Entanglement branes, Modular flow, and Extended quantum field theory

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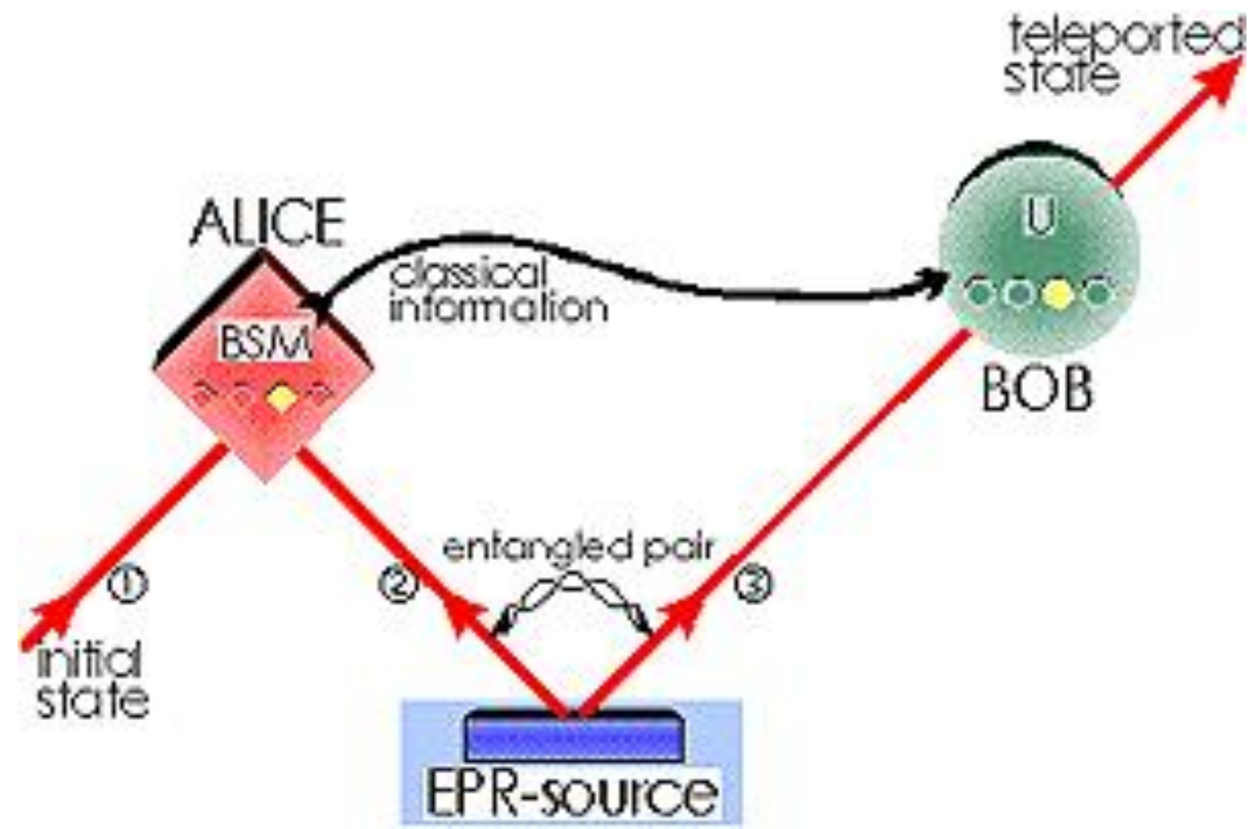
Based on collaboration with William Donnelly: hep-th 1811.10785

Quantum Entanglement is a **non local** feature of Quantum Mechanics

Unentangled $|0\rangle|0\rangle$

Entangled EPR $\frac{1}{\sqrt{2}}(|1\rangle|0\rangle - |0\rangle|1\rangle)$

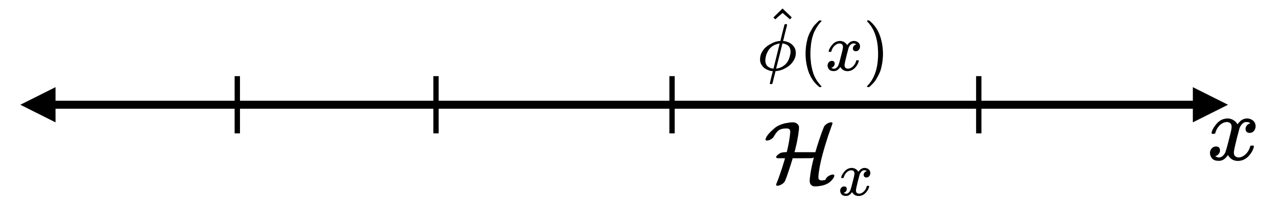


- Quantum Teleportation via EPR Pairs
- Non-local order parameter in Topological phases
- Many body localization
- Quantum gravity: Emergent smooth spacetimes from quantum entanglement

What do we mean by **locality in** quantum mechanics?

Hilbert Space factorization

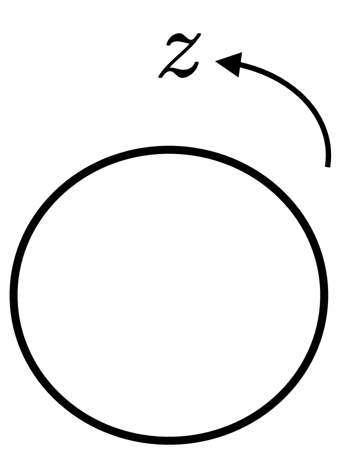
$$\mathcal{H} = \otimes_x \mathcal{H}_x$$



Different notions of locality can be assigned to the same Hilbert space

$$\mathcal{H} = \{\text{Anti-symmetric wavefunctions } \psi(z_1 \ldots, z_N),\ z_j = e^{i\theta_j}\}$$

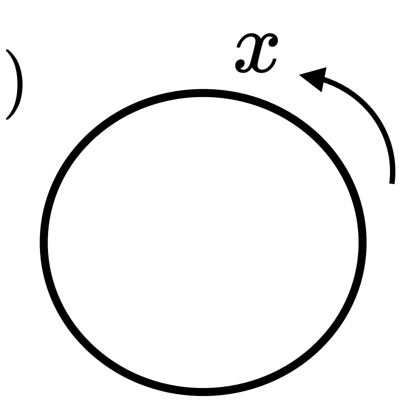
N non-relativisitic fermions in on a spatial circle



U(N) Yang Mills on a spatial circle .

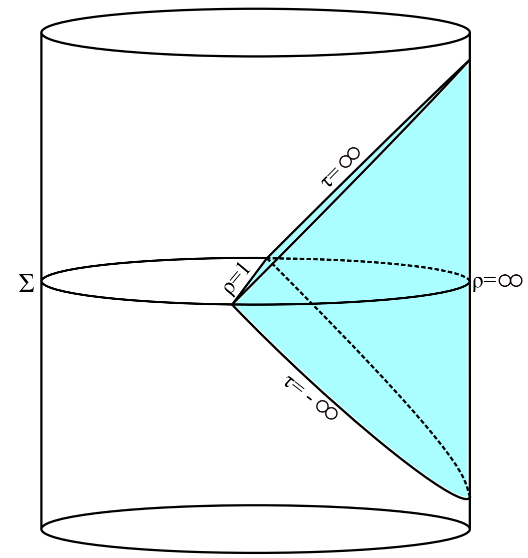
$$U = \mathrm{P}\exp(i \int A_x \, dx)$$

$$= \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix}$$



AdS/CFT and Bulk locality

- AdS/CFT provides a QG Hilbert space at asymptotic infinity
- How is the local bulk spacetime encoded in the CFT Hilbert space at infinity?



A free fermion Hilbert space in N=4 SYM

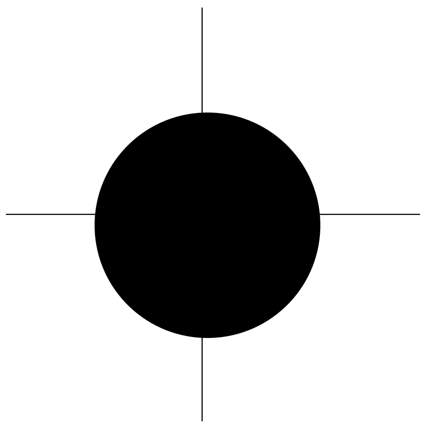
Ten- Dimensional Geometry for the IIB string

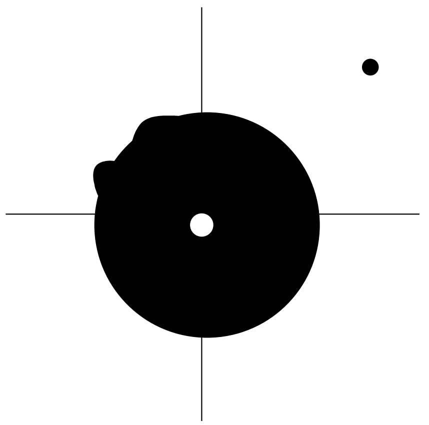
(Lin, Lunin, Maldacena)

$$
\begin{aligned}
ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \\
h^{-2} &= 2y \cosh G, \\
y\partial_y V_i &= \epsilon_{ij}\partial_j z, \qquad y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z \\
z &= \frac{1}{2}\tanh G \\
F &= dB_t \wedge (dt + V) + B_t dV + d\hat{B}\ , \\
\tilde{F} &= d\tilde{B}_t \wedge (dt + V) + \tilde{B}_t dV + d\hat{\tilde{B}} \\
B_t &= -\frac{1}{4}y^2 e^{2G}, \qquad\qquad \tilde{B}_t = -\frac{1}{4}y^2 e^{-2G} \\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d\left(\frac{z+\frac{1}{2}}{y^2}\right)\ , \qquad d\hat{\tilde{B}} = -\frac{1}{4}y^3 *_3 d\left(\frac{z-\frac{1}{2}}{y^2}\right)
\end{aligned}
$$

1/2 BPS sector of N=4 SYM

Free Fermions in a Magnetic field in the LLL





The extended Hilbert space construction

Hilbert Space factorization

$$\mathcal{H} = \otimes_x \mathcal{H}_x$$

Two obstructions

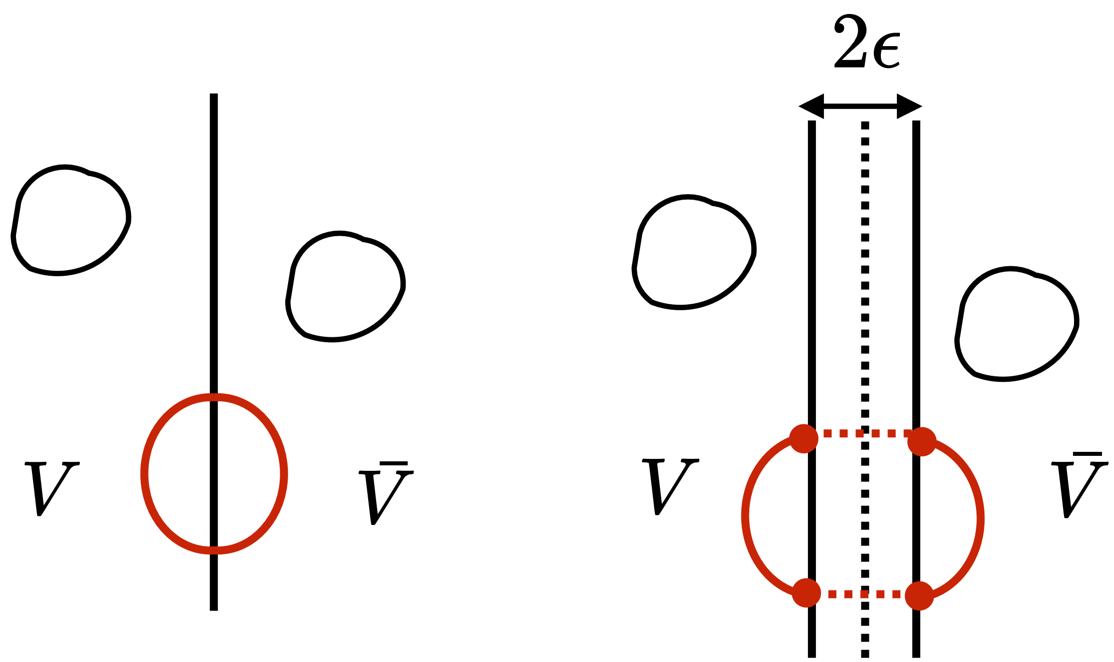
1 A subregion has a a boundary and therefore edge modes, even for a scalar field! (Agon, HeadrickJefferis,Kasko) (Campaglia, Freidal,et al)

2 Degrees of freedom in subregions are not independent

- continuity in a quantum field theory
- Gauss Law constraint in gauge theory. Even on a lattice !

The extended Hilbert space construction provides a solution by combining 1 and 2 (Donnelly, Freidel, Buividovich ,…)

Extended Hilbert space for gauge theories



$$\mathcal{H}_{\text{physical}} \subset \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$$

Contains edge modes transforming under boundary symmetry group $G_L \times G_R$

Gauss law

$$Q|\psi\rangle = 0 \quad \text{for} \quad |\psi\rangle \in \mathcal{H}_{\text{Physical}}$$

Invariance under $G = \text{Diag}\ (G_L \times G_R)$

Entangling product

$$\mathcal{H}_{\text{physical}} = \mathcal{H}_V \otimes_G \mathcal{H}_{\bar{V}}$$

Reduced density matrix

$$\rho_V = \text{tr}_{\bar{V}} |\psi\rangle\langle\psi|$$

Entanglement Entropy

$$S_V = -\text{tr}\ \rho_V \log\ \rho_V$$

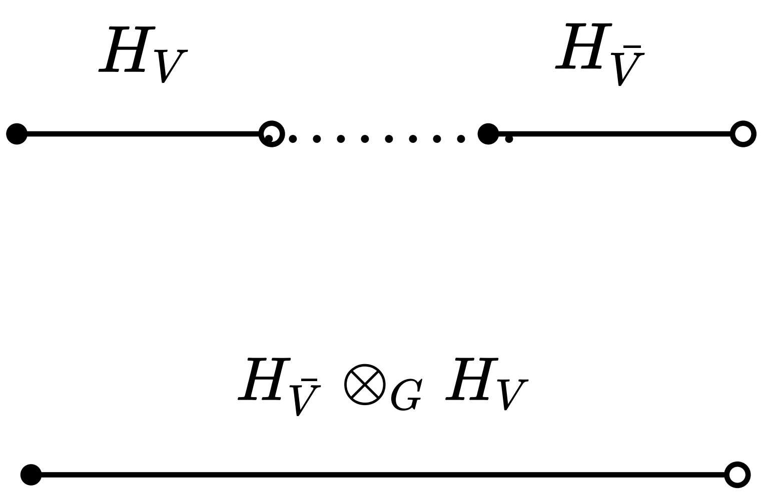
Extended Hilbert space and extended TQFT

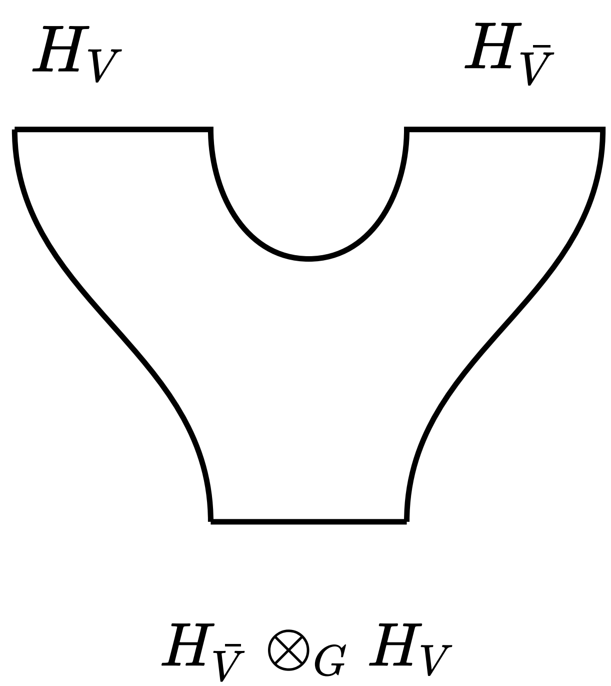
- Edge modes are not unique e.g. in quantum hall states (Cano, Cheng, Mulligan, …et. al) (Fliss,Wen,Parrikar,..et. al.)

- What are the rules for determining the “correct” edge modes and their gluings?

- In 2D, we provide constraints on the Hilbert space extension using the frame work of extended topological quantum field theory

- Key insight: View the entangling product as a spacetime process=cobordism

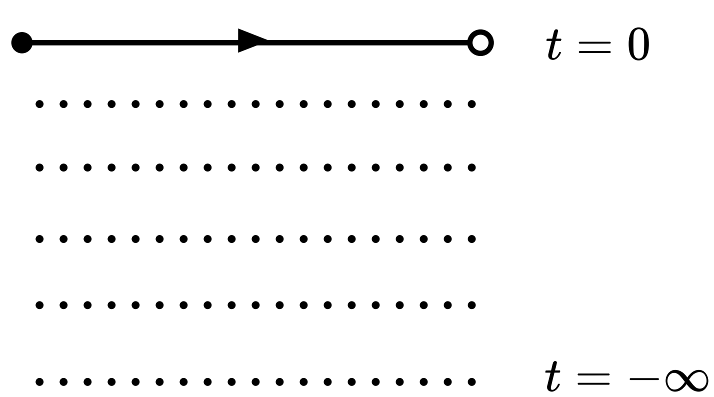




Entangling product from the path integral

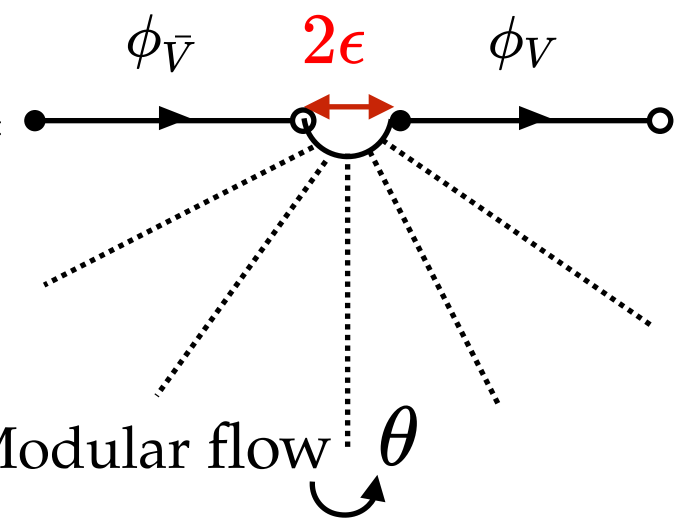
Euclidean path integral prepares the (unnormalized) vacuum

$\langle \phi(x)|0\rangle =$



Angular quantization

$\langle \phi_V\, \phi_{\bar V}|0\rangle =$



State-Channel duality

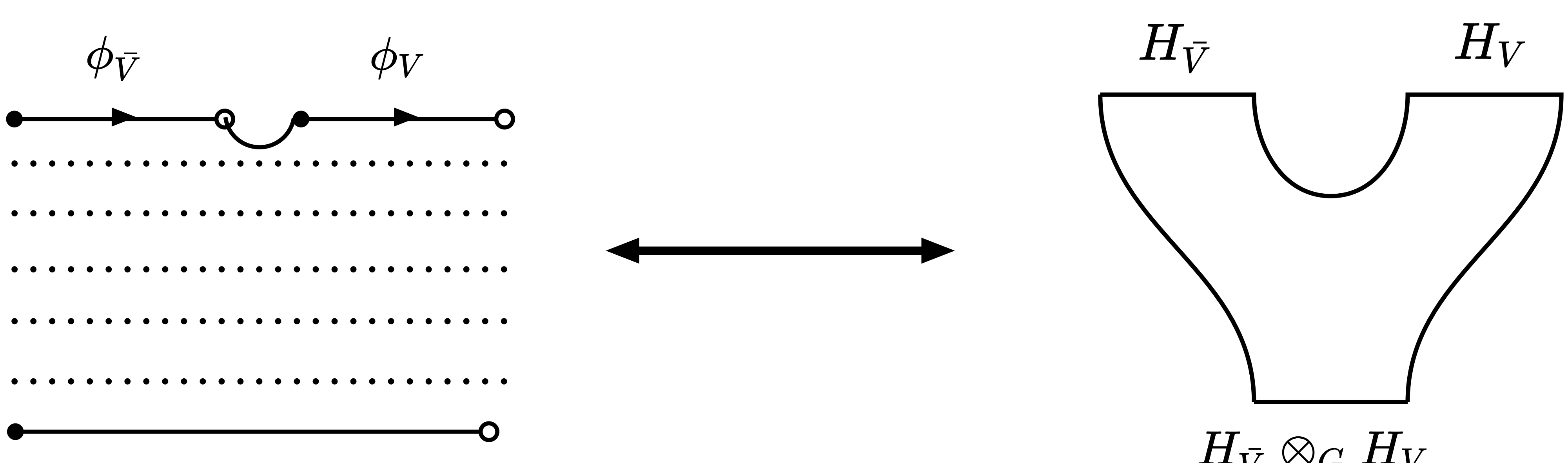
Modular Hamiltonian

CPT

$$\langle \phi_V\, \phi_{\bar V}|0\rangle = \langle \phi_V|\exp\left(\frac{-H_V}{2}\right)\mathcal{J}|\phi_{\bar V}\rangle \;=\; \sum_n \exp\left(-\frac{E_n}{2}\right)\langle \phi_V|n\rangle\langle \phi_{\bar V}|\bar n\rangle$$

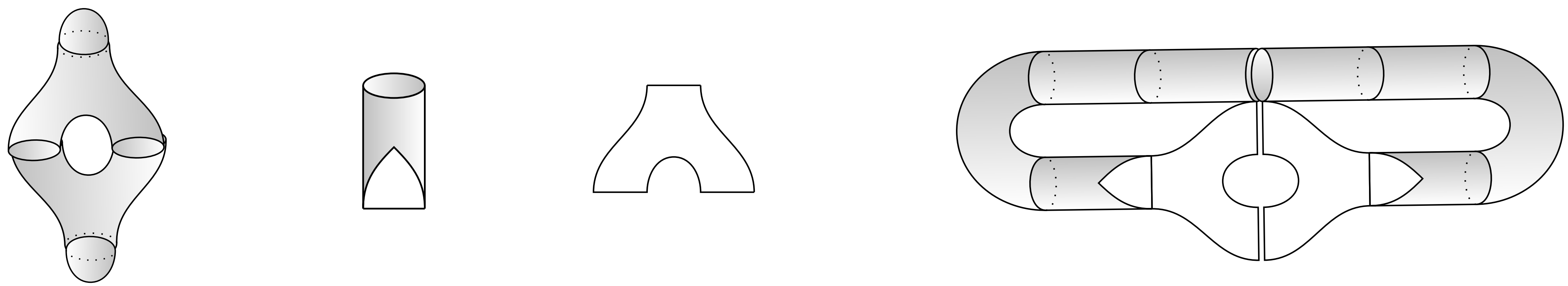
Entangling product in CS gauge theory (Wong 2018)

$\langle \phi_V\, \phi_{\bar V}|0\rangle =$



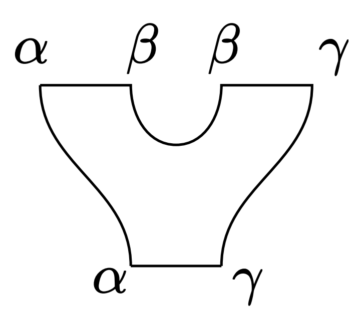
Locality in Extended TQFT

Cut path integral along surfaces of increasing codimension (Atiyah, Segal, Freed, Baez,...)

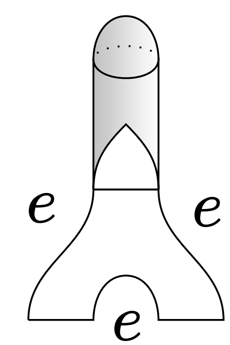


Moore-Segal gave sewing constraints for cutting and gluing path integrals.
These rules determine allowed boundary conditions=D branes.

D branes



E branes



What we did:

- Introduce the Entanglement brane boundary condition
- Interpret Moore-Segal as constraints for extended Hilbert space and edge modes
- Formulate 2D Yang Mills as an extended TQFT a la Moore-Segal
- Compute multi-interval modular flows, EE, negativity

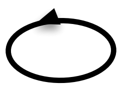
Outline

- Open-closed extended TQFT (Moore-Segal)
- Entanglement brane
- Multi-interval Modular flows, EE
- 2DYM as an open-closed TQFT
- Future works: CFT, higher dimensions, holography

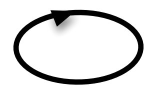
Atiyah's formulation of Axiomatic TQFT

In 2D, a TQFT is a rule assigning

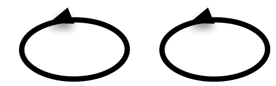
1-dim closed manifolds =Hilbert space over $\mathbb{C}$



$\mathcal{H}_{S_1}$



$\mathcal{H}_{-S^1} = \mathcal{H}^*_{S^1}$

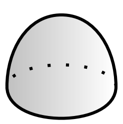


$\mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1}$

$\emptyset$

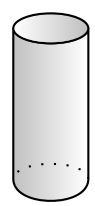
$\mathbb{C}$

Cobordism between circles = Linear maps (quantum evolution)



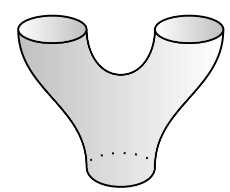
$\mathbb{C} \longrightarrow \mathcal{H}_{S^1}$

Wavefunction $\Psi[\phi(x)]$



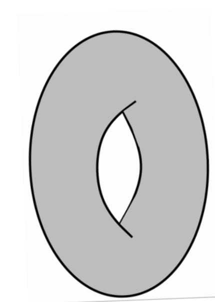
$\mathcal{H}_{S^1} \xrightarrow{e^{-tH}} \mathcal{H}_{S^1}$

Propagator=Identity



$\mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \longrightarrow \mathcal{H}_{S^1}$

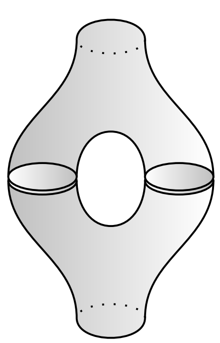
Fusion/Multiplication



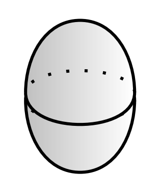
$\mathbb{C} \longrightarrow \mathbb{C}$

Partition function

Gluing Cobordisms =Composing linear maps



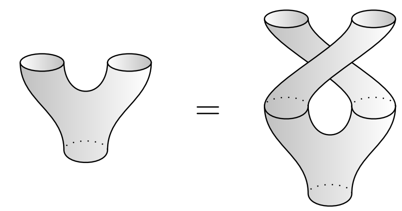
$\mathcal{H}_{S^1} \longrightarrow \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \longrightarrow \mathcal{H}_{S^1}$



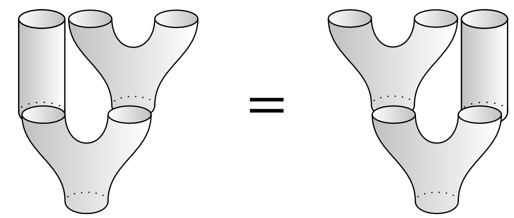
$$\int D\phi(x)\, \Psi[\phi(x)]\Psi^*[\phi(x)] = \langle \Psi | \Psi \rangle$$

A 2D Closed TQFT is a commutative Frobenius algebra

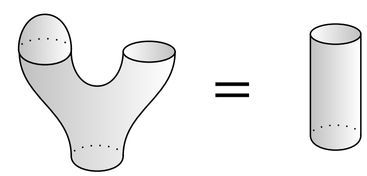
Commutative



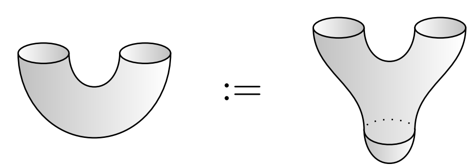
Associative



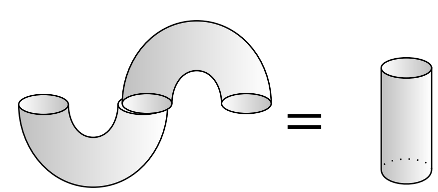
Unit



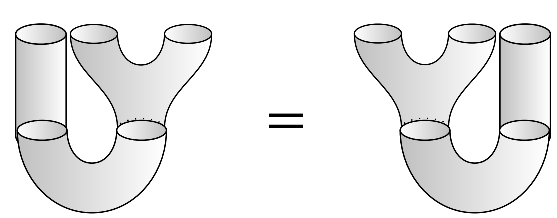
Symmetric bilinear form



Invertible



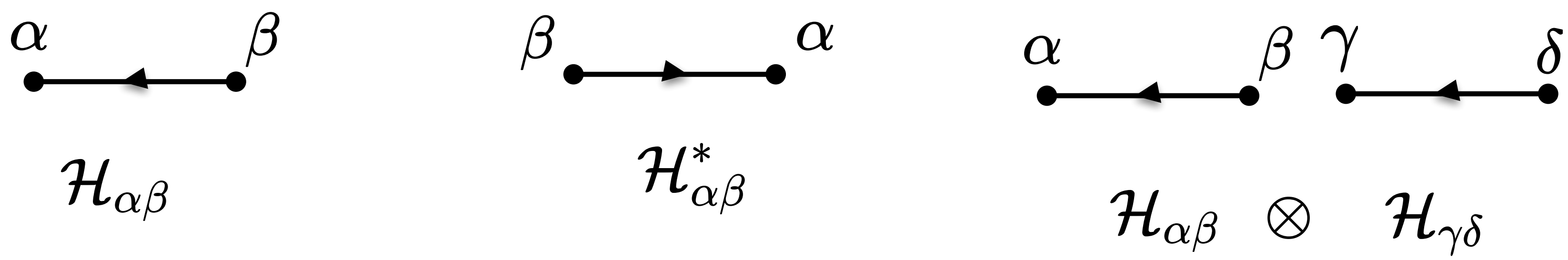
Invariance



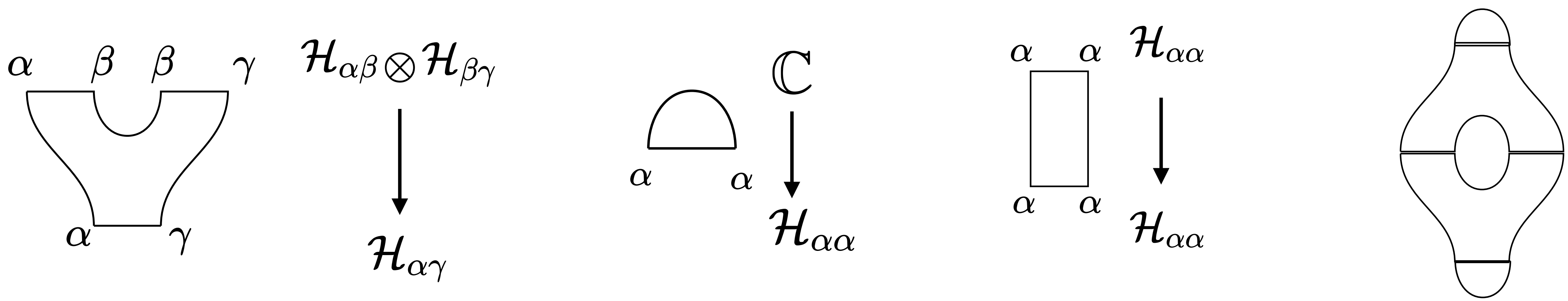
Open TQFT is a symmetric Frobenius Algebra

An open TQFT assigns

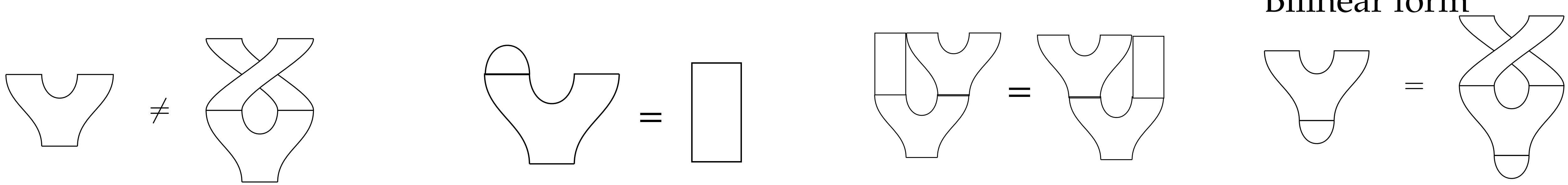
Hilbert space to oriented intervals with boundary conditions :



Open cobordisms to linear maps

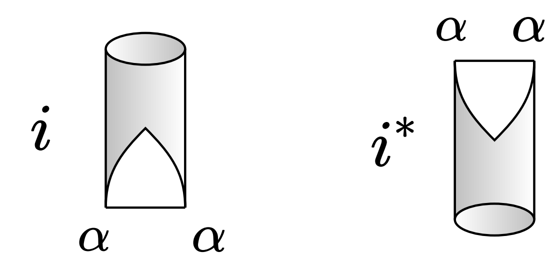


Non-commutative mult. Unit Associative Invariant symmetric Bilinear form

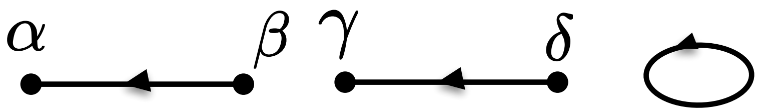


Open closed TQFT

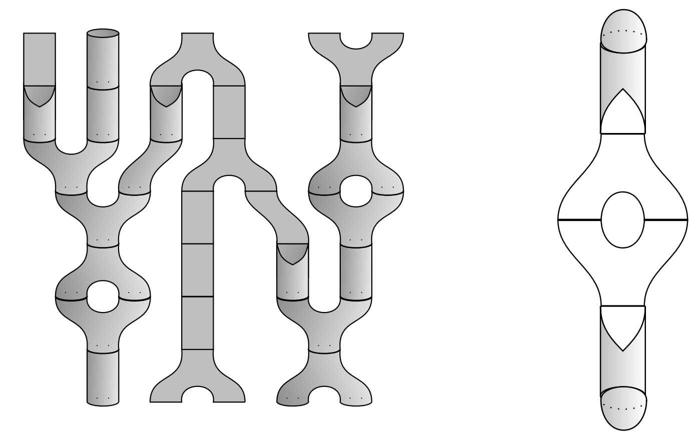
The zipper relates the closed and open algebra...



Open-closed Hilbert spaces and cobordisms:

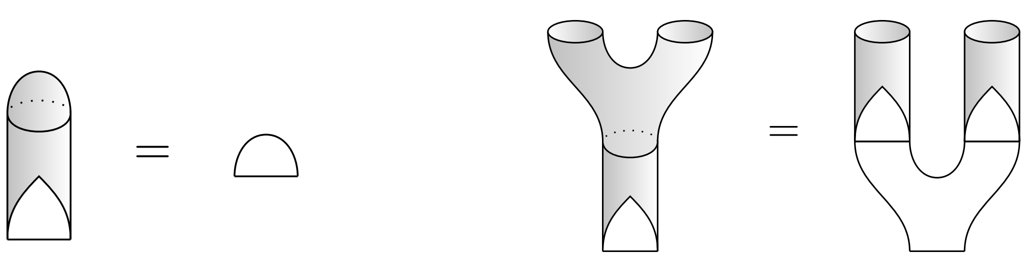


$\mathcal{H}_{\alpha\beta} \otimes \mathcal{H}_{\gamma\delta} \otimes \mathcal{H}_{S_1}$

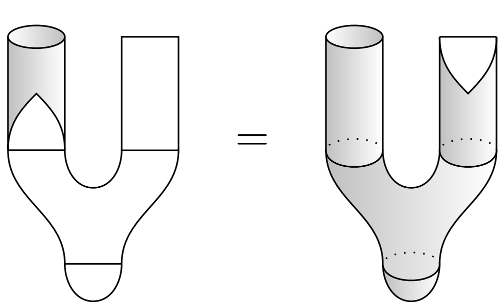


Moore-Segal Sewing rules : Ensures compatibility of gluing

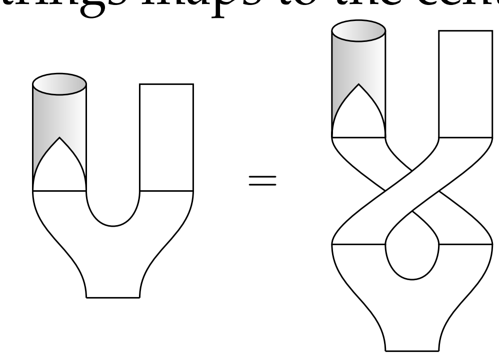
i is an algebra homomorphism



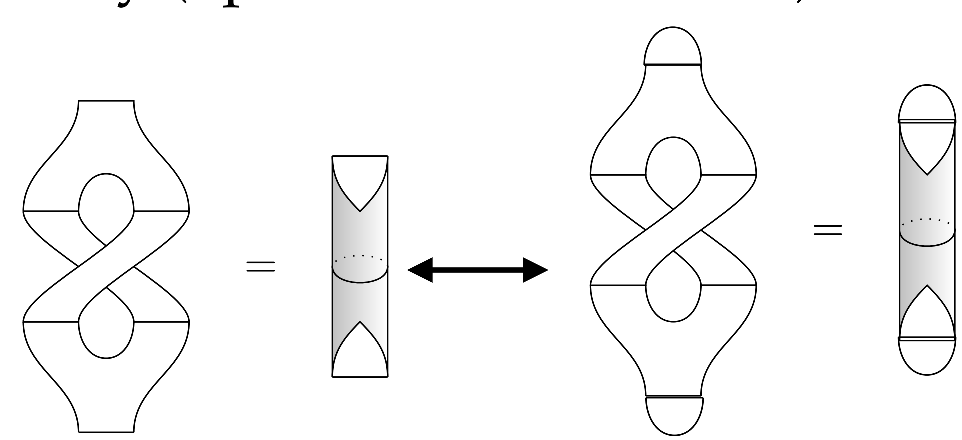
i is the adjoint of i^*



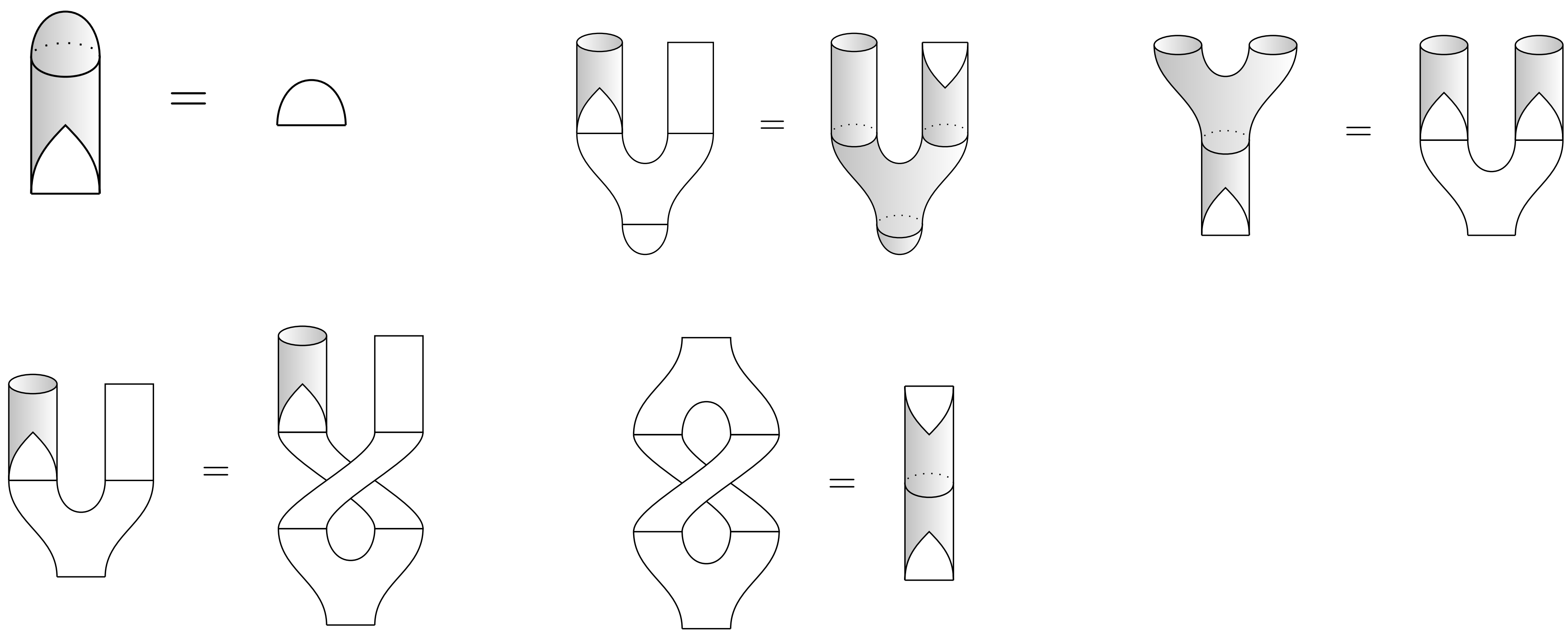
Closed strings maps to the center of open strings



Cardy (Spacetime covariance)



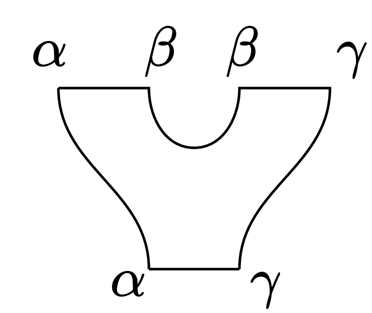
Moore-Segal Sewing relations



Moore-Segal:

Q: Given a closed string theory, what are the possible boundaries, i.e. D Branes?

A: D branes correspond to extensions to an open string algebra satisfying these constraints.



For us: **Open string algebra ~ choice of Hilbert space extension i.e. edge modes**

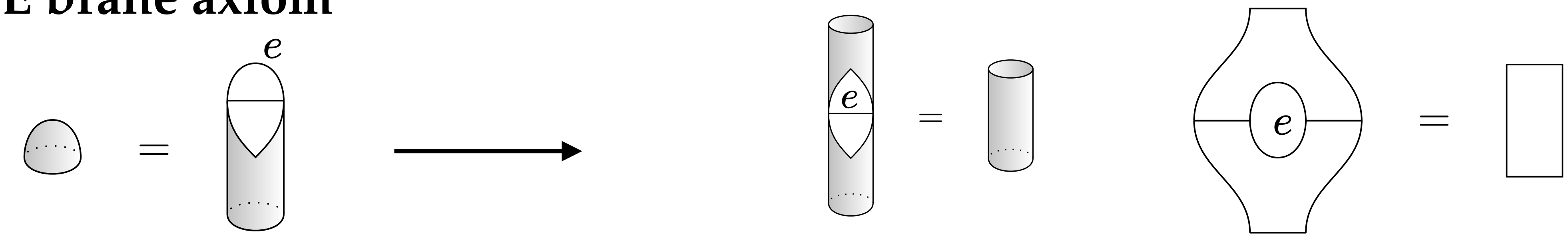
Outline

- Open-closed extended TQFT (Moore-Segal)
- Entanglement brane
- Multi-interval Modular flows,EE
- 2DYM as an open closed TQFT
- Future works: CFT, higher dimensions, holography

The Entanglement Brane axiom

Holes originating from splitting the Hilbert space can be sewed up

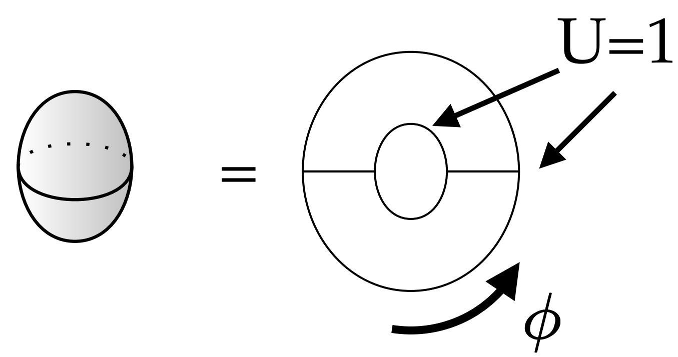
E brane axiom



e= choice of (possibly nonlocal) boundary conditions

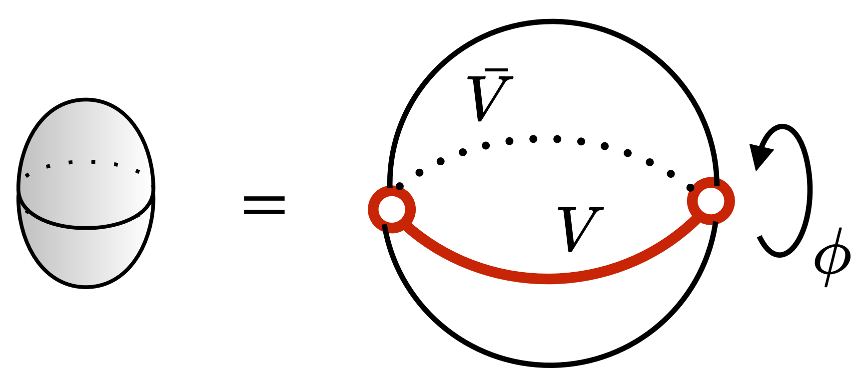
In 2D Yang Mills: e = trivial holonomy along boundary circles
~sum over electric boundary conditions.

$$U = \mathrm{P}\exp(i \int A_x \, dx)$$



Implies correlations are preserved under reduction to V:

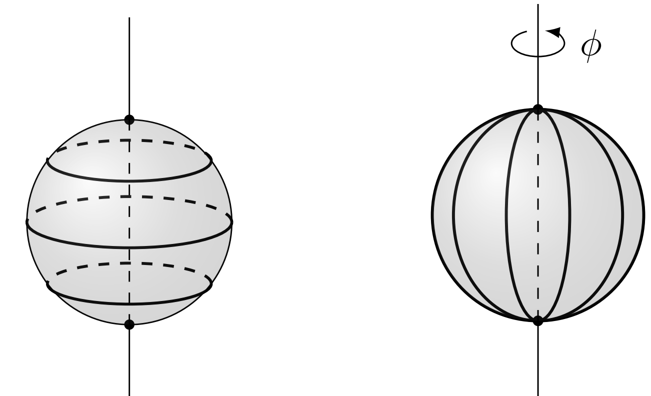
$$\langle O_V \rangle = \mathrm{tr}(\rho_V O_V)$$



The Entanglement Brane in a toy string theory

2D Yang Mills = Closed String theory (Gross-Taylor)

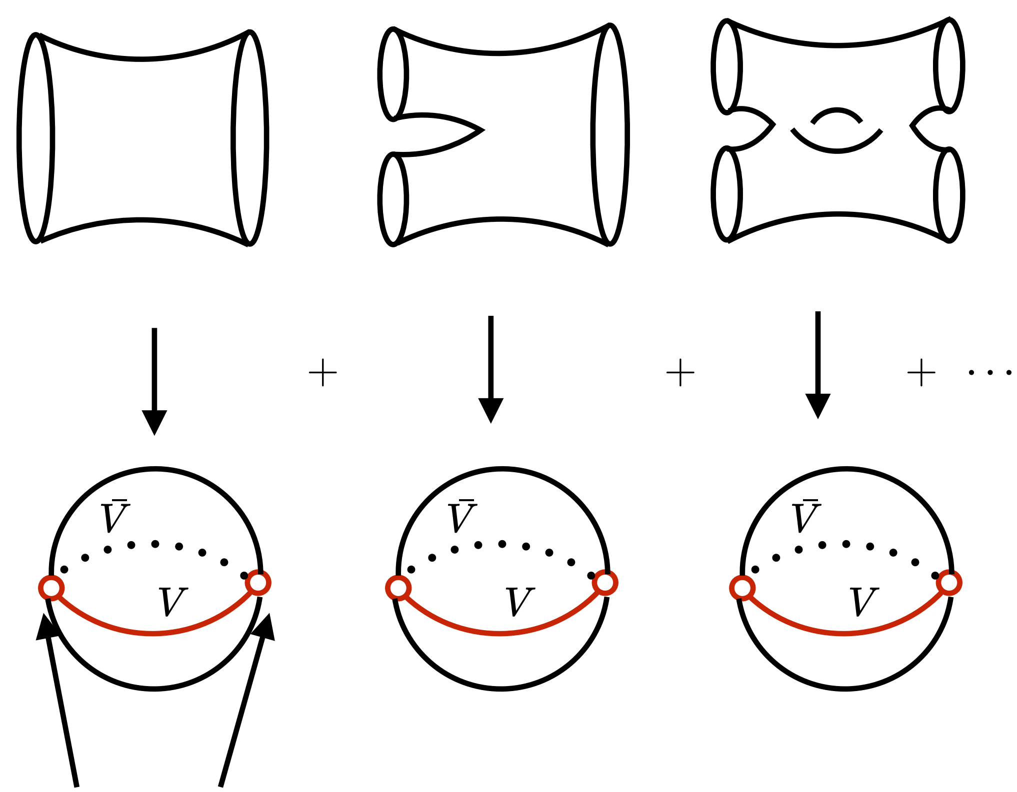
Entanglement brane axiom relates open and closed sector:



c.f. Susskind-Uglum

Cutting a closed string results in N= $\dfrac{1}{g_{\text{string}}}$ **Chan-Paton factors.**

$$Z_{\text{YM}}(S^2) = \text{tr}_V\, \rho_V =$$



N D0-branes

$$N = \frac{1}{g_{\text{string}}}$$

Outline

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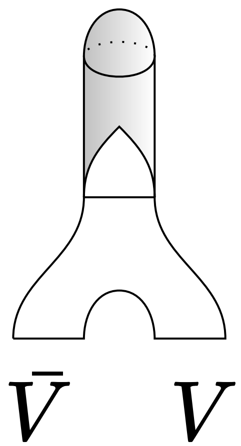
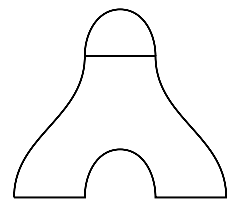
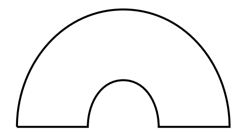
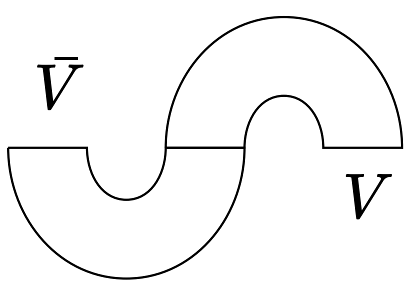
Single interval Modular flow

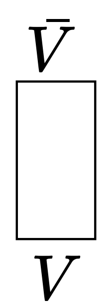
Tensor product factorization

$|\psi\rangle =$ 

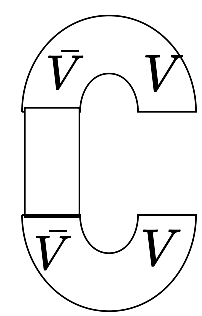
$=$  $=$ 

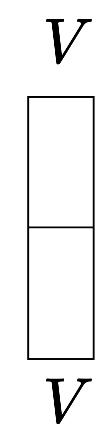
State-Channel duality

$\hat{\psi} =$ 

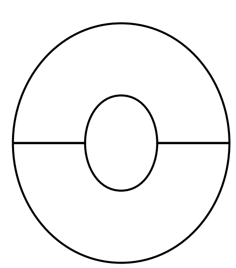
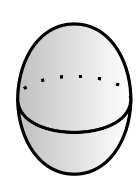
$=$ 

Unnormalized reduced density matrix

$\rho_V =$ 

$= \hat{\psi}\hat{\psi}^\dagger =$ 

Effective partition function

$Z = \mathrm{tr}_V\, \rho_V =$  $=$ 

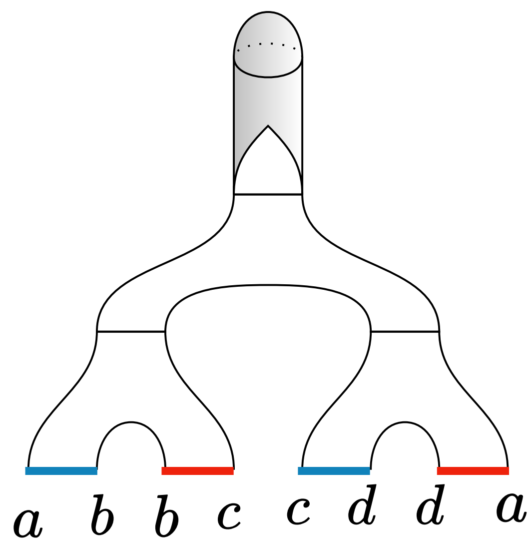
Entanglement entropy

$$S = -\mathrm{tr}_V \frac{\rho_V}{Z} \log \frac{\rho_V}{Z}$$

Multi-interval Modular flow

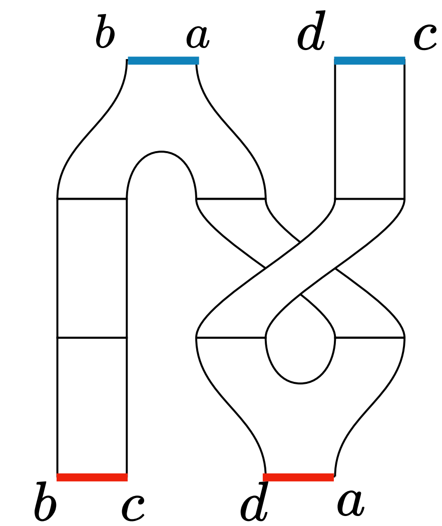
Tensor product factorization

$$|\psi\rangle =$$

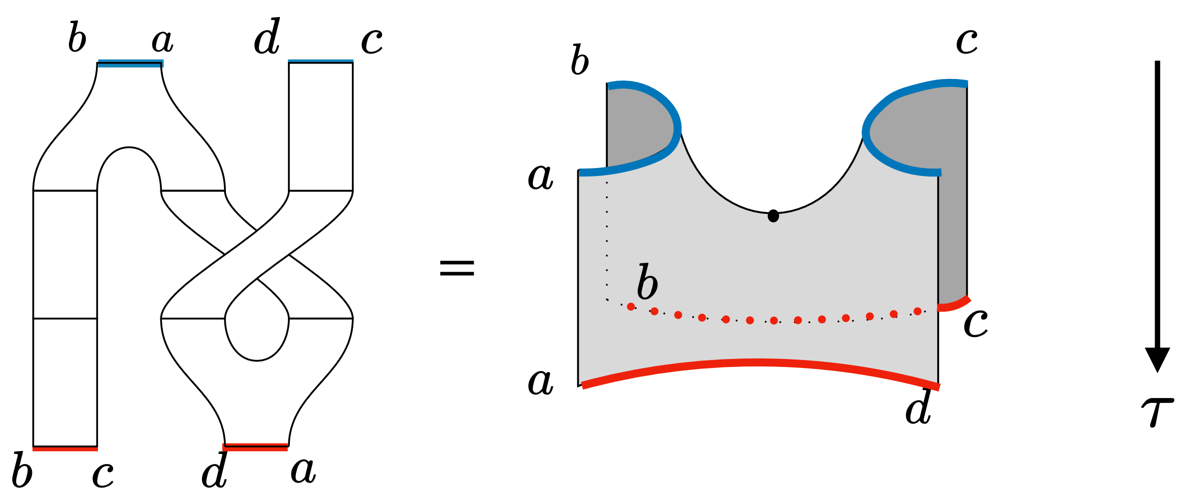


State-Channel duality

$$\psi =$$



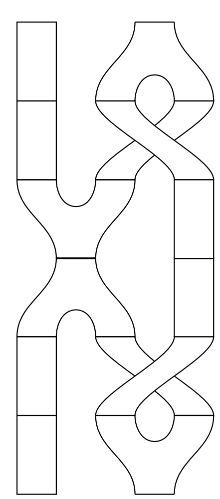
Saddle point



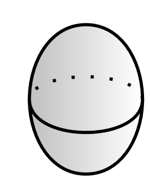
Modular time τ is a morse function !

Reduced density matrix

$$\rho = \psi\psi^{\dagger} =$$



$$Z = \mathrm{tr}\rho =$$



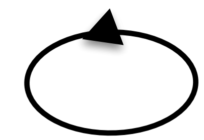
Outline

- Open-closed extended TQFT (Moore-Segal)
- Entanglement brane
- Multi-interval Modular flows, EE, and Negativity
- 2DYM as an open-closed TQFT
- Future works: CFT, higher dimensions, holography

2DYM as a closed TQFT

Configuration space

$$U = \mathcal{P}\exp\left(i\int_0^L dx\, A_x(x)\right) \in G$$



$$x \sim x + L$$

Hilbert space on a circle = Class functions on G

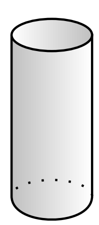
Representation Basis

$$\Psi[U] = \Psi[gUg^{-1}]$$

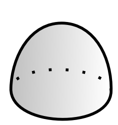
$$\langle U|R\rangle = \mathrm{Tr}_R(U)$$

Hamiltonian ~ $\mathrm{tr}(E^2)$ = Casimir

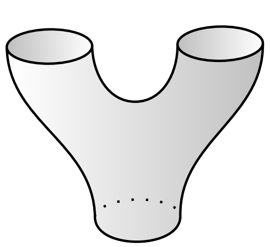
$$H|R\rangle = \frac{g_{\mathrm{YM}}^2 L}{2} C_2(R)|R\rangle$$



$$= \sum_R e^{-AC_2(R)}|R\rangle\langle R|$$



$$= \sum_R \dim R\, e^{-AC_2(R)}|R\rangle$$



$$= \sum_R \frac{1}{\dim R} e^{-AC_2(R)}|R\rangle\langle R|\langle R|$$

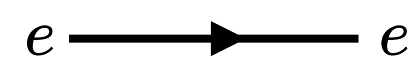
Euler characteristic

$$Z(M) = \sum_R (\dim R)^{\chi(M)} e^{-AC_2(R)}$$

2DYM as an open TQFT

Configuration space

$$U = \mathrm{P}\exp \int_a^b A_x\, dx$$



Hilbert space on an interval

General functions on gauge group G

Basis

$$\langle U | Rab \rangle = \sqrt{\dim R}\; R_{ab}(U)$$

Edge modes $\quad a, b = 1, .. \dim R$

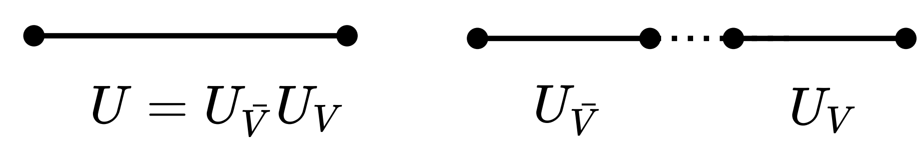
Boundary symmetry :

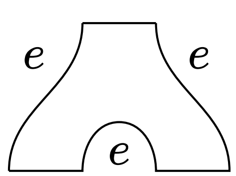
$$U \to g(a) U g^{-1}(b)$$

$$R \to G R G^{-1}$$

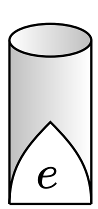
Entangling product = Matrix Multiplication

$$R_{ac}(U_{\bar{V}} U_V) = \sum_b R_{ab}(U_{\bar{V}}) R_{bc}(U_V)$$

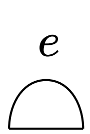




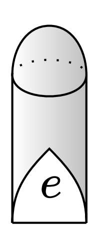
$$= \sum_{R,a,b,c} \frac{e^{-AC_2(R)}}{\sqrt{dimR}} |Rab\rangle |Rbc\rangle \langle Rac|$$



$$= \sum_{R,a} \frac{e^{-AC_2(R)}}{\sqrt{\dim(R)}} |Raa\rangle \langle R|$$



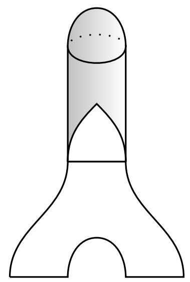
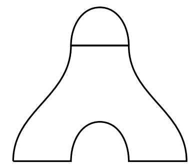
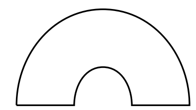
$=$



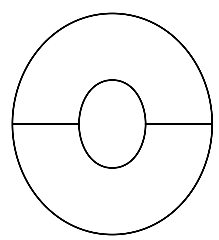
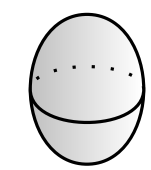
$$= \sum_{R,a} \sqrt{\dim R}\; e^{-AC_2(R)} |Raa\rangle$$

Single interval Modular flow and EE

Tensor product factorization

$|\psi\rangle =$  $=$  $=$  $= \sum_{R,a,b} e^{\frac{-AC_2(R)}{2}} |Rab\rangle |Rba\rangle$

Effective partition function

$Z = \mathrm{tr}_V \, \rho_V =$  $= \sum_{R,a,b} e^{-AC_2(R)} = \sum_R (\dim R)^2 e^{-AC_2(R)} =$ 

Entanglement entropy in terms of $P(R) = \frac{(\dim R)^2 e^{-AC_2(R)}}{Z}$

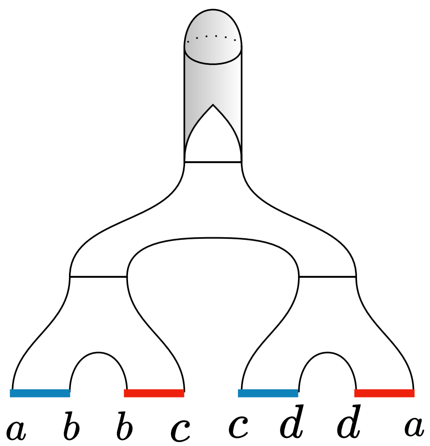
Edge modes

$$S = -\mathrm{tr}_V \frac{\rho_V}{Z} \log \frac{\rho_V}{Z} = \sum_R -P(R) \log P(R) \; +2P(R) \log \dim R$$

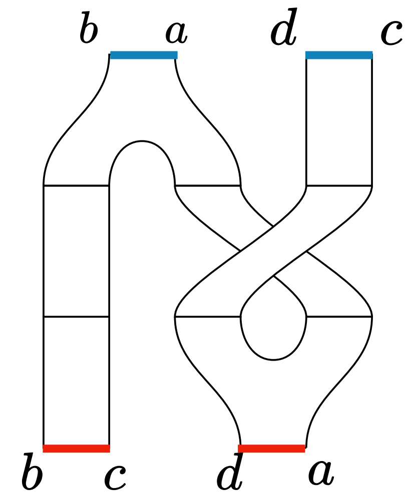
Multi-interval Modular flow

State-Channel duality

$|\psi\rangle =$



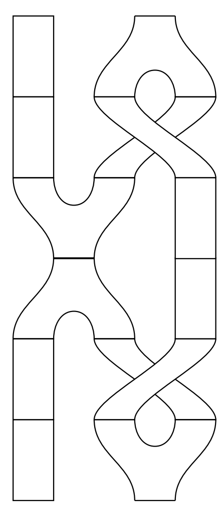
$\psi =$



$$= \sum_{R,a,b,c,d} \frac{e^{\frac{-AC_2(R)}{2}}}{\dim R} |Rbc\rangle |Rcd\rangle \langle Rba| \langle Rdc|$$

Density matrix

$\rho_V = \psi\psi^\dagger =$



$$= \sum_{R,a,b,c,d} \frac{e^{-AC_2(R)}}{(\dim R)^2} |Rbc\rangle |Rda\rangle \langle Rbc| \langle Rda|$$

Entropy

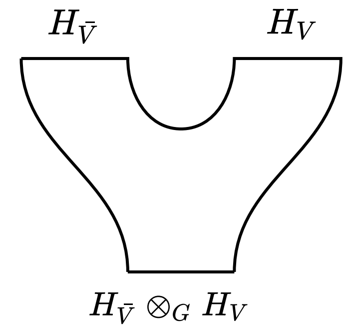
$$P(R) = \frac{(\dim R)^2 e^{-AC_2(R)}}{Z}$$

Number of entangling surfaces

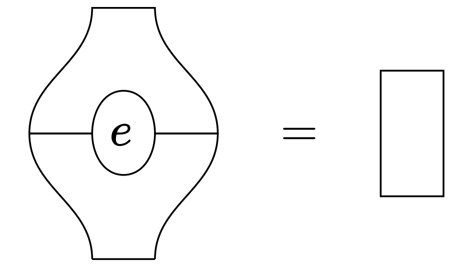
$$S = \sum_R -P(R) \log P(R) \; +4 \sum_R P(R) \log \dim R$$

Summary

- **Entanglement probes the structure of extended QFT**
  e.g. extension defines an open string algebra

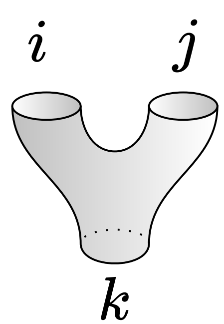


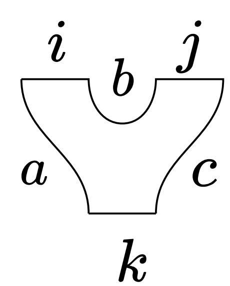
- **The extension satisfies the E-brane axiom**

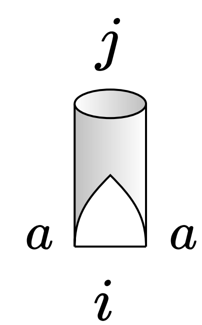


In Progress: Entanglement and Extended CFT

OPE's of a BCFT

$C_{ijk} =$ 

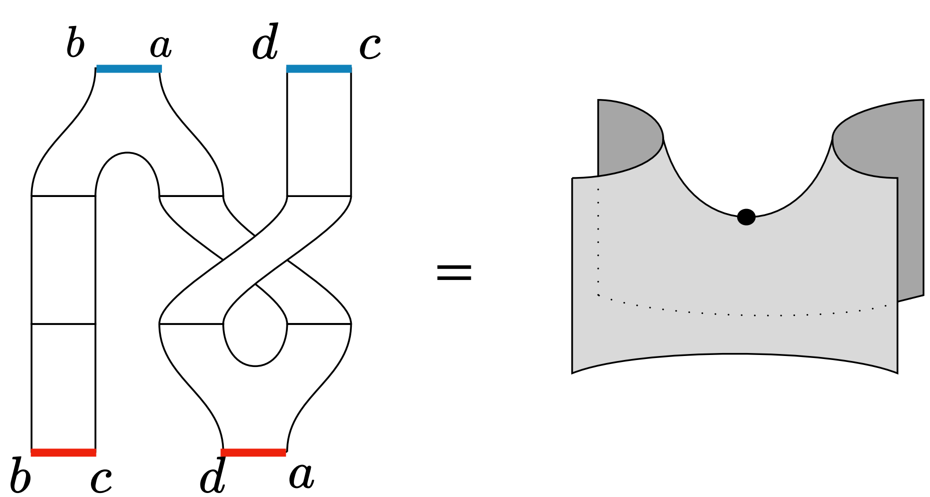
$C^{abc}_{ijk} =$ 

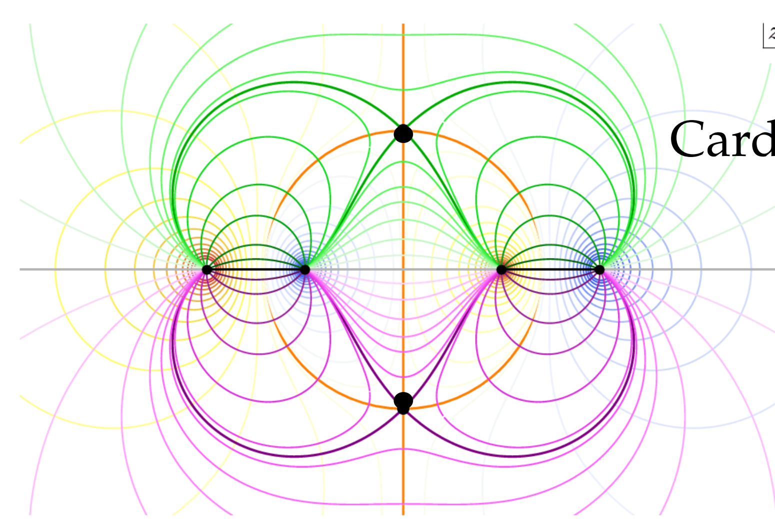
$C^{a}_{ij} =$ 

E brane boundary condition ~ Conformally Inv. BC

Fusion Rule ~ Entangling product ?

A hint from free fermions





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Extra slides