Data-driven Inference, Reconstruction, and Observational Completeness of Quantum Devices

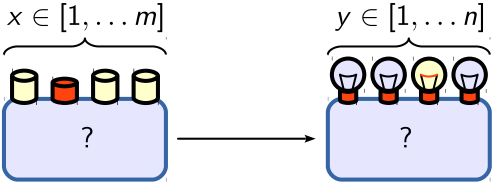
Based on [arXiv:1805.01159, 1812.08470, 1905.04895]

Michele Dall'Arno and Valerio Scarani (CQT, NUS)
Francesco Buscemi (Nagoya University)
Alessandro Bisio and Alessandro Tosini (Pavia University)

QIST 2019
13 June 2019

Informal Introduction

Aim: to make an inference about a quantum device (in this talk, the measurement):



Given:

- a set of data generated by the device (in this talk, the probability distributions $\{\mathbf{p}_x\}$ on the outcomes $\{y\}$, that is $[\mathbf{p}_x]_y := p(y|x)$)
- some prior information about the state space $\mathbb{S}$

Part I: Range Inversion

How much can probability distributions $\{\mathbf{p}_x\}$ tell us about the measurement that generated them?

Measurement Range

Any system of any linear physical theory specifies a **state** space $\mathbb{S}$.

Any **measurement** M acts as a linear transformation from the state space $\mathbb{S}$ to the probability space.

For any state $\rho_x \in \mathbb{S}$, $\mathbf{p}_x := M\rho_x$ is a probability distribution over the outcomes of $M := \{\pi_y\}$.

For example in quantum theory:

$$[\mathbf{p}_x]_y := \mathrm{Tr}\left[\rho\pi_y\right]$$

The **range** $M\mathbb{S}$ of measurement M is the set of all probability distributions that M can generate upon the input of any state.

Range Inversion

Theorem (M. D., Buscemi, Bisio, Tosini, arXiv:1812.08470)

For any informationally complete measurement M, the range $M\mathbb{S}$ identifies M up to gauge symmetries.

Gauge symmetry: any linear transformation L such that $L\mathbb{S} = \mathbb{S}$.

Disclaimer: our results are general, but in this talk we assume informational complete measurements to avoid technicalities.

Range Inversion for Qubit Measurements

Qubit gauge symmetries: (anti)–unitaries.

Corollary (M. D., Brandsen, Buscemi, et al., PRL 118 ('17))

For any informationally complete **qubit** *measurement M, the range $M\mathbb{S}$ identifies M up to (anti)–unitaries.*

For qubits, range inversion identifies the measurement up to a choice of the **reference frame**, which is the optimum achievable in general by a data–driven approach.

Range of qubit measurements

Problem: the previous theorem guarantees that range inversion is possible, but does not provide an explicit construction.

Theorem (M. D., Brandsen, Buscemi, et al., PRL 118 ('17))

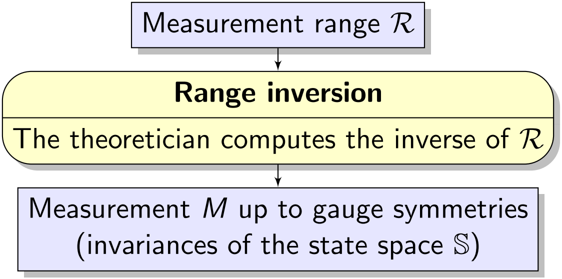
For (hyper)–spherical state space $\mathbb{S}$, the measurement range $M\mathbb{S}$ is the (hyper)–ellipsoid of all probability distributions $\mathbf{p}$ *such that*

$$(\mathbf{p}-\mathbf{t})^T Q^+(\mathbf{p}-\mathbf{t}) \leq 1, \quad and \quad (\mathbb{1} - QQ^+)(\mathbf{p}-\mathbf{t}) = 0,$$

where

$$Q_{x,y} = \frac{1}{2}\operatorname{Tr}[\pi_x \pi_y] - \frac{1}{4}\operatorname{Tr}[\pi_x]\operatorname{Tr}[\pi_y], \quad and \quad t_x = \frac{1}{2}\operatorname{Tr}[\pi_x].$$

Algorithmic Representation of Range Inversion



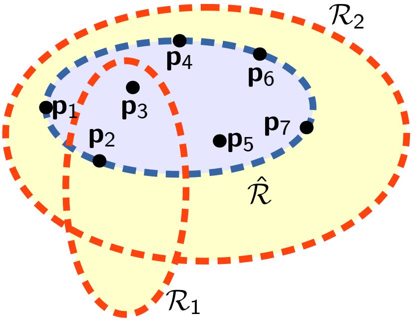
Part II: Data–driven Inference

Given a set $\{\mathbf{p}_x\}$ of probability distributions, what measurement range should be inferred?

Example: bit and qubit systems

Prior information about state-space: $\mathbb{S}$ is a (hyper)–sphere

Observed distributions: $\{\mathbf{p}_x\}$.



Range $\mathcal{R}_1$: not compatible with $\{\mathbf{p}_x\}$

Range $\mathcal{R}_2$: compatible with $\{\mathbf{p}_x\}$, but not minimum volume

Range $\hat{\mathcal{R}}$: compatible with $\{\mathbf{p}_x\}$ and minimum volume

Data-driven Inference

Why **volume**? The inference must not change under linear transformations, and that is all we care about in linear theories.

Why **minimal**? We want to infer the measurement that explains the probability distributions $\{\mathbf{p}_x\}$ and as little else as possible.

For given state space $\mathbb{S}$, what is the **least committal** measurement $\hat{M}$ **consistent** with a given set $\{\mathbf{p}_x\}$ of probability distributions?

Definition (Data-driven inference of measurements)

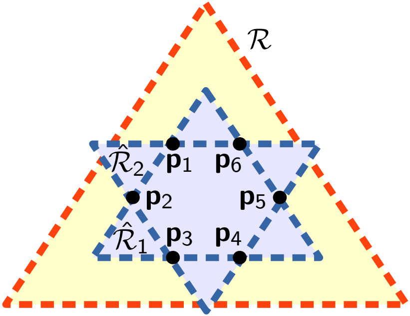
We introduce ddi *as the algorithm that, upon the input of* $\{\mathbf{p}_x\}$*, outputs the least committal measurement range* $\hat{M}\mathbb{S}$ *compatible with* $\{\mathbf{p}_x\}$*, that is*

$$\mathrm{ddi}\left(\{\mathbf{p}_x\}|\mathbb{S}\right) := \underset{\{\mathbf{p}_x\}\subseteq M\mathbb{S}}{\underset{M}{\mathrm{argmin}}}\ \mathrm{vol}\left(M\mathbb{S}\right)\mathbb{S}.$$

Example: classical trit system

Prior information about state-space: $\mathbb{S}$ is a simplex

Observed distributions: $\{\mathbf{p}_x\}$.



Range $\mathcal{R}$: compatible with $\{\mathbf{p}_x\}$, but not minimum volume

Ranges $\hat{\mathcal{R}}_1$ and $\hat{\mathcal{R}}_1$: compatible with $\{\mathbf{p}_x\}$ and minimum volume

Uniqueness of Data-Driven Inference

Theorem (M. D., Buscemi, Bisio, Tosini, arXiv:1812.08470)

The output of $\mathrm{ddi}(\{\mathbf{p}_x\}|\mathbb{S})$ *is a singleton for any set* $\{\mathbf{p}_x\}$ *of probability distributions if and only if* $\mathbb{S}$ *is a (hyper)-sphere.*

Foundational digression: if we postulate the uniqueness of the inference as an epistemic principle, we recover the elementary systems of classical and quantum theories.

This rules out commonly considered general probabilistic theories, such as the sqit (or square bit), for which the state space is a square.

Machine Learning of Quantum Measurements

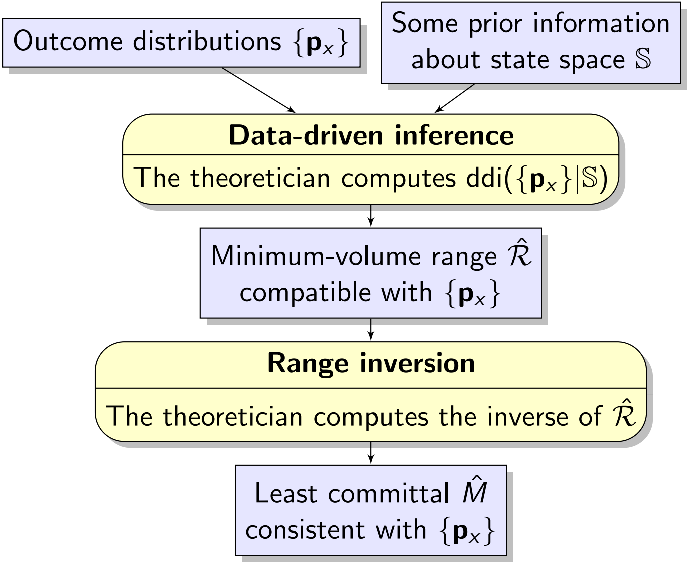
Data–driven inference can be regarded as an algorithm for the machine learning of quantum measurements.

Theorem (M.D., Ho, Buscemi, Scarani, arXiv:1905.04895)

For (hyper)–spherical state space $\mathbb{S}$*, the map* ddi *represents a convex programming problem.*

Connection with John's theory of minimum volume enclosing ellipsoids (MVEE).

Algorithmic Representation of Data-Driven Inference



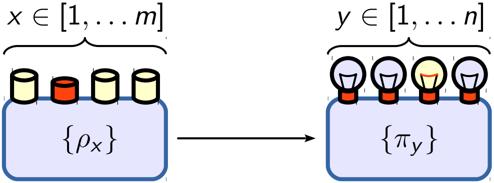
Part III: Observational Completeness

What set $\mathcal{S}$ of states should be fed into measurement M if we require that data–driven inference returns the range of measurement M?

Data-Driven Reconstruction

Suppose the set $\{\mathbf{p}_x\}$ of probability distributions is generated by asympthotically many runs of the following experiment:

1. upon the input of x, preparing a state $\rho_x \in \mathbb{S}$ of a system with state space $\mathbb{S}$
2. measuring it with a measurement M and collecting input y



For example, in quantum theory $[\mathbf{p}_x]_y = \mathrm{Tr}\left[\rho_x \pi_y\right]$.

Observational completeness of states

Data–driven inference returns the **least committal** range, which may or may not coincide with the range of the actual measurement used in the experiment.

What set of states $\mathcal{S} := \{\rho_x\}$ should be prepared in order for the data-driven inference of $\{\mathbf{p}_x\}$ to coincide with the range $M\mathbb{S}$?

Definition (Observational completeness)

A set $\mathcal{S}$ of states is **observationally complete** *(OC) for measurement M if and only if applying the map* ddi *to the set $M\mathcal{S}$ of probability distributions returns the range $M\mathbb{S}$, that is*

$$\mathrm{ddi}\left(M\mathcal{S}|\mathbb{S}\right) = \left\{M\mathbb{S}\right\}.$$

Characterization of Observational Completeness

Remark: the observational completeness of a set $\mathcal{S}$ of states depends on measurement M.

Problem: can we have a measurement independent characterization of observationally complete sets of states?

Theorem (M.D., Buscemi, Bisio, Tosini, arXiv:1812.08470)

A set $\mathcal{S}$ of states is **observationally complete** *for any informationally complete measurement if and only if the minimum volume linear transformation of the state space $\mathbb{S}$ that encloses $\mathcal{S}$ is the state space $\mathbb{S}$ itself, that is*

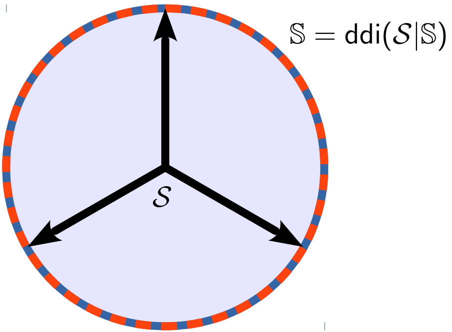
$$\mathrm{ddi}\left(\mathcal{S}|\mathbb{S}\right) = \left\{\mathbb{S}\right\}.$$

Remark: the map ddi has been extended by linearity from the probability space to the entire linear space.

Example: Observationally Complete Set of Qubit States

Prior information about state–space: $\mathbb{S}$ is a (hyper)-sphere

Set $\mathcal{S}$ of states: regular simplex



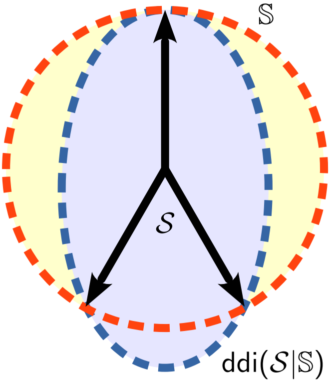
$\mathcal{S}$ is OC since the state–space $\mathbb{S}$ coincides with $\mathrm{ddi}(\mathcal{S}|\mathbb{S})$, and

$\mathcal{S}$ is IC since its linear span is the entire linear space

Example: Observationally Incomplete Set of Qubit States

Prior information about state–space: $\mathbb{S}$ is a (hyper)-sphere

Set $\mathcal{S}$ of states: irregular simplex



$\mathcal{S}$ is not OC since the state–space $\mathbb{S}$ differs from ddi($\mathcal{S}|\mathbb{S}$), but

$\mathcal{S}$ is IC since its linear span is the entire linear space

Informational Completeness

A set $\mathcal{S}$ of states is **informationally complete** if and only if it allows for the tomographic reconstruction of any measurement, that is, its linear span is the entire linear space.

Corollary (M.D., Buscemi, Bisio, Tosini, arXiv:1812.08470)

Informational completenes is **necessary** *but* **not sufficient** *for a set of states $\mathcal{S}$ to be observationally complete for any informationally complete measurement.*

Characterization of Observational Completeness for Qubit

Theorem (M.D., Ho, Buscemi, Scarani, arXiv:1905.04895)

A set $\mathcal{S}$ of **qubit** *states is* **observationally complete** *for any informationally complete measurement if and only if $\mathcal{S}$ support a spherical 2-design.*

A set $\mathcal{S} := \{\rho_x\}$ of states supports a **spherical t-design** if and only if there exists a probability distribution **p** such that the ensemble $\{p_x, \rho_x\}$ is undistinguishable from the uniform distribution on the sphere given t copies, that is

$$\sum_x p_x \rho_x^{\otimes t} = \int \psi^{\otimes t} d\psi,$$

where $d\psi$ is the uniform measure on the sphere.

SIC and MUBS as Minimal Cardinality OCs

A set $\mathcal{S}$ of states is **symmetric informationally complete** if and only if it is a **regular symplex**.

Corollary (M.D., Buscemi, Bisio, Tosini, arXiv:1812.08470)

For (hyper)-spherical state space $\mathbb{S}$*, the* **unique** *minimal cardinality OC set of states is the* **symmetric informationally complete** *set.*

A **qubit** set $\mathcal{S}$ of states is a **mutual unbiased bases** if and only if it is a **regular octahedron**.

Corollary (M.D., Ho, Buscemi, Scarani, arXiv:1905.04895)

For qubit state space $\mathbb{S}$*, the* **unique** *minimal cardinality OC set of* **bases** *are the* **mutually unbiased bases**.

Algorithmic Representation of Data-Driven Reconstruction

Measurement M

↓

Observational completeness

The experimentalist feeds in M a set $\mathcal{S}$ of states which is OC for M

↓

Outcome distributions $\{\mathbf{p}_x\} = M\mathcal{S}$

↓

Data-driven inference

The theoretician computes $\mathrm{ddi}(\{\mathbf{p}_x\}|\mathbb{S})$

↓

Minimum-volume range $\hat{\mathcal{R}}$ compatible with $\{\mathbf{p}_x\}$

↓

Range inversion

The theoretician computes the inverse of $\hat{\mathcal{R}}$

↓

Measurement M up to gauge symmetries

Summary

1. **Range inversion** The measurement range $M\mathbb{S}$ identifies measurement M up to **gauge symmetries**
   - For qubit systems, gauge symmetries are (anti)–unitaries
   - Closed–form range inversion for (hyper)–spherical state space
2. **Data–driven inference** Given a set $\{\mathbf{p}_x\}$ of probability distributions, outputs the **least committal** range **consistent** with $\{\mathbf{p}_x\}$
   - Unique if and only if the state space is (hyper)–spherical
   - Convex programming for (hyper)–spherical state space
3. **Observational completeness** A set $\mathcal{S}$ of states produces $M\mathbb{S}$ if and only if it is observationally complete.
   - Characterization of observational completeness
   - Closed–form characterization in terms of spherical 2-designs for (hyper)–spherical state space

References

On the inference of measurements: M. D. , F. Buscemi, A. Bisio, and A. Tosini, *"Data-Driven Inference, Reconstruction, and Observational Completeness of Quantum Devices"*, arXiv:1812.08470.

On the inference of channels: F. Buscemi and M. D., *"Data-Driven Inference of Physical Devices: Theory and Implementation"*, arXiv:1805.01159.

On the inference of states: M. Dall'Arno, A. Ho, F. Buscemi, and V. Scarani, *"Data-driven inference and observational completeness of quantum devices"*, arXiv:1905.04895.

On measurement ranges: M. D., S. Brandsen, F. Buscemi, et al., Phys. Rev. Lett. **118**, 250501 ('17).