An Operational Meaning for Entanglement Entropy in Continuum QFT

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Warning

This talk will be

1. Mathematically trivial.
2. Obvious to many.
3. Extremely imprecise.
4. Hopefully, accessible to the entire audience.

Introduction 1

In finite-dimensional Hilbert spaces, entanglement entropy has operational meaning:
the number of Bell pairs that the state can be converted to/from using LOCC.

In continuum field theory, EE not well-defined
$\leftrightarrow$ EE has UV divergence.

But it still has non-universal operational meaning:
EE in particular cutoff = no. of Bell pairs that can be extracted in particular smearing scheme.

Part 1:
I will show this in one particular regularisation = smearing scheme.

Introduction 2

In lattice gauge theories, a part of the EE comes from edge modes, and it can't be extracted into Bell pairs
(it's not true entanglement).

But they're still important to get the right answer for universal bits of the EE.

This non-extractibility is a lattice-scale effect —
and one might wonder if there's any purely continuum description of them.

Part 2:
Yes. Edge modes are certain non-smearable symmetries of the modular Hamiltonian.

Entanglement-Shentanglement

In finite-dimensional Hilbert spaces, we define reduced density matrix as partial trace.

Interesting for us: *modular Hamiltonian*, defined as

$$H_{mod} = -\log \rho_{red}\,, \quad \rho_{red} = e^{-H_{mod}}\,. \tag{1}$$

Thing that's nice in QFT: two-sided modular Hamiltonian, $H_{mod,L} - H_{mod,R}$.

Entanglement entropy (EE) is

$$\begin{aligned} S_E &= -\operatorname{tr} \rho_{red} \log \rho_{red} \\ &= \operatorname{tr}\left(H_{mod} e^{-H_{mod}}\right) \\ &= \langle\psi|H_{mod}|\psi\rangle\,. \end{aligned} \tag{2}$$

We'll think of S_E roughly as
no. of independent Bell inequalities that can be violated.

An Important Bell Inequality[1]

Take two harmonic oscillators in thermofield double state

$$|\psi\rangle = \sqrt{1-x^2} e^{x a_L^\dagger a_R^\dagger} |0\rangle_L |0\rangle_R = \sqrt{1-x^2} \{|00\rangle + x |11\rangle\} + \ldots, \quad x \in [0,1). \tag{3}$$

Thinkiing of two-dimensional subspace as qubits L, R, we have

$$\sqrt{2} \langle\psi|\sigma_L^z \sigma_R^z + \sigma_L^x \sigma_R^x|\psi\rangle = \sqrt{2}(1+x)^3(1-x). \tag{4}$$

This violates the CHSH inequality

$$2\sqrt{2} \geq \sqrt{2} \langle\psi|\sigma_L^z \sigma_R^z + \sigma_L^x \sigma_R^x|\psi\rangle > 2 \quad \text{for} \quad x \in (.2, .7). \tag{5}$$

For other values of x, more complicated inequalities.
But saturation possible only for $x > .42$.

[1] S. Raju 1809.10154.

Continuum-Shontinuum

In continuum field theory,
the Hilbert space doesn't tensor-factorise
into Hilbert spaces for subregions.

because tensor factorisation means
$\exists$ operators that create states in which subregions are completely uncorrelated.
$\Rightarrow$ kinky configurations.

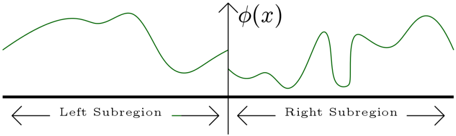


Figure: An abomination caused by tensor factorisation.

But these operators have infinite energy
since Hamiltonian usually contains derivative terms, like $(\boldsymbol{\nabla}\phi)^2$.

An Important Subtlety

We can't create states that are pure kink
but every finite-energy state has support on
arbitrarily kinky configurations.

In fake equation,[2]

$$\delta > 0 \quad \Rightarrow \quad \langle\text{configuration with kink at scale } \delta|0\rangle \neq 0. \qquad (6)$$

Imagine these kinks at boundary between subregions.
$\Rightarrow$ state has support on infinitely many configurations (one for every δ)
$\Rightarrow$ entanglement diverges!

[2] Real 'equation:' Reeh-Schlieder theorem. Ask me offline, or read Witten1803.

Subregions aren't Subsystems let's go home?

Even though subregions don't correspond to tensor factors,
there's an important sense in which they're subsystems:
there's a local algebra ($=$ set of operators) associated to each one.

Formally,
algebra of a region L is the set of all operators

$$\phi[f] = \int f(x)\phi(x)$$

where f is smooth and has support only within L.

Notice:
this automatically rules out any operator that has support on *all* of L,
because f has to go to 0 *smoothly* at ∂L.
E.g.s of such non-existent operators: ρ_{red}, H_{mod}.

Modular Operator to the Rescue

Even though reduced density matrix and one-sided modular Hamiltonian don't exist,
it turns out that the two-sided modular Hamiltonian $H_L - H_R$ does.
It's not in the L algebra, but it *does* know about the split into L and R.

To define entanglement, we write

$$H_L - H_R = H_{L,reg} - H_{R,reg} \tag{7}$$

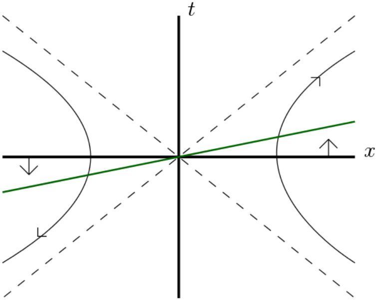
where the regularised one-sided modular Hamiltonians *are* in the respective algebras
and define entanglement

$$S_E = \langle\psi|H_{L,reg}|\psi\rangle \,. \tag{8}$$

Notice that it depends on regularisation, so not 'universal.'

The Paradigmatic Modular Hamiltonian

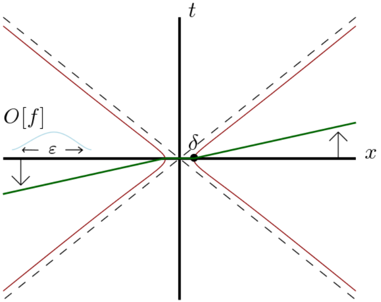
In case when we're calculating entanglement between two halves of space in the vacuum,
$L = \{x < 0\}, \ R = \{x > 0\}$,
two-sided modular Hamiltonian is boost generator.



Clearly, H_R does something very discontinuous at origin.
$H_{R,reg}$ is generator that does the same thing far away from origin but smoothly goes to 0 at origin.

Picking a Regularisation

We'll pick a very discontinuous regularisation, in which $H_{L,reg} - H_{R,reg}$ generates



This doesn't change modular evolution $e^{iH_L} O e^{-iH_L}$ when O is smeared over scale $\varepsilon \gg \delta$ and entirely contained within left.

An Auxiliary Vacuum

Fun trick:
This regularised modular Hamiltonian is the real modular Hamiltonian for
an auxiliary state in an auxiliary Hilbert space.

In free massless 2d scalar field theory, this state is

$$|0\rangle_{reg} = \left\{ \bigotimes_{n=0}^{\infty} \sqrt{1 - e^{-2\pi\omega_n}} e^{e^{-2\pi\omega_n} a^\dagger_{L,\omega_n} a^\dagger_{R,\omega_n}} \right\} |0\rangle_L \, |0\rangle_R \, , \quad \omega_n \sim \frac{n - 1/2}{\log \delta} . \tag{9}$$

Expectation values of operators smeared over scale $\varepsilon \gg \delta$ and contained within one region
are same in $|0\rangle$ and $|0\rangle_{reg}$.

Main result 1: Bell inequalities

This regularised state is just a bunch of harmonic oscillator TFDs.
So, Bell inequalities mentioned earlier are violated by this state.

Saturation is only possible for $\omega_n < .13$,
i.e. $n - \frac{1}{2} < .13 \log \delta$.

Smearing over $\varepsilon \sim$ writing operators that have support on $\log \delta / \log \varepsilon$ tensor factors above.
So, approximately $\log \varepsilon$ independent Bell inequalities are saturated!

This matches with the behaviour of the EE!
More detailed matching requires better specification of regularisation scheme and EE calculation.
And also, counting Bell inequalities is not a terribly precise operational count.

Edge Modes on the Lattice

In free Maxwell theory,
physical states satisfy Gauss' law as operator eqn, $\nabla \cdot E \, |\psi\rangle = 0$.

At an entangling boundary, this implies that at every point $x \in bd$,

$$E \cdot \hat{n}_{in}(x) \, |\psi\rangle = -E \cdot \hat{n}_{out}(x) \, |\psi\rangle \, . \tag{10}$$

So, we have operators that are effectively in both algebras,
edge modes.

This induces a decomposition for EE[3]

$$S_E = -\sum_k \left(p_k \log p_k + p_k \rho_k \log \rho_k \right) . \tag{11}$$

The first term is edge mode entropy and can't be extracted into Bell pairs.[4]

[3] e.g. Buividovich, Polikarpov 2008.

[4] RMS, S. P. Trivedi 1510.07455; K. van Acoleyen et al 1511.04369.

Can we make this difference precise in the continuum?

Yes. Edge mode entropy comes entirely from operators that can't be smeared.

More precisely,
if we try to take an edge mode operator and smear it,
we end up with an operator that is no longer an edge mode operator.

Intuitively, Gauss' law relates left and right normal electric fields *on the boundary*.
So, this part of the entropy can only be seen in operators that have most of their support at the boundary.
Which is *not* the sort of smeared operator we've been talking about.

Calculation

Consider two-point function of normal electric field in 4d Maxwell, where boundary is the 2d plane $z = 0$.
The electric field two-point fn is

$$\langle 0|E_i(x_1)E_j(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3} k \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) e^{ik\cdot(x_1 - x_2)}. \qquad (12)$$

Putting one operator at $z = 0$ and one at $z = \Delta$ and fourier transforming the other two directions, we find[5]

$$\langle 0|E_z(k_{\|,1}, 0)E_z(k_{\|,2}, \Delta)|0\rangle \sim \delta(k_{\|,1} - k_{\|,2}) \int \frac{dk_z}{\sqrt{k_z^2 + k_\|^2}} e^{ik_z\Delta} \qquad (13)$$

whose behaviour depends sensistively on relative value of Δ and $k_\|$.
Correct entropy only found when $\Delta \ll k_\|^{-1} \leq \varepsilon$.

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[5]U. Moitra, RMS, S. P. Trivedi 1811.06986; Upamanyu's poster.

Conclusions

As expected,
EE in continuum QFT continues to have operational meaning, and the fact that EE is regularisation dependent is because operational question needs regularisation.

Edge modes continue to be non-extractable in the continuum also, since they can't be smeared.