QIST 2019 June 6th, 2019

Entanglement of Purification in Many Body Systems and Symmetry Breaking

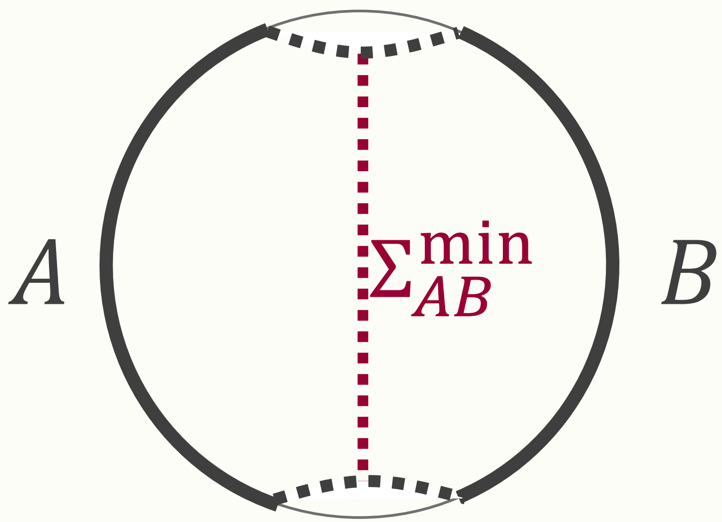
Koji Umemoto (YITP)

Based on

Arpan Bhattacharyya (YITP), Alexander Jahn (Freie U.)
Tadashi Takayanagi (YITP) and KU [1902.02369]

$E_W = E_P$ conjecture

Takayanagi-KU '17, Nguyen-Devakul-Halbasch-Zaletel-Swingle '17

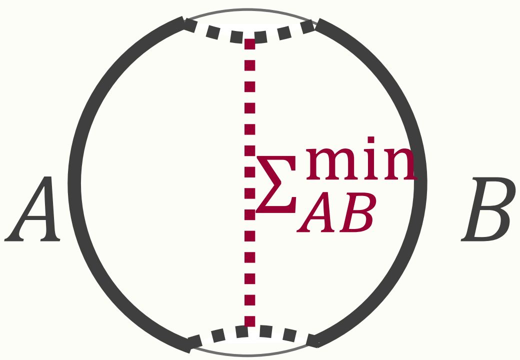


Entanglement wedge cross section

$$E_W(\rho_{AB}) := \min_{\Sigma_{AB}} \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$

$E_W = E_P$ conjecture

Takayanagi-KU '17, Nguyen-Devakul-Halbasch-Zaletel-Swingle '17



$E_W(\rho_{AB})$

? =

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S_{AA'}$$

Entanglement of purification

Kudler-Flam and Ryu '18

$$\mathcal{E}_N(\rho_{AB}) := \log\left|\rho_{AB}^{T_A}\right|_1$$

Logarithmic negativity ($\frac{3}{2}E_W$)

Tamaoka '18

$$S_o(\rho_{AB}) := \lim_{n_o:odd\to 1} \frac{\left[\mathrm{Tr}\left(\rho_{AB}^{T_A}\right)^{n_o} - 1\right]}{1-n_o}$$

Odd entanglement entropy ($E_W + S_{AB}$)

Dutta and Faulkner '19

$$S_R(\rho_{AB}) := S(AA^*)_{\sqrt{\rho_{AB}}}$$

Reflected entropy ($2E_W$)

Entanglement of Purification (EoP)

Definition:

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S(\rho_{AA'}) \qquad (\rho_{AA'} := \mathrm{Tr}_{BB'}[|\psi\rangle\langle\psi|_{AA'BB'}])$$

Entanglement of Purification (EoP)

Definition:

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S(\rho_{AA'}) \qquad (\rho_{AA'} := \mathrm{Tr}_{BB'}[|\psi\rangle\langle\psi|_{AA'BB'}])$$

- In practice, hard to compute
- Thus we still don't know much about EoP in physical many body systems e.g. CFTs

Related papers: Terhal-Horodecki-Leung-DiVincenzo '02, Chen-Winter '12
Nguyen-Devakul-Halbasch-Zaletel-Swingle '17, ...

Our goal

$$E_P(\rho_{AB}) \coloneqq \min_{|\psi\rangle_{AA'BB'}} S(\rho_{AA'})$$

To compute EoP in many body systems (on a lattice) by numerically performing the minimization

Our Targets

- 2d (massless) free scalar field theory on a lattice

  Method: Minimal Gaussian purification ansatz

  Ground state reduced matrix ρ_{AB} is Gaussian

  Purifications $|\Psi\rangle_{AA'BB'}$ is assumed to be Gaussian with $|A'B'| := |AB|$

- 2d transverse-field (critical) Ising model

  Method: Full minimization without ansatz

  A theorem [Ibinson-Linden-Winter '06] guarantees that
  it is sufficient to search $\mathrm{dim}\mathcal{H}_{A'} \leq \mathrm{rank}\rho_{AB}, \ \ \mathrm{dim}\mathcal{H}_{B'} \leq \mathrm{rank}\rho_{AB}$
  for minimizing $S_{AA'}$

  E.g. 2 qubits

  A ● ● B ▶ $A \cup B$ ● ● ● ● ● ● $A' \cup B'$

  $\mathrm{rank}\rho_{AB} = 4$ $\mathrm{dim}\mathcal{H}_{A'} = \mathrm{dim}\mathcal{H}_{B'} = 4$

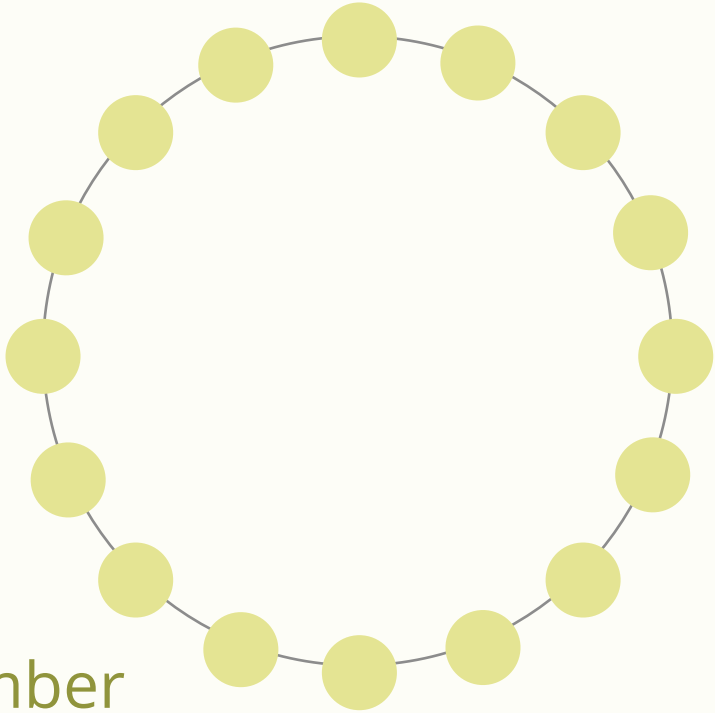
Our Targets

EoP behaves similarly in both models.

Common results

- EoP can increase with the physical distance.
- Even if $\rho_{AB} = \rho_{BA}$, the optimal purification can break its symmetry i.e. $|\psi^*\rangle_{AA'BB'} \neq |\psi^*\rangle_{BB'AA'}$.

Setup



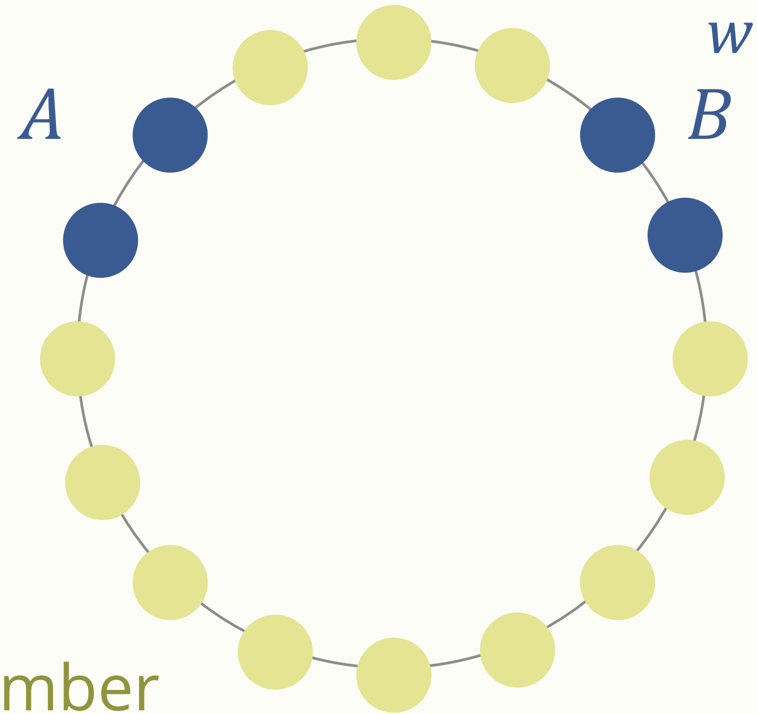
N: 16

N: total sites number

[1+1d, vacuum, periodic boundary condition]

Setup

w: size of subsystem



N: 16
w: 2

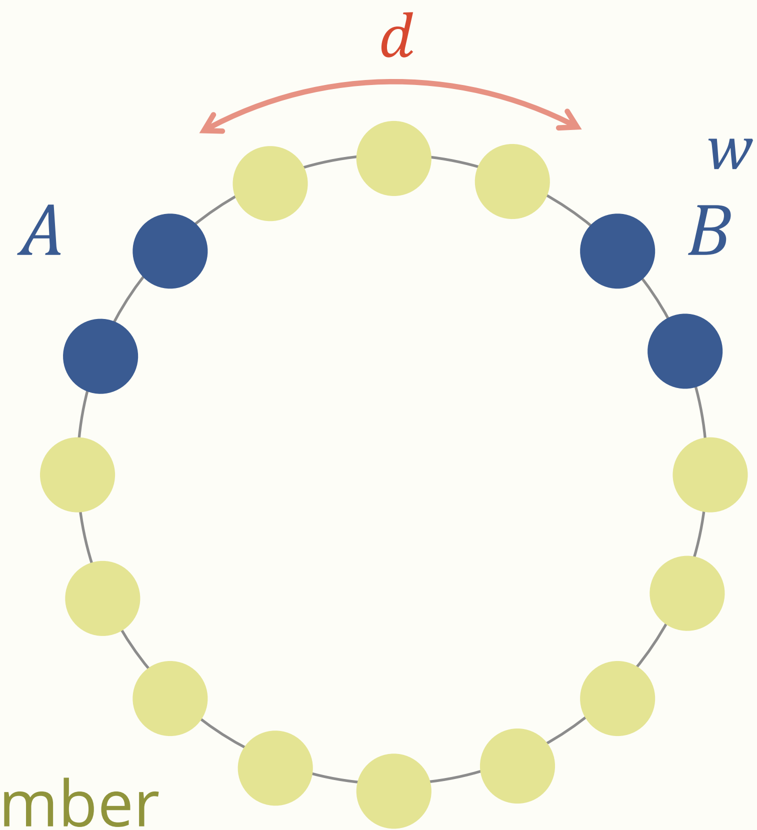
N: total sites number

[1+1d, vacuum, periodic boundary condition]

Setup

d: distance between A and B

w: size of subsystem



N: 16
w: 2
d: 3

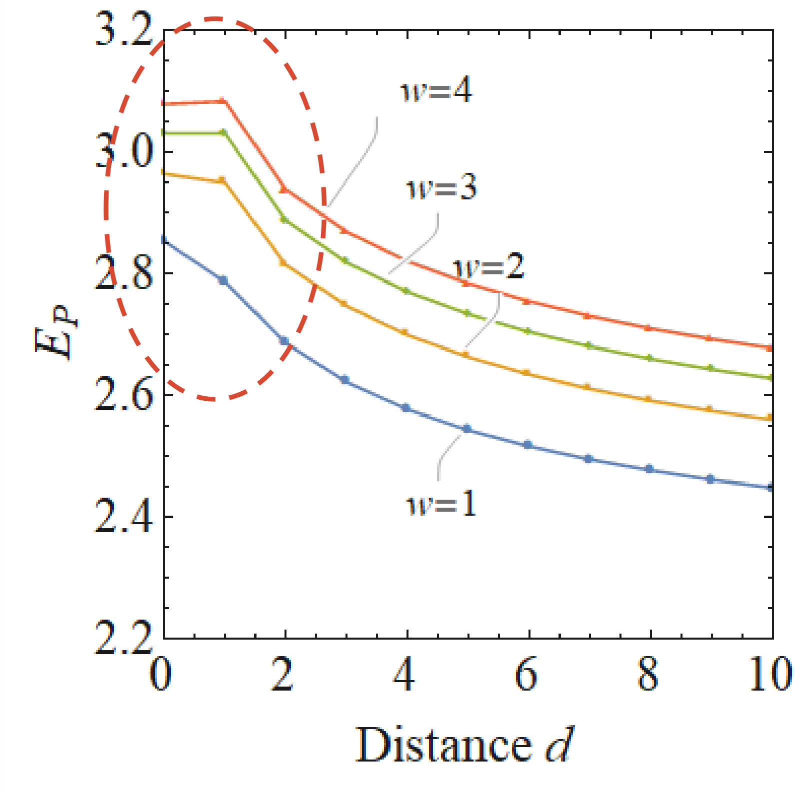
N: total sites number

[1+1d, vacuum, periodic boundary condition]

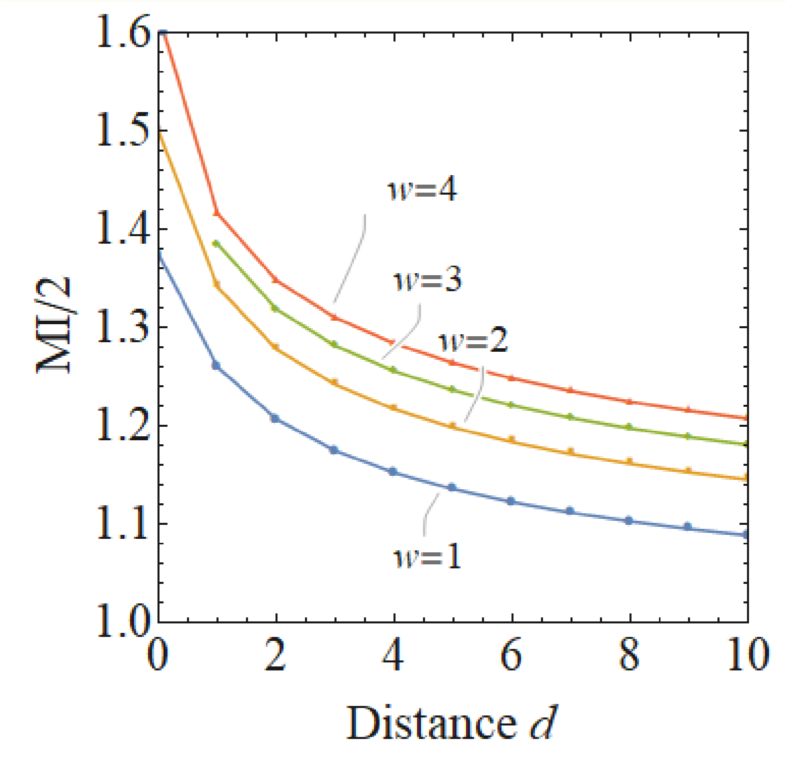
Results

E.g. $N = 60$ free scalar

Plateau-like behavior EoP



Half of mutual information

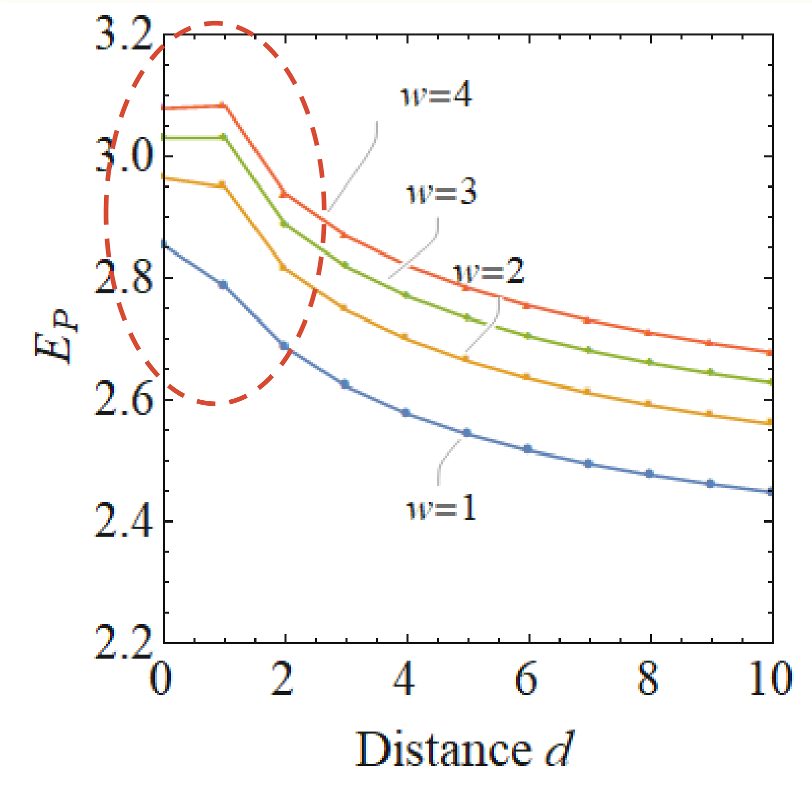


(Mutual information $I(A:B) = S_A + S_B - S_{AB}$)

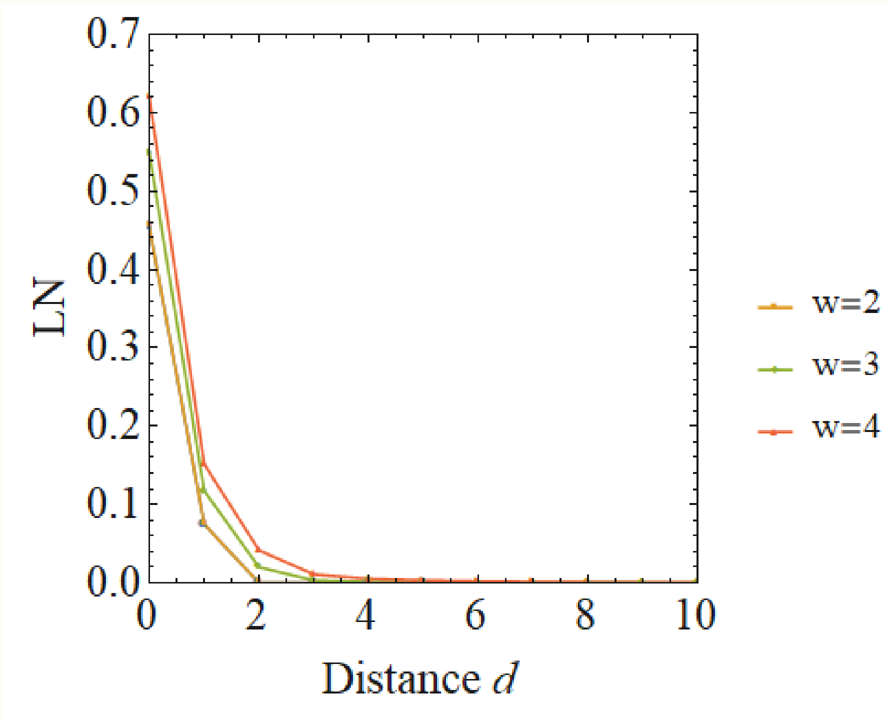
Results

E.g. $N = 60$ free scalar

Plateau-like behavior EoP



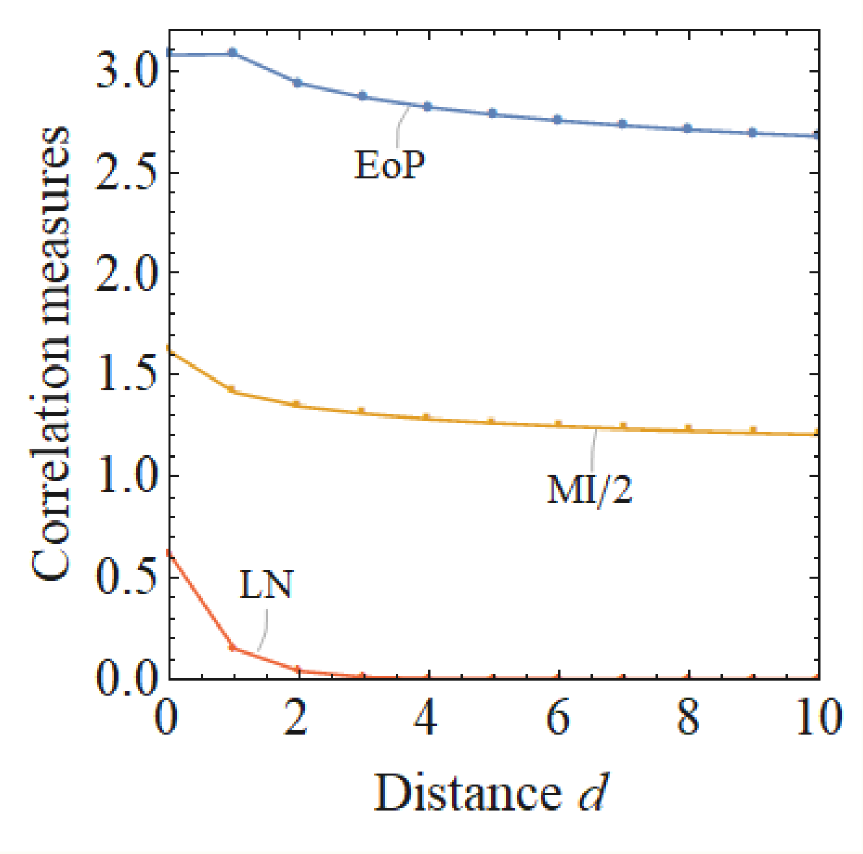
Log negativity



(Log negativity $E_N = \log\left|\rho_{AB}^{T_A}\right|_1$)

Results

E.g. $N = 60$ free scalar

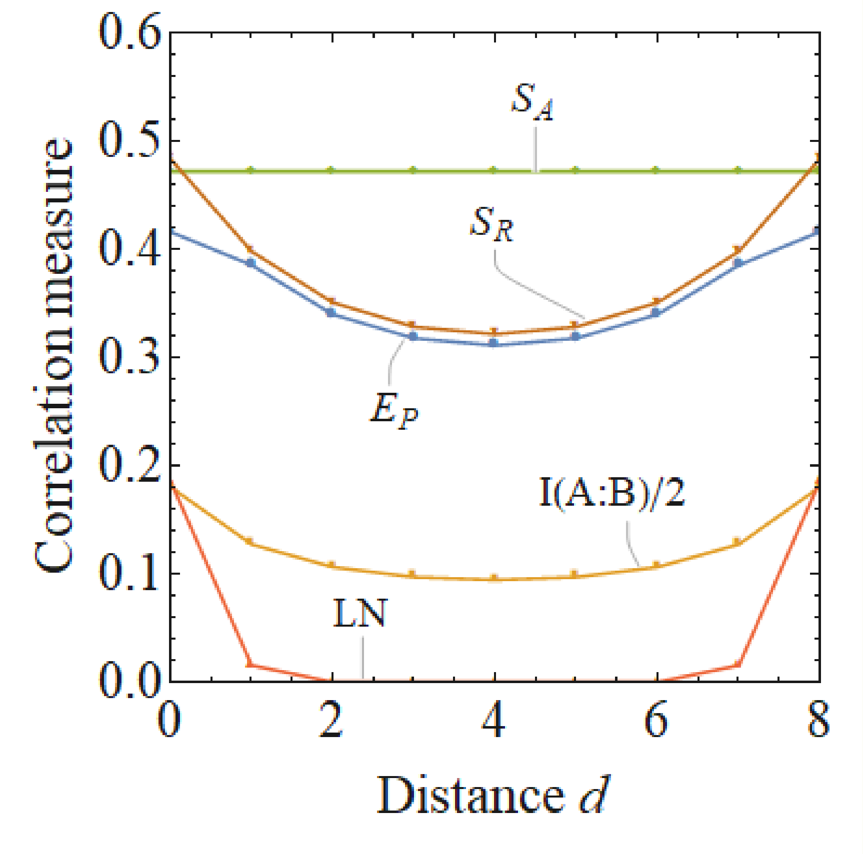


$(w = 4)$

Results

E.g. $N = 10$ Ising

(periodic b.c.)



$(w = 1)$

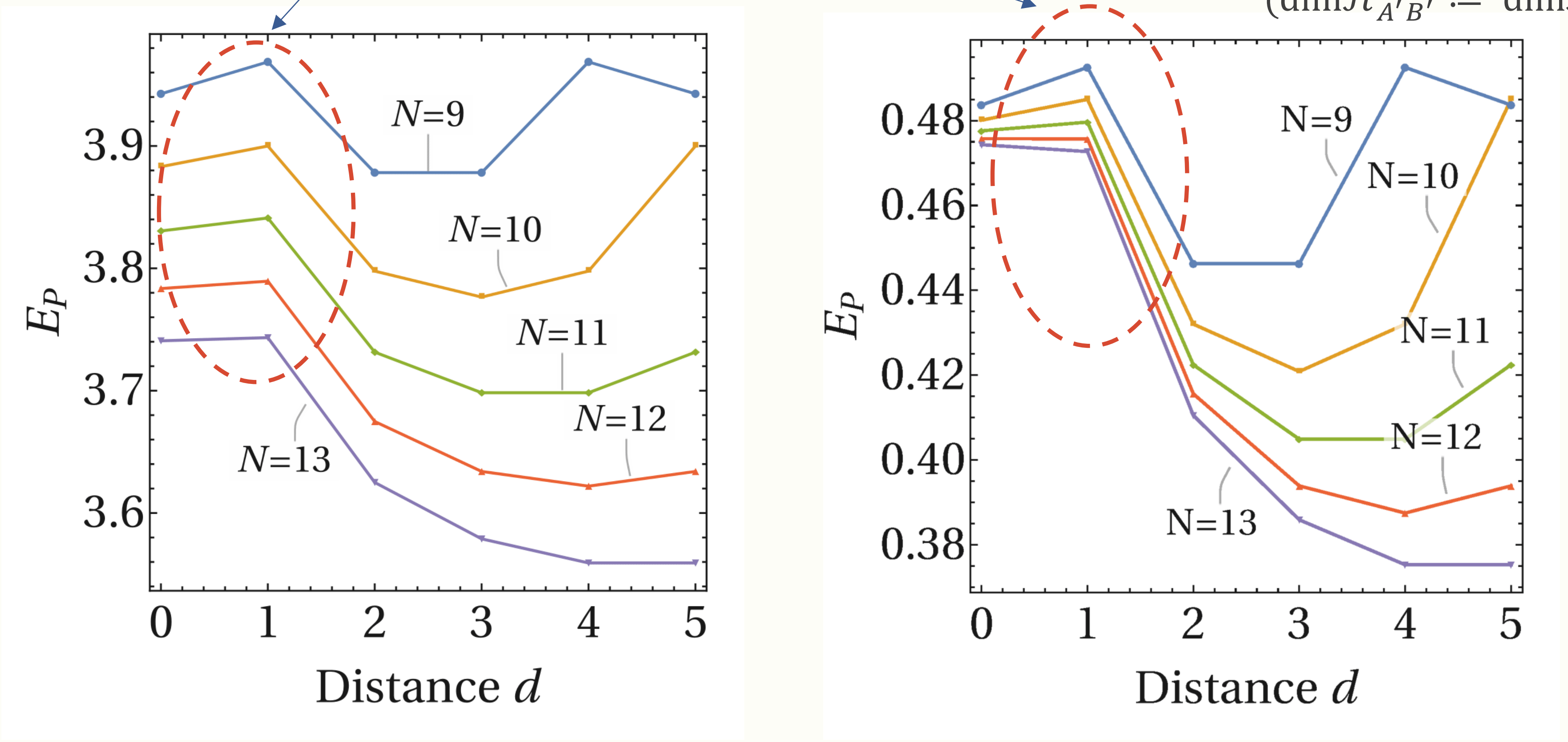
Non-monotonicity

EoP increases with the distance d (For small N, at $d = 1$)

Free scalar $w = 2$

Ising model $w = 2$

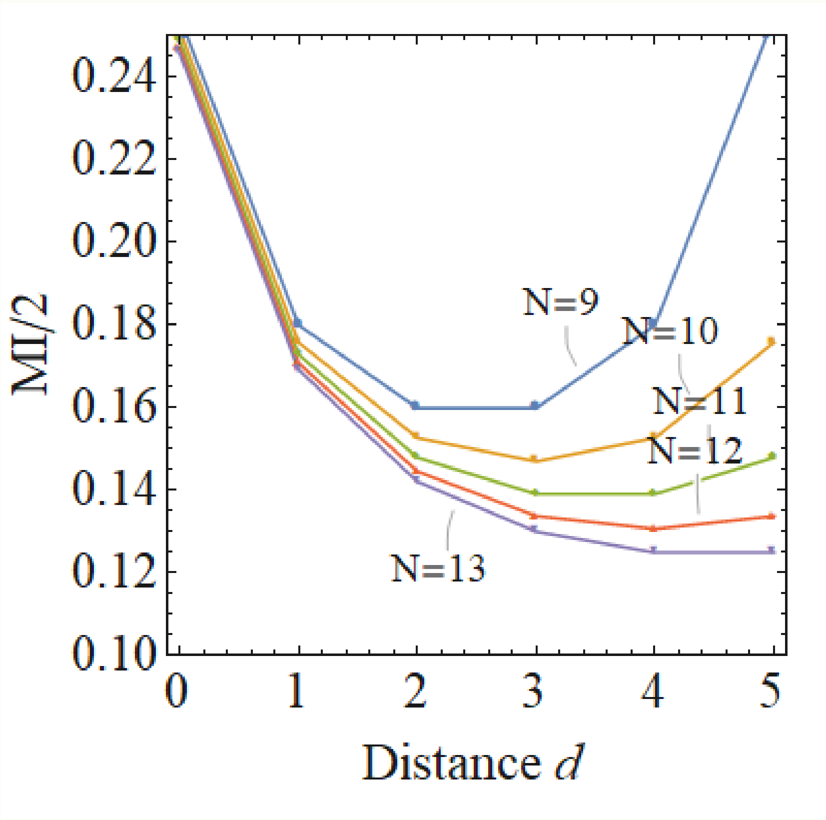
$(\dim\mathcal{H}_{A'B'} \coloneqq \dim\mathcal{H}_{AB})$



Non-monotonicity

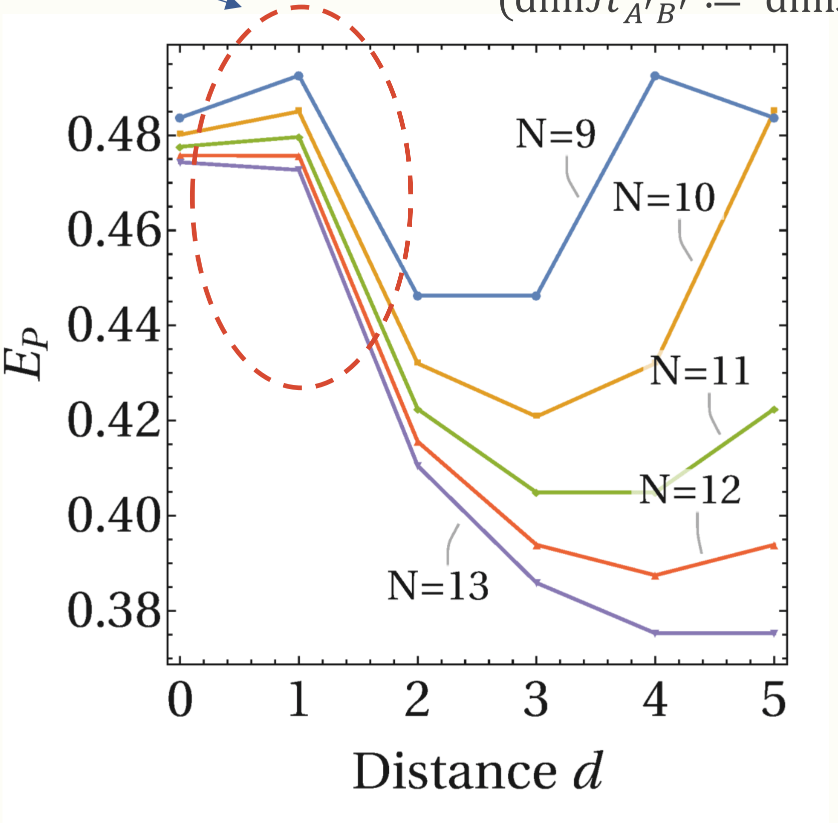
EoP increases with the distance d (For small N, at $d = 1$)

Mutual information



Ising model $w = 2$

$(\dim\mathcal{H}_{A'B'} \coloneqq \dim\mathcal{H}_{AB})$

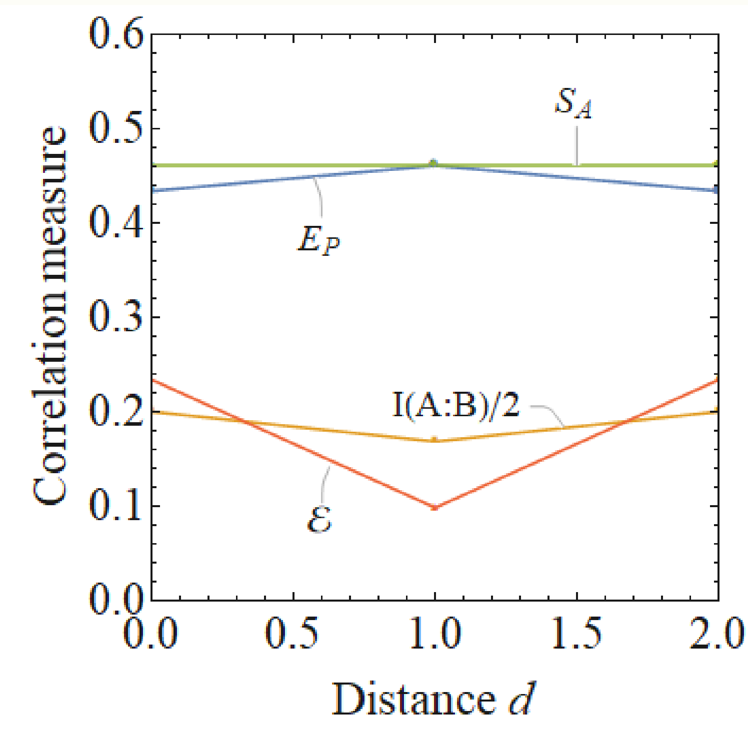


Non-monotonicity

It's so weird… Perhaps the minimization does not work well?

- We can show this behavior analytically in some cases

E.g) in $N = 4$ Ising model

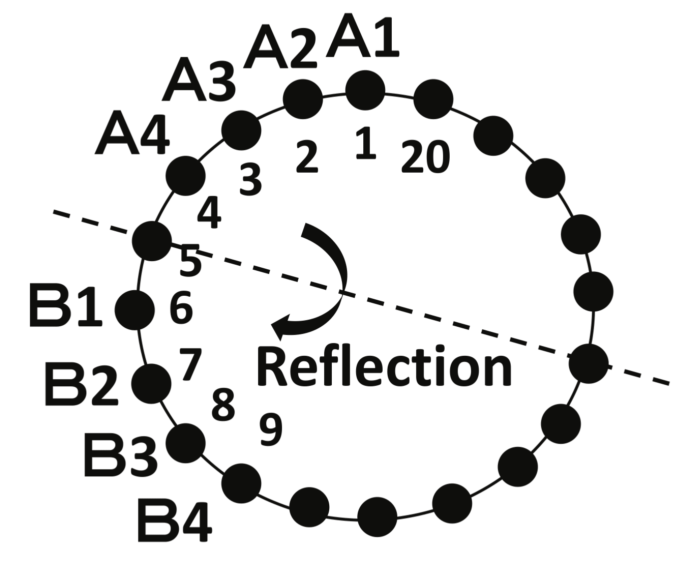


We can show $E_P(d = 1) = S_A = S_B$ by a thm.
and $E_P(d = 0) < S_A$ by numerics

Theorem Christandl-Winter '05
If ρ_{AB} has support only on the (anti-) symmetric subspace of $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, then $E_P(A\!:B) = S_A = S_B$.

Z2 symmetry breaking

The optimal purifications do not necessarily have the exchange symmetry $(AA' \leftrightarrow BB')$



$$\rho_{AB} = \rho_{BA}$$

But In some cases,

- $|\psi^{\mathrm{opt.}}\rangle_{AA'BB'} \neq |\psi^{\mathrm{opt.}}\rangle_{BB'AA'}$

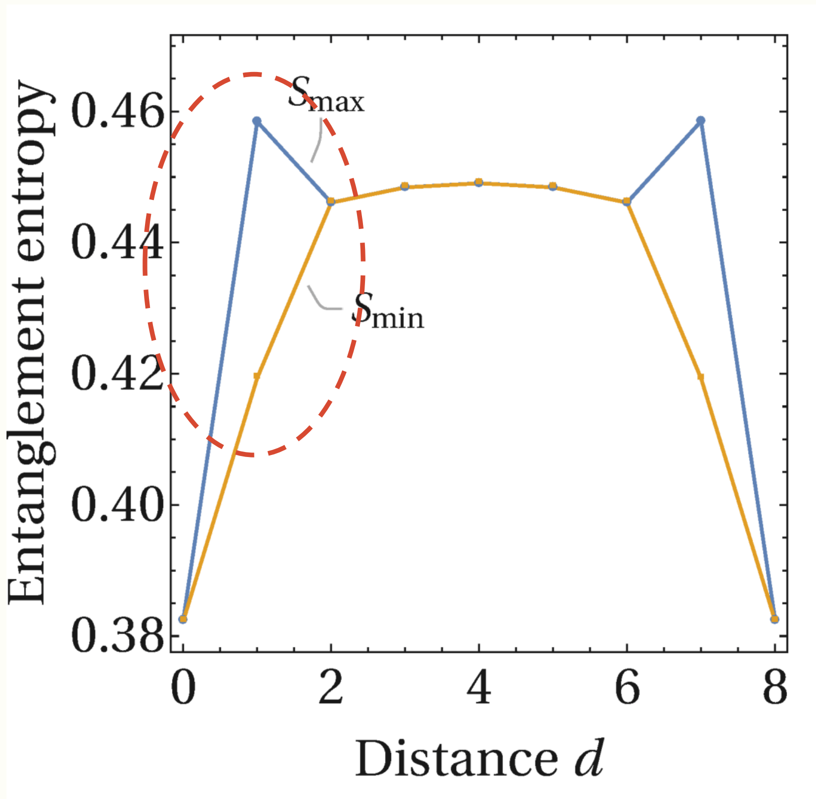
Optimal purification

Z2 symmetry breaking

The optimal purifications do not necessarily have the exchange symmetry $(AA' \leftrightarrow BB')$

E.g. Ising model, $N = 10, w = 1$

$S(\rho_{A'}^{\text{opt.}}) \neq S(\rho_{B'}^{\text{opt.}})$ at $d = 1$



$$S_{\text{max}} := \max\{S_{A'}, S_{B'}\},$$
$$S_{\text{min}} := \min\{S_{A'}, S_{B'}\}$$

A qualitative interpretation

Try to understand the qualitative aspects of results

- Interplay between quantum entanglement and classical correlations

  Classical correlations: typically in separable states

$$\rho_{AB} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i$$

  Suppose that total correlation, half of MI, is just a sum of them

(Of course not precise)

$$\frac{I(A:B)}{2} \sim E(A:B) + C(A:B)$$

Remainings ($C := \frac{I}{2} - E_{sq}$)

Squashed entanglement E_{sq} will be a good candidate ($\because E_{sq} \leq I/2$)

A qualitative interpretation

Q. What is the coefficients for EoP?

$$E_P(A:B) \sim a\, E(A:B) + b\, C(A:B)$$

1) EoP coincides with S_A for pure states

When $C(A:B) = 0,\ E_P(A:B) = S_A = E(A:B)$

$$\therefore a = 1$$

2) EoP is at least as large as $I(A:B)$ for separable states

When $E(A:B) = 0,\ E_P(A:B) \geq I(A:B) = 2C(A:B)$

Terhal-Horodecki-Leung-DiVincenzo '02

$$\therefore b \geq 2$$

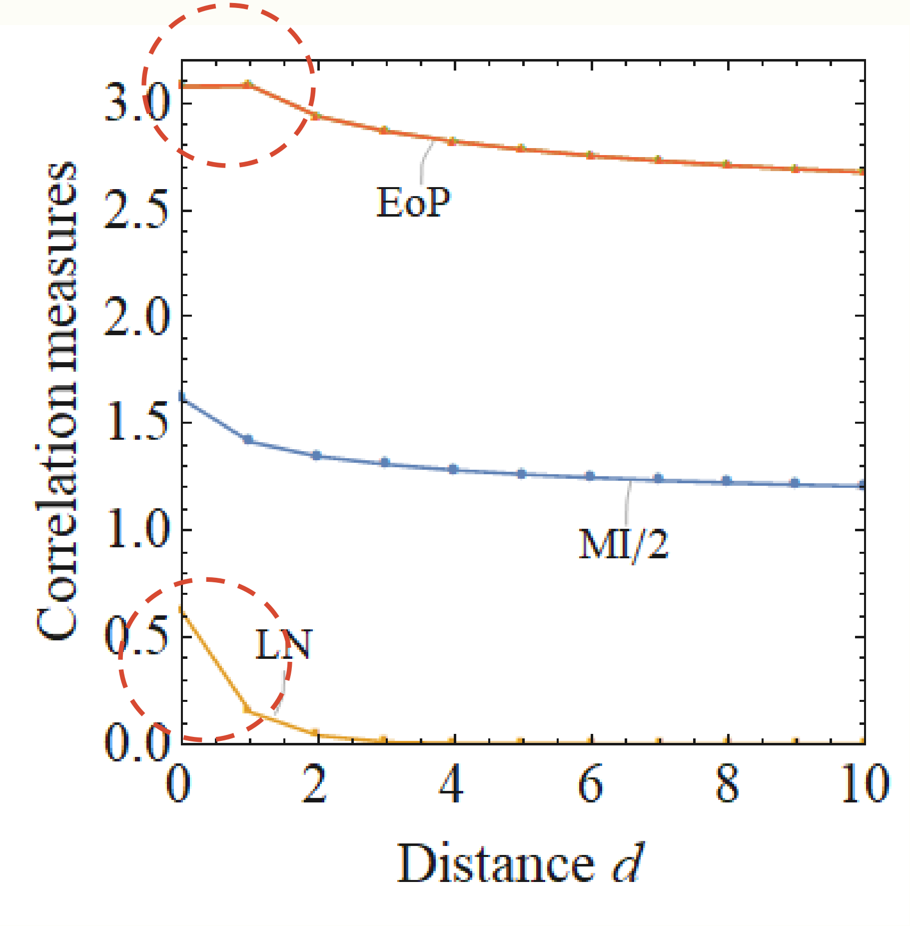
Claim: $E_P(A:B) \gtrsim 1\, E(A:B) + 2\, C(A:B)$

A qualitative interpretation

$$E_P(A:B) \gtrsim 1\, E(A:B) + 2\, C(A:B)$$

$$\frac{I(A:B)}{2} \sim 1\, E(A:B) + 1\, C(A:B)$$

Entanglement is strong



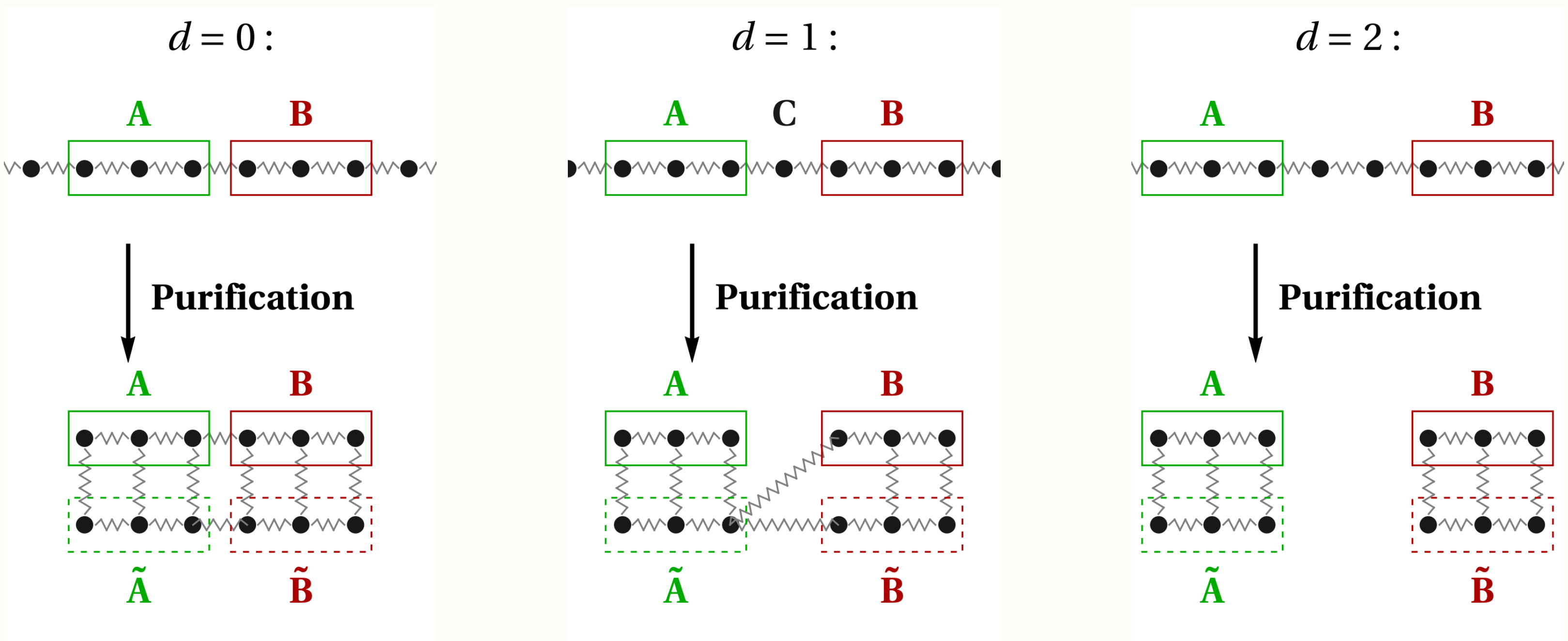
Classical correlation is strong

A qualitative interpretation

A toy model explaining why Z_2 symmetry is broken only at $d = 1$:

Focus on the nearest neighbor entanglement

- EoP converts classical correlation into entanglement in the purified system



Summary

- We computed entanglement of purification in 2d free scalar field and 2d Ising model by numerically performing the minimization.
- We found that EoP can increase with the physical distance. It is quite different from other measures such as mutual information.
- The optimal purifications are not necessarily symmetric under exchange $AA' \leftrightarrow BB'$ even if the original state satisfies $\rho_{AB} = \rho_{BA}$
- Both can be interpreted as an interplay between entanglement and classical correlation

Appendices

Entanglement of Purification (EoP)

Definition Terhal-Horodecki-Leung-DiVincenzo '02

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S(\rho_{AA'}) \qquad (\rho_{AA'} := \mathrm{Tr}_{BB'}[|\psi\rangle\langle\psi|_{AA'BB'}])$$

- It monotonically decreases under local operations
- $E_P \geq 0$ and $E_P = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$

∴A measure of total correlation (not just entanglement)

Cf. mutual information $I(A\colon B) := S_A + S_B - S_{AB}$

EoP for free scalar field

- Vacuum is Gaussian state

d.o.f. on the sites

$$\Psi^0_{\text{total}}(\vec{\phi}) \propto \exp\left(-\frac{1}{2}\vec{\phi}^T W \vec{\phi}\right)$$

Lattice cutoff = 1

where $W_{nn'} = \frac{1}{N}\sum_{k=1}^{N}\sqrt{4\sin^2(\frac{\pi k}{N}) + m^2 a^2}\, e^{\frac{2\pi i k(n-n')}{N}}$

(small) mass $\sim 10^{-4}$

$$=: \exp\left(-\frac{1}{2}(\vec{\phi}_{AB}, \vec{\phi}_{\text{other}})^T \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} (\vec{\phi}_{AB}, \vec{\phi}_{\text{other}})\right)$$

Minimal Gaussian Purification ansatz

$$\rho_{AB}(\vec{\phi}_{AB},\vec{\phi}'_{AB}) \propto \exp\left(-\frac{1}{2}(\vec{\phi}_{AB},\vec{\phi}_{AB})^T \begin{pmatrix} P-\frac{1}{2}QR^{-1}Q^T & -\frac{1}{2}QR^{-1}Q^T \\ -\frac{1}{2}QR^{-1}Q^T & P-\frac{1}{2}QR^{-1}Q^T \end{pmatrix} (\vec{\phi}_{AB},\vec{\phi}'_{AB})\right)$$

Minimal Gaussian Purification ansatz

$$|AB| = |A'B'|$$

$$\Psi^{\text{Gaussian}}_{AA'BB'}(\vec{\phi}) \propto \exp\left(-\frac{1}{2}(\vec{\phi}_{AB},\vec{\phi}_{A'B'})^T \begin{pmatrix} J & K \\ K^T & L \end{pmatrix} (\vec{\phi}_{AB},\vec{\phi}_{A'B'})\right)$$

- $\mathrm{Tr}_{A'B'}|\Psi^G\rangle\langle\Psi^G|_{AA'BB'} = \rho_{AB}$ $\Rightarrow$ only K is free parameters
- Perform the minimization of $S_{AA'}(K)$ over the minimal Gaussian purification ansatz by changing the components of K

(We can further reduce the numbers of parameters by using a symmetry of EE)

EoP for Ising model

$$H_{\text{total}} = -\sum_{<i,j>} \sigma_i^z \otimes \sigma_j^z - \sum_i \sigma_i^x$$

- The critical 2d Ising model

- Total vacuum state $|\Omega\rangle_{\text{total}} \to \rho_{AB} \to |\psi_0\rangle_{AA'BB'}$ (any purification)

- All possible purifications = All isometry maps (embedding + unitary)

$$\rho_{AB} \to I_{AB} \otimes V_{A'B'}^{iso} |\psi_0\rangle_{AA'BB'}$$

- Minimize $S_{AA'}(V_{A'B'}^{iso})$ without any ansatz

EoP for Ising model

We do not need to consider arbitrary large Hilbert space $\mathcal{H}_{A'B'}$

<u>Theorem</u> Ibinson-Linden-Winter '06
In a finite dimensional case, the minimum of $S_{AA'}$ can be achieved by

$$\mathrm{dim}\mathcal{H}_{A'} \leq \mathrm{rank}\rho_{AB} \quad \text{and} \quad \mathrm{dim}\mathcal{H}_{B'} \leq \mathrm{rank}\rho_{AB}$$

E.g. 2 qubits

A ● ● B ▶ $A \cup B$ ● ● ● ● ● ● $A' \cup B'$

$\mathrm{rank}\rho_{AB} = 4$ $\mathrm{dim}\mathcal{H}_{A'} = \mathrm{dim}\mathcal{H}_{B'} = 4$

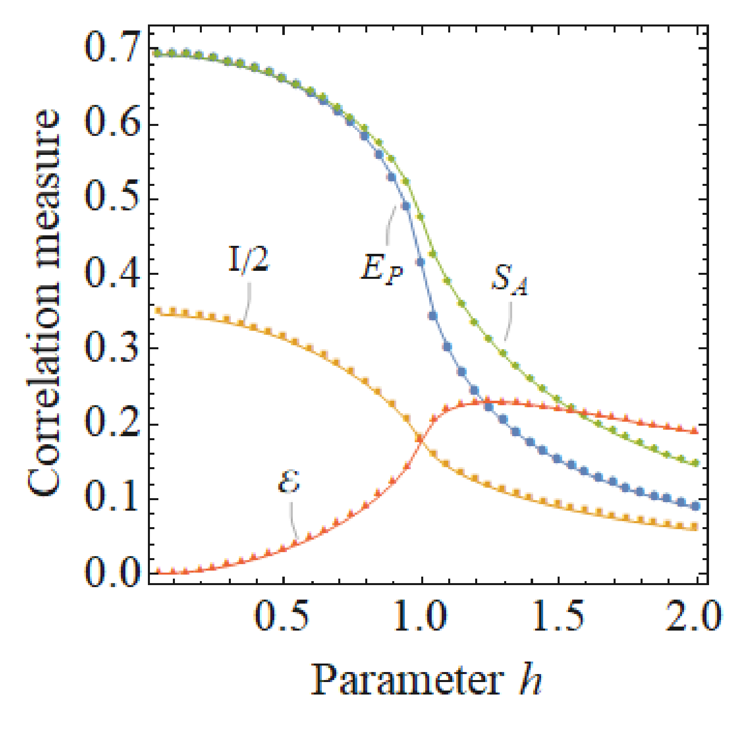
Z2 symmetry breaking

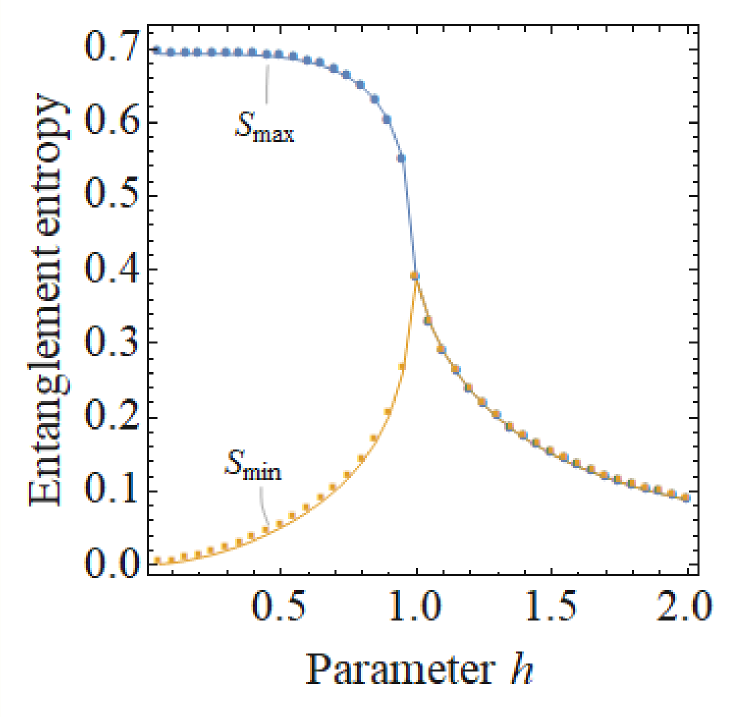
The Z2 symmetry breaking and quantum phase transition

$$H_{\text{total}} = -\sum_{<i,j>} \sigma_i^z \otimes \sigma_j^z - h\sum_i \sigma_i^x$$

($N = \infty$, thermal ground state $|\Omega\rangle = \lim_{\beta\to\infty} e^{-\beta H}/Z(\beta)$)

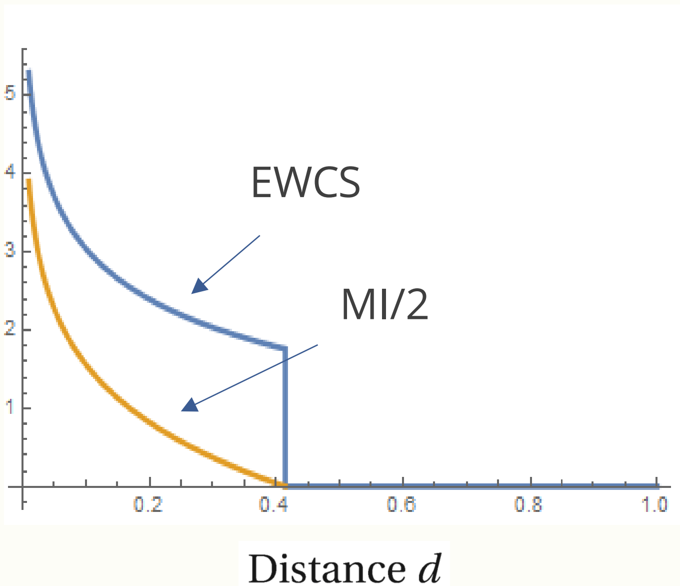
$(w = 1, d = 1)$

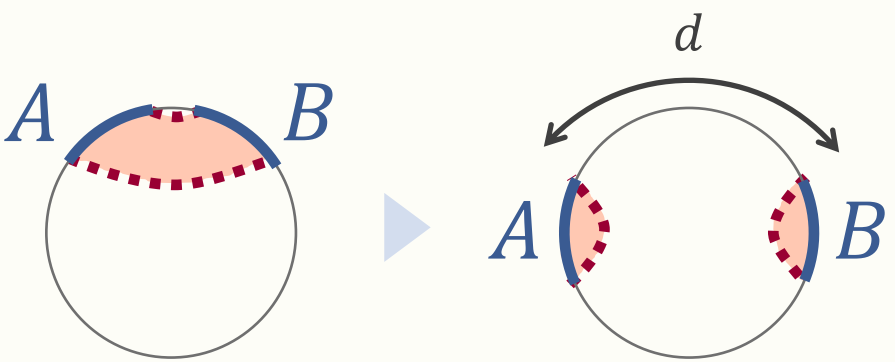




Implications to holography

- We know that EWCS behaves differently from MI around the transition point





- Reflection symmetry could also break in excited states or $O(1)$ correction