mathbb{Z}_K$ flux attached to each charge, in the general case

Back to EE

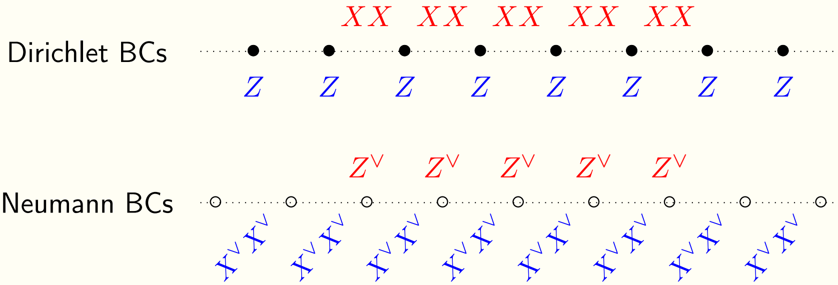
- Now that we know how **all** operators dualize, we can understand how a given reduced density matrix dualizes
- Operator approach to EE: specify a **subalgebra**, not just a region
  [Zanardi, Lidar, Lloyd; Casini, Huerta + collaborators; ĐR; many others]
- The reduced density matrix is the unique element ρ_A of the subalgebra $\mathcal{A}_A$ that reproduces expectations of all $\mathcal{O} \in \mathcal{A}_A$ via

$$\langle \mathcal{O} \rangle = \mathrm{Tr}_A(\rho_A \, \mathcal{O})$$

- Given a region, there are **many** choices of subalgebras that do not live on any smaller region. Among them are the subalgebras that correspond to different **boundary conditions** at the region edge

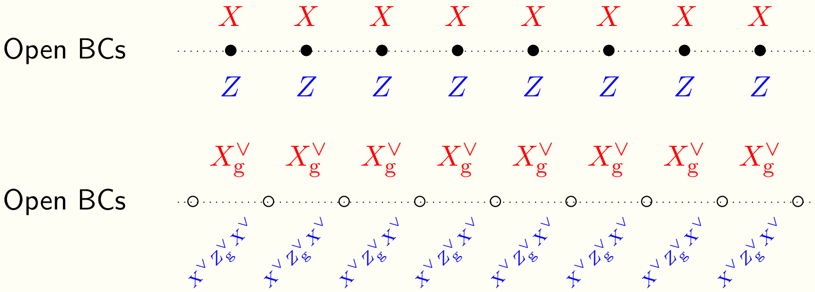
Example: boundary conditions of a spin chain

Example: local Kramers-Wannier duality



- Two dual subalgebras, neither of them maximal on their regions
- Same reduced density matrices, different entropies (different dim$\mathcal{H}$)
- This difference is canonical! (log 2 in the above case)

Example: global Kramers-Wannier duality



($X_{\mathrm{g}}^{\vee}$ and $Z_{\mathrm{g}}^{\vee}$ denote operators of the $\mathbb{Z}_2$ gauge theory)

- This extends to all exact dualities presented here
- Analogous in the continuum $\implies$ BCs are important in replica trick

Concluding remarks

- Well-known exact dualities are all within the same systematization
- Connections between dualities and anomalous kinematic symmetries, nontrivial topologies, and boundaries can all be transparently understood using Hamiltonian methods
- What QFT dualities can be deduced from these lattice constructs?
- Can we calculate EEs with nontrivial boundary conditions, in CFT or AdS? Lessons for bulk reconstruction?

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Thank you!