“QIST 2019” @YITP,Kyoto 2019/06/05

Path-Integral Optimization & Complexity in CFT

(+an attempt in BCFT)

Kento Watanabe (U.Tokyo)

Based on

- Phys.Rev.Lett. 119 (2017) no.7, 071602
- JHEP 1711 (2017) 097

w/ Pawel Caputa, Nilay Kundu, Masamichi Miyaji and Tadashi Takayanagi

(+ work just started w/ Yoshiki Sato(IPMU))

In this talk, I will review

Path-Integral Optimization & the “Complexity” in CFT

[Caputa-Kundu-Miyaji-Takayanagi-KW’17] 2 years ago…

Maybe useful appetizer to listen Michal’s talk on complexity

Masamichi’s talk on EoP

Then, I will try to show you our recent attempt in BCFT…

Preliminary observation
just started w/ Yoshiki Sato

AdS/CFT & Quantum Information

Modern Perspective of AdS/CFT ⟶ A "Geometrization" of Quantum States

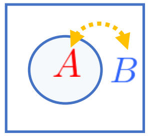
Quantum Entanglement ⟷ Emergent Geometry

AdS/CFT
Tensor Network

"Gravity"
"Spacetime"

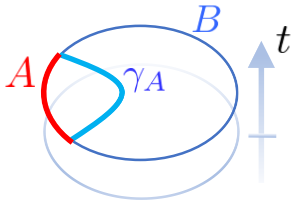
Ex: Entanglement Entropy (EE) ⟷ Min. Area of codim.-2 Surface

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A$$

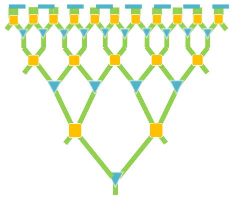


$$\min_{\gamma_A}\left[\frac{Area(\gamma_A)}{4G_N}\right]$$

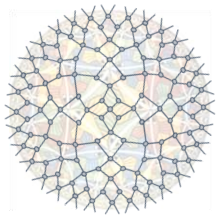
[Ryu-Takayanagi 06]



MERA network
(for CFT vacuum)
[Vidal 05,06]



⟶ Hyperbolic geometry (a time slice of AdS)
[Swingle 09]



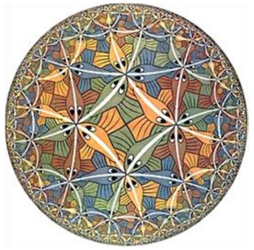
But still mysterious mechanism… ⟶
- More direct way to dual geometries from quantum states?
- Entanglement is NOT enough…Better probe? Complexity?

Motivation & Proposal (1)

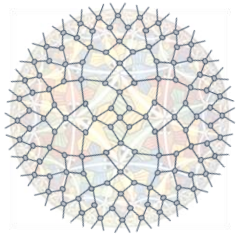
AdS/CFT → 20 years old! but still mysterious…

Direct or Systematic Way to Get Information about Dual Geometries?

Quantum State ←AdS/CFT→ Geometry (a time slice of AdS)



- MERA (Tensor Network) —AdS/TN→ Geometry
- Euclidean Path-Integral → MERA (Tensor Network)
- Euclidean Path-Integral → Geometry: Direct way ?



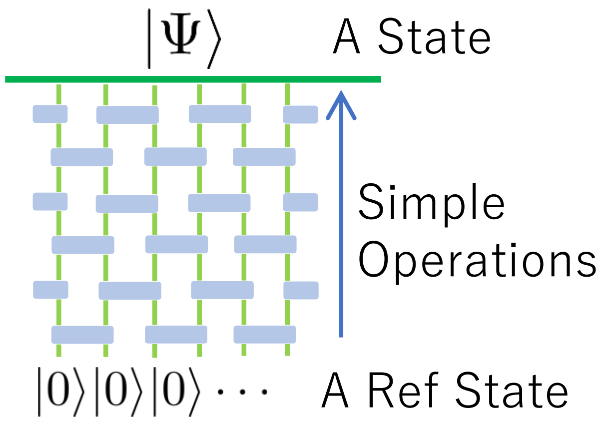
A toy model of AdS/CFT

Our Proposal [Caputa-Kundu-Miyaji-Takayanagi-KW’17]

“Optimization” of Euclidean Path-Integral for Wave Functional in CFTs

In 2d CFT → “Minimization” of the Liouville Action of the Back-ground Metric

Motivation & Proposal (2)

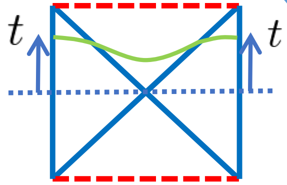


CFT Analogue of "Complexity of Quantum State" ?

$C_{\Psi}=$ min[#(Operations)]

- Holographic Complexity → A new probe for dual spacetime beyond HEE

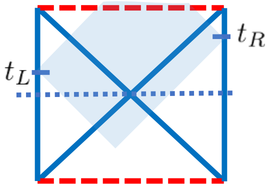
Complexity ⟺ "Volume"



[Susskind, +Stanford 14]

(Max. codim-1 surface)

or "Action"



[Brown-Roberts-Susskind-Swingle-Zhao 15]

(Action on WDW patch("codim-0"))

- CFT analogue? → No Definition of the complexity in CFTs so far…

A few developing attempts…
(including ours)

[Susskind, + collaborators…] [Chapman-Heller-Marrochio-Pastawski 17]
[Jefferson-Myers 17] [Yang, + collaborators 17]
[Caputa-Magan 18]…etc

Michal's talk?

Motivation & Proposal (2)

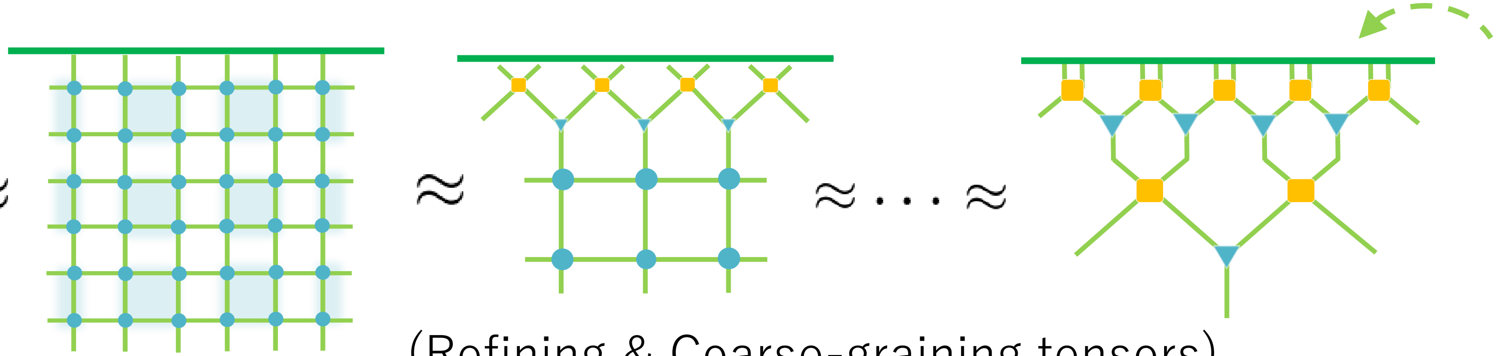
CFT Analogue of "Complexity of Quantum State" ?

Our Proposal [Caputa-Kundu-Miyaji-Takayanagi-KW'17]

Complexity of States in CFTs ⟷ "Optimized Action"

In 2d CFT, "Liouvlille Action"

Hint Tensor Network Renormalization (TNR) Procedure → "Optimization"

$\Psi[\tilde{\varphi}(x)] \approx$ 

(Refining & Coarse-graining tensors)

"Optimized" TN

Ex: MERA (CFT vacuum)

→ Complexity $\sim$ min#[tensors]

Optimization of Euclidean Path-Integrals & Complexity in CFTs

[Caputa-Kundu-Miyaji-Takayanagi-KW'17]

More detail···

Basic Rules for Our "Optimization" Procedure

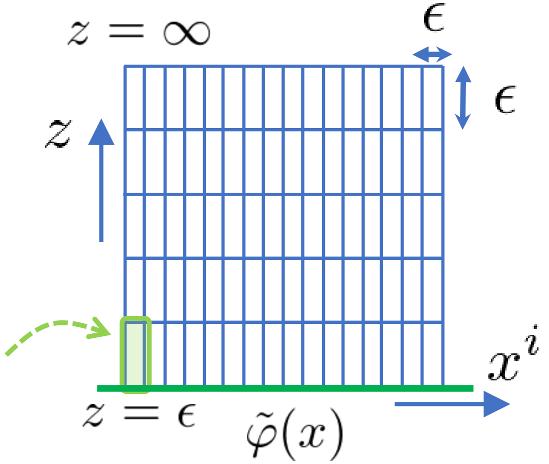
(CFT Analogue of TNR Procedure)

- Discretization of Euclidean path-integral

  (Regularization)

$$\Psi[\tilde{\varphi}(x)] = \int \left(\prod_x \prod_{\epsilon<z<\infty} D\varphi(z,x) \right) e^{-S_{CFT}(\varphi)} \cdot \prod_x \delta(\varphi(\epsilon,x) - \tilde{\varphi}(x))$$

→ Metric (one cell = unit area) $ds^2 = \epsilon^{-2} \cdot (dz^2 + dx^i dx^i)$



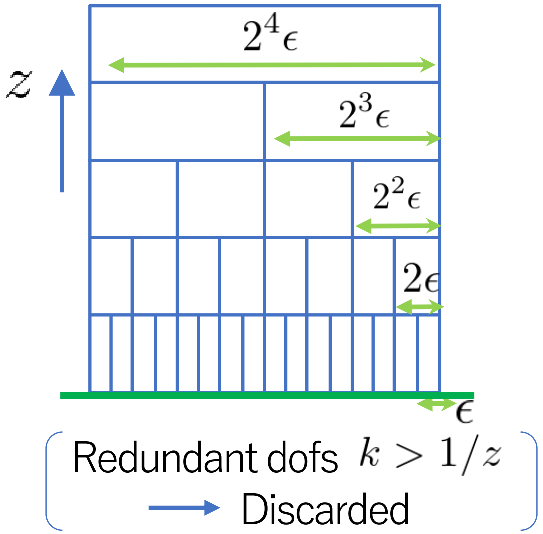
- "Optimization" of the path-integral

  → Changing the geometry of the lattice regularization

  → Modifying the back-ground metric for the path-integral

  with fixing the UV bdy condition

$$g_{zz}(z=\epsilon,x) = \epsilon^{-2}, \quad g_{ij}(z=\epsilon,x) = \delta_{ij} \cdot \epsilon^{-2} \qquad g_{iz}(z=,x) = 0$$



Basic Rules for Our “Optimization” Procedure

(CFT Analogue of TNR Procedure)

- After optimization, reproduce the correct wave functional up to a normalization

(Ansatz)

$$\longrightarrow \quad \Psi_{g_{ab}}[\tilde{\varphi}(x)] = N_{(g,\delta)} \cdot \Psi_{\delta_{ab}}[\tilde{\varphi}(x)]$$

Estimate redundant dofs → Minimize this!!

- “Optimization” of the path-integral

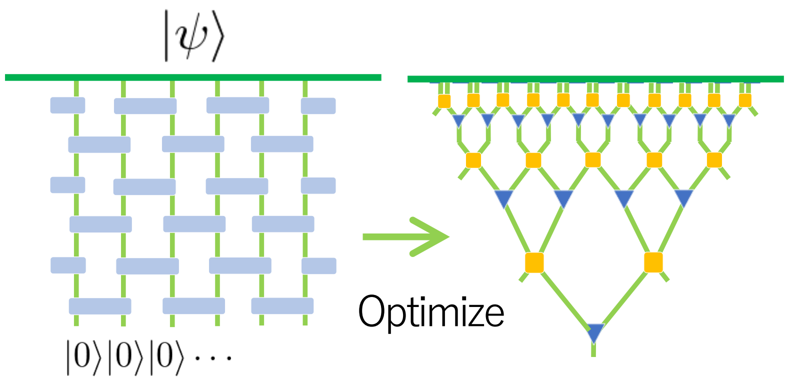
→ Minimizing “# of lattice points”

(= “# of tensors in a TN”)

→ Our Conjecture for Complexity in CFT

$$\boxed{N_{opt} \approx \exp(\mathrm{Complexity})}$$

→ Min[#(Operations)]



For CFT vacuum,
MERA network

2d CFT case

- In 2d CFT, we can diagonalize the general back-ground metric

Weyl Scaling $ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$ with $e^{2\phi(z=\epsilon,x)} = \frac{1}{\epsilon^2}$

- The change of the measure is characterized by the Liouville action

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}$$

UV regularization

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz[(\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu e^{2\phi} + R_0\phi]$$

→ # of unitaries in TN

Conformal Anomaly → # of isometries

[Czech 17]

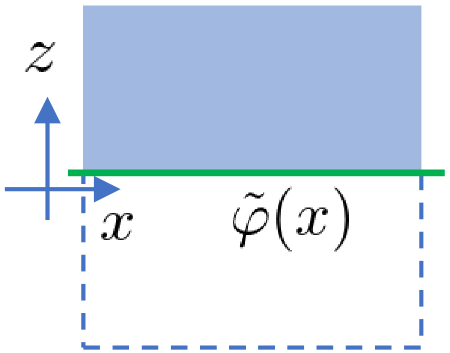
$$\longrightarrow \quad \Psi_{g_{ab}=e^{2\phi}\delta_{ab}}[\tilde{\varphi}(x)] = e^{\boxed{S_L[\phi]-S_L[0]}} \cdot \Psi_{g_{ab}=\delta_{ab}}[\tilde{\varphi}(x)]$$

Minimize this !!

$$\boxed{\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})}$$

Vacuum on Plane (Poincare AdS$_3$)

$w = z + ix$



$e^{2\phi(z=\epsilon,x)} = \frac{1}{\epsilon^2}$

EOM : $\partial_w \partial_{\bar{w}} \phi = \frac{\mu}{4} e^{2\phi} \longrightarrow e^{2\phi} = \frac{4}{\mu} \cdot \frac{A'(w)B'(\bar{w})}{(1 - A(w)B(\bar{w}))^2}$

Especially, $A(w) = w, \quad B(\bar{w}) = -1/\bar{w}$

a time slice of Poincare AdS$_3$

$$\longrightarrow e^{2\phi} = \frac{4}{\mu} \cdot \frac{1}{(w+\bar{w})^2} = \frac{1}{\mu} \cdot \frac{1}{z^2} \longrightarrow ds^2 = \frac{dx^2 + dz^2}{z^2} \quad : H_2$$

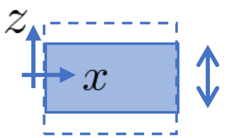
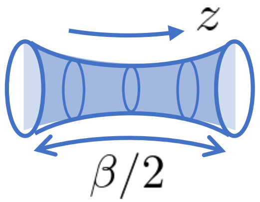
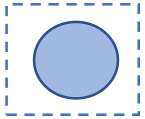
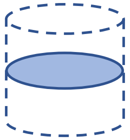
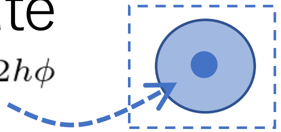
This solution clearly minimizes the Liouville action : $\quad e^{2\phi(z=\infty,x)} = 0 \qquad L = \int dx$

$$S_L = \frac{c}{24\pi} \int dxdz \left[(\partial_x \phi)^2 + (\partial_z \phi + \sqrt{\mu} e^{\phi})^2 \right] - \frac{c}{12\pi} \int dx [\sqrt{\mu} e^{\phi}]_{z=\epsilon}^{z=\infty} \geq \frac{c\sqrt{\mu}L}{12\pi\epsilon}$$

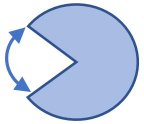
Volume divergence!!

Other Examples

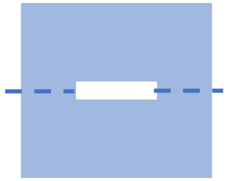
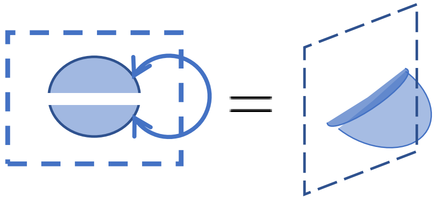
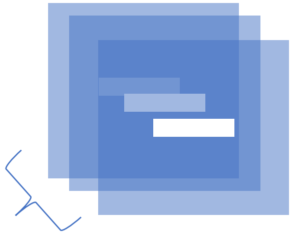
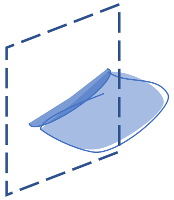
Work well for simples examples!!

- Finite T or TFD state  → Wormhole  (A time slice of BTZ BH)
- Vacuum on disk  → Hyperbolic disk H_2  (A slice of global AdS3)
- Primary State  → Conical Singular Geometry (Match to AdS3/CFT2 for $h \ll c$)

  $O(w, \bar{w}) \propto e^{-2h\phi}$

  Deficit angle 

Similarly ,

- $\rho_A = \mathrm{Tr}_{A^c} |\Psi\rangle\langle\Psi|$  → Entanglement Wedge 
- $S_A = -\mathrm{Tr}\rho_A \log \rho_A$  → Holographic EE $S_A \propto$ Min[Area] 

Examples : Our "Complexity" in CFTs

2d CFT

$$\boxed{\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})}$$

- Vacuum on plane (Poincare AdS3)

$$S_L = \frac{cL}{12\pi\epsilon}$$

Volume divergence !!

$$L = \int dx$$

- Vacuum on disk (Global AdS3)

$$S_L = \frac{c}{6} \cdot \left(\frac{1}{\epsilon} - 1 \right)$$

- TFD (BTZ BH)

$$S_L = \frac{c}{3} \cdot \left(\frac{1}{\epsilon} - \frac{\pi^2}{2\beta} \right)$$

With a naïve extension…

3d CFT

- (Global AdS4) $N = \frac{1}{8\pi G_N}$

$$S_L = 4\pi N \cdot \left(\frac{1}{\epsilon^2} + \frac{1}{2} + \log\left(\frac{2}{\epsilon}\right) \right)$$

4d CFT

- (Global AdS5) $N = \frac{3}{16\pi G_N}$

$$S_L = 2\pi^2 N \cdot \left(\frac{2}{3\epsilon^3} + \frac{1}{\epsilon} - \frac{5}{12} \right)$$

→
- [The volume law leading divergence & The divergence structure] agree with the holographic complexities !!
- The relative coefficients are different in general…

[Chapman-Marrochio-Myers 16]
[Lehner-Poisson-Myers-Sorkin 16]
[Carmi-Myers-Rath 16] [Reynolds-Ross16]

Summary [Caputa-Kundu-Miyaji-Takayanagi-KW'17]

Metric on a time slice of dual spacetime

- A proposal to define complexity of states in CFTs using "Optimization" of Euclidean Path-Integrals
  - Some checks for some states in 2d CFTs, ρ_A , S_A (EW, HEE)
- "Complexity" = "Optimal Action" = "the Liouville Action" (2d) → $\boxed{\min_{\phi}[S_L(\phi) - S_L(0)] \approx (\text{Complexity})}$
- (Qualitative) Matching to the Holographic Conjectures!!

Further Works Developing in past 2 years!!

- Higher dims, Time depend., Relation to Holographic Complexity
- Non-CFT case
- RG flow
- Application to EoP → Masamichi's talk

⋯ etc

→ • How about CFT w/ defect or bdy ?

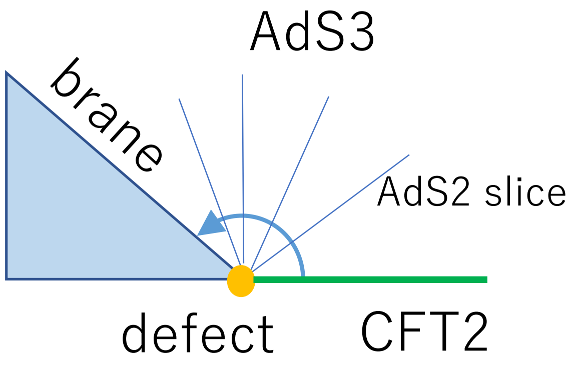
Defects distinguish Holographic Proposals?

[Chapman-Ge-Policastro'18]
(cf[Flory'17] for CV)

A toy model of AdS3/D(or B)CFT2
[Azeyanagi-Karch-Takayanagi-Thompson'07]

CFT2 w/ a defect (or bdy) ⟷ AdS3 w/ a brane on a AdS2 slice

$$I_{\text{bulk}} + I_{\text{brane}} = \frac{1}{16\pi G_N}\int d^3x\sqrt{-g}\left(R - \frac{2}{R^2_{\text{AdS}}}\right) + \lambda\int d^2x\sqrt{-h}$$



[Chapman-Ge-Policastro'18]

→ $C_V - C_{V,\text{vac}} = \frac{8c}{3}\sinh y^* \cdot \log\left(\frac{2L_{\mathcal{B}}}{\delta}\right)$ $\propto$ defect brane tension

$C_A - C_{A,\text{vac}} = 0$ No defect contribution ! (scheme dependent?)

→ Which is better as the complexity? Need the CFT counterpart…

Path-Integral Optimization in BCFT (Preliminary⋯)

(Just started working w/ Yoshiki Sato⋯)

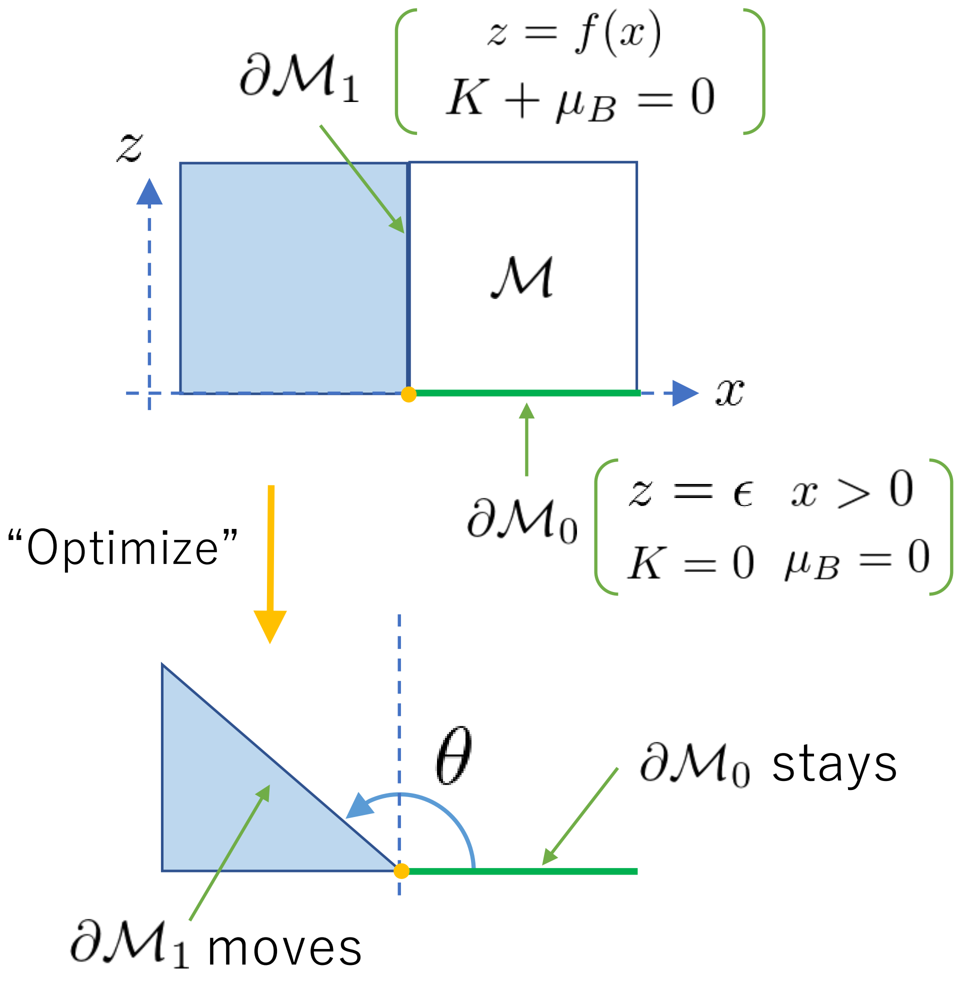
Boundary Liouville field theory

$$S_L[\phi] = \frac{c}{24\pi}\int_{\mathcal{M}}((\partial\phi)^2 + R\phi + \mu e^{2b\phi}) + \frac{c}{12\pi}\boxed{\int_{\partial\mathcal{M}}(K\phi + \mu_B e^{b\phi})}$$

EOM w/ b.c. on $\partial\mathcal{M}_0$ & $\partial\mathcal{M}_1$

$$C_L = \frac{1}{2}\cdot C_{L,\mathrm{vac}} \boxed{+\, C_{L,\mathrm{bdy}}(\mu_B)}$$

Might appear bdy contribution
(or corner contribution?)



Check the bdy entropy, g-thm⋯

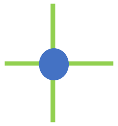
Work harder! Please discuss!

Thank you !

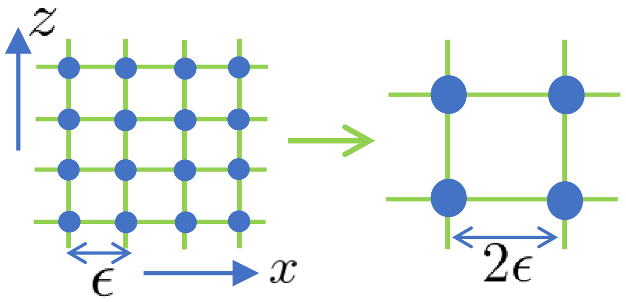
Naïve Estimation: Liouville Action From TNR

[Czech 17] [Caputa-Kundu-Miyaji-Takayanagi-KW17]

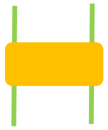
- Suppose each tensor has unit area in the original square lattice



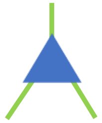
$$\frac{dxdz}{\epsilon^2} = e^{2\phi(z,x)}dxdz \xrightarrow{\text{Coarse-graining}} \frac{dxdz}{(2\epsilon)^2}$$



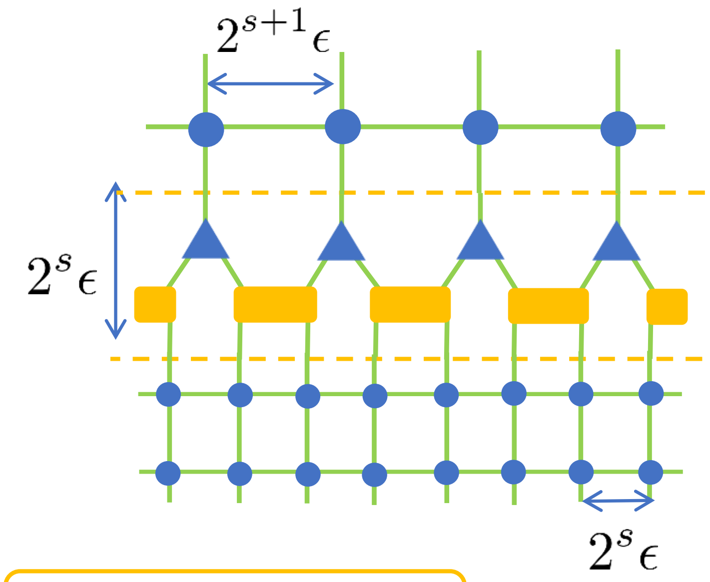
- For the s-th layer of MERA network, per unit cell,



$$\frac{dxdz}{(2^s\epsilon)^2} \longrightarrow e^{2\phi}$$



$$\frac{dxdz}{(2^{s-1}\epsilon)(2^s\epsilon)} \longrightarrow \partial\phi$$



(s+1)-th layer

$\phi = -\log(2^{s+1}\epsilon)$

MERA layer

s-th layer

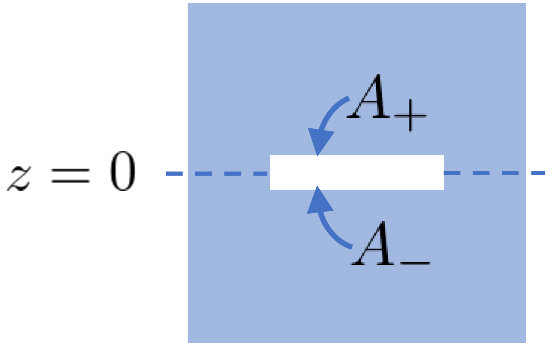
$\phi = -\log(2^s\epsilon)$

- Total # of tensors in the optimal network

$$\longrightarrow S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty}dx\int_{\epsilon}^{\infty}dz[(\partial\phi)^2+\mu e^{2\phi}] \sim \boxed{\text{Complexity !!}}$$

Optimization for Density Matrix

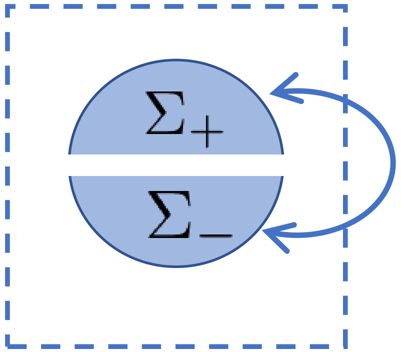
$$\rho_A = \mathrm{Tr}_{A^c} |\Psi\rangle\langle\Psi|$$



Optimize S_L

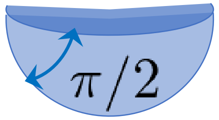
$$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$$

$$e^{\phi}|_{A_\pm} = \frac{1}{\epsilon}$$



Identify along $\partial\Sigma_\pm$

=



Same as holographic EE !!

- The shape of $\partial\Sigma_\pm$ is given by extremizing

$$S_{Lb} = \frac{c}{12\pi}\int_{\partial\Sigma_\pm} ds[K_0\phi + \mu_B e^{\phi}] \longrightarrow \mu_B = 0$$

(tension less)

- Half circle $x^2 + z^2 = l^2$

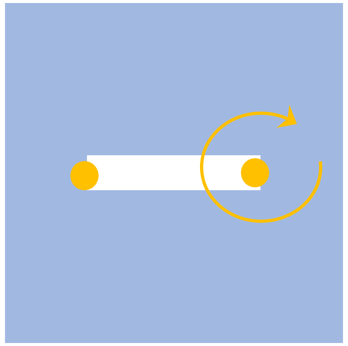
- Neumann bdy condition $\longrightarrow e^{\phi}K = n^a\partial_a\phi + K_0 = 0$

$\longrightarrow$ Entanglement wedge

Entanglement Entropy from Optimization

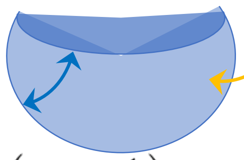
n-sheeted geometry
w/ deficit angle $2\pi(1-n)$

ρ_A^n



Optimize

$|n-1| \ll 1$



$\partial\Sigma_\pm$

$\frac{\pi}{2}+\pi(n-1)$

- Bdy condition & shape of $\partial\Sigma_\pm$ changes

$$K+\mu_B=0 \longrightarrow \begin{array}{l} K=\pi(n-1) \\ \mu_B=\pi(1-n) \end{array}$$

Same as holographic EE !!

- $$S_A=-\partial_n\left[\frac{c\mu_B}{12\pi}\int_{\partial\Sigma_+} ds\ e^\phi+\frac{c\mu_B}{12\pi}\int_{\partial\Sigma_-} ds\ e^\phi\right]_{n=1}=\frac{c}{6}\int_{\partial\Sigma_+} ds\ e^\phi=\frac{c}{3}\log\frac{2l}{\epsilon},$$

$\propto$ Length of the extremal surface