Unitary designs from statistical mechanics in random quantum circuits

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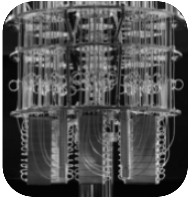
Based on: NHJ, 1905.12053

Random quantum circuits are efficient implementations of randomness and are a solvable model of chaotic dynamics.

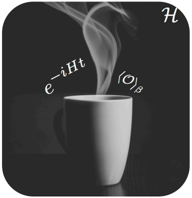
As such, RQCs are a valuable resource in quantum information:

Decoupling

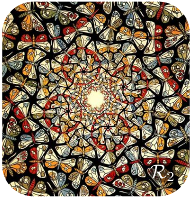
Randomness


Quantum advantage

and in quantum many-body physics:



Thermalization

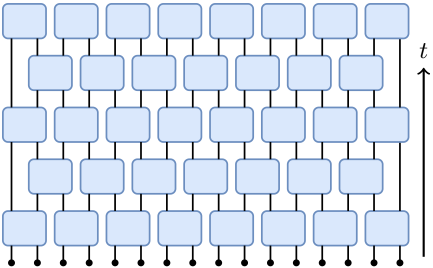


Quantum chaos

Transport

Random quantum circuits

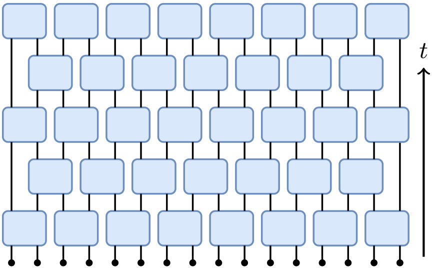
Consider local RQCs on n qudits of local dimension q, evolved with staggered layers of 2-site unitaries, each drawn randomly from $U(q^2)$



where evolution to time t is given by $U_t = U^{(t)} \dots U^{(1)}$

Our goal

Study the convergence of random quantum circuits to **unitary k-designs**



where we start approximating moments of the unitary group

Unitary k-designs

Haar: (unique L/R invariant) measure on the unitary group $U(d)$

For an ensemble of unitaries $\mathcal{E}$, the k-fold channel of an operator $\mathcal{O}$ acting on $\mathcal{H}^{\otimes k}$ is

$$\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) \equiv \int_{\mathcal{E}} dU \, U^{\otimes k}(\mathcal{O}) U^{\dagger \otimes k}$$

An ensemble of unitaries $\mathcal{E}$ is an exact k-design if

$$\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) = \Phi_{\mathrm{Haar}}^{(k)}(\mathcal{O})$$

e.g. $k=1$ and Paulis, $k=2,3$ and the Clifford group

Unitary k-designs

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exact k-design: $\Phi_{\mathcal{E}}^{(k)}(\mathcal{O}) = \Phi_{\text{Haar}}^{(k)}(\mathcal{O})$

but for general k, few exact constructions are known

Definition (Approximate k-design)

For $\epsilon > 0$, an ensemble $\mathcal{E}$ is an ϵ-approximate k-design if the k-fold channel obeys

$$\left\| \Phi_{\mathcal{E}}^{(k)} - \Phi_{\text{Haar}}^{(k)} \right\|_{\diamond} \leq \epsilon$$

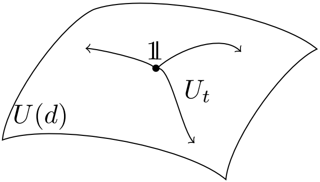
$\rightarrow$ designs are powerful

Intuition for k-designs

(eschewing rigor)

How random is the time-evolution of a system compared to the full unitary group $U(d)$?

Consider an ensemble of time-evolutions at a fixed time t: $\mathcal{E}_t = \{U_t\}$
e.g. RQCs, Brownian circuits, or $\{e^{-iHt},\ H \in \mathcal{E}_H\}$ generated by disordered Hamiltonians



quantify randomness:
when does $\mathcal{E}_t$ form a k-design?
(approximating moments of $U(d)$)

Previous results

RQCs form approximate unitary k-designs

- Harrow, Low ('08): RQCs form 2-designs in $O(n^2)$ steps
- Brandão, Harrow, Horodecki ('12): RQCs form approximate k-designs in $O(nk^{10})$ depth

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Moreover, a lower bound on the k-design depth is $O(nk)$

Furthermore,

- [Harrow, Mehraban] showed higher-dimensional RQCs form k-designs in $O(n^{1/D}\mathrm{poly}(k))$ depth
- [Nakata, Hirche, Koashi, Winter] considered a random (time-dep) Hamiltonian evolution, forms k-designs in $O(n^2 k)$ steps up to $k = o(\sqrt{n})$

as well as many other papers studying the convergence properties of RQCs:
[Emerson, Livine, Lloyd], [Oliveira, Dahlsten, Plenio], [Žnidarič], [Brown, Viola], [Brandão, Horodecki], [Brown, Fawzi], [Ćwikliński, Horodecki, Mozrzymas, Pankowski, Studziński]

Frame potential

The frame potential is a more tractable measure of Haar randomness, where the k-th frame potential for an ensemble $\mathcal{E}$ is defined as [Gross, Audenaert, Eisert], [Scott]

$$\mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V \in \mathcal{E}} dU dV \left| \mathrm{Tr}(U^\dagger V) \right|^{2k}$$

(2-norm distance to Haar-randomness)

k-th frame potential for the Haar ensemble: $\mathcal{F}_{\mathrm{Haar}}^{(k)} = k!$ for $k \leq d$

For any ensemble $\mathcal{E}$, the frame potential is lower bounded as

$$\mathcal{F}_{\mathcal{E}}^{(k)} \geq \mathcal{F}_{\mathrm{Haar}}^{(k)} ,$$

with $=$ if and only if $\mathcal{E}$ is a k-design

Frame potential

k-th frame potential : $\mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V\in\mathcal{E}} dU\,dV \left|\mathrm{Tr}(U^\dagger V)\right|^{2k}$

where: $\mathcal{F}_{\mathcal{E}}^{(k)} \geq \mathcal{F}_{\mathrm{Haar}}^{(k)}$ and $\mathcal{F}_{\mathrm{Haar}}^{(k)} = k!$ (for $k \leq d$)

Related to ϵ-approximate k-design as

$$\boxed{\left\|\Phi_{\mathcal{E}}^{(k)} - \Phi_{\mathrm{Haar}}^{(k)}\right\|_{\diamond}^{2} \leq d^{2k}\left(\mathcal{F}_{\mathcal{E}}^{(k)} - \mathcal{F}_{\mathrm{Haar}}^{(k)}\right)}$$

Frame potential

k-th frame potential : $\quad \mathcal{F}_{\mathcal{E}}^{(k)} = \int_{U,V\in\mathcal{E}} dU\,dV \left|\mathrm{Tr}(U^\dagger V)\right|^{2k}$

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Related to ϵ-approximate k-design as

$$\left\|\Phi_{\mathcal{E}}^{(k)} - \Phi_{\mathrm{Haar}}^{(k)}\right\|_\diamond^2 \leq d^{2k}\left(\mathcal{F}_{\mathcal{E}}^{(k)} - \mathcal{F}_{\mathrm{Haar}}^{(k)}\right)$$

The frame potential has recently become understood as a diagnostic of quantum chaos [Roberts, Yoshida], [Cotler, NHJ, Liu, Yoshida], ...

Our approach

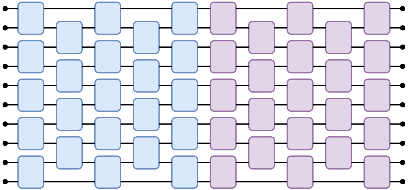
- Focus on 2-norm and analytically compute the frame potential for random quantum circuits
- Making use of the ideas in [Nahum, Vijay, Haah], [Zhou, Nahum], we can write the frame potential as a lattice partition function
- We can compute the $k = 2$ frame potential exactly, but for general k we must sacrifice some precision
- We'll see that the decay to Haar-randomness can be understood in terms of domain walls in the lattice model

Frame potential for RQCs

The goal is to compute the FP for RQCs evolved to time t:

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \int_{U_t, V_t \in \mathrm{RQC}} dU\, dV \left|\mathrm{Tr}(U_t^\dagger V_t)\right|^{2k}$$

Consider one $U_t^\dagger V_t$:

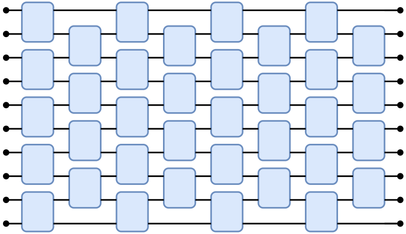


Frame potential for RQCs

The goal is to compute the frame potential for RQCs:

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \int dU \left|\mathrm{Tr}(U_{2(t-1)})\right|^{2k}$$

simply moments of traces of RQCs, with depth $2(t-1)$



Haar integrating

Recall how to integrate over monomials of random unitaries.
For the k-th moment [Collins], [Collins, Śniady]

$$\int dU\, U_{i_1 j_1} \dots U_{i_k j_k} U^\dagger_{\ell_1 m_1} \dots U^\dagger_{\ell_k m_k} \\ = \sum_{\sigma,\tau \in S_k} \delta_\sigma(\vec{\imath}|\vec{m}) \delta_\tau(\vec{\jmath}|\vec{\ell}) \mathcal{W}g^U(\sigma^{-1}\tau, d)\,,$$

where

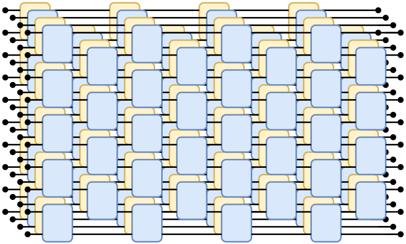
$$\delta_\sigma(\vec{\imath}|\vec{\jmath}) = \delta_{i_1 j_{\sigma(1)}} \dots \delta_{i_k j_{\sigma(k)}}$$

and where $\mathcal{W}g(\sigma, d)$ is the unitary Weingarten function.

Lattice mappings for RQCs

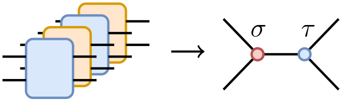
[Nahum, Vijay, Haah], [Zhou, Nahum]

Consider the k-th moments of RQCs, k copies of the circuit and its conjugate:



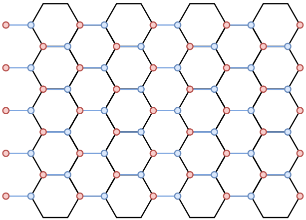
Lattice mappings for RQCs

Haar averaging the 2-site unitaries gives



where we sum over $\sigma, \tau \in S_k$. The frame potential is then

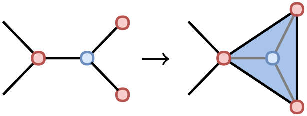
$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = \sum_{\{\sigma,\tau\}}$$

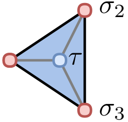


with pbc in time, where the diagonal lines are index contractions between gates, given as the inner product of permutations $\langle \sigma | \tau \rangle = q^{\ell(\sigma^{-1}\tau)}$, and the horizontal lines are $\mathcal{W}g(\sigma^{-1}\tau, q^2)$.

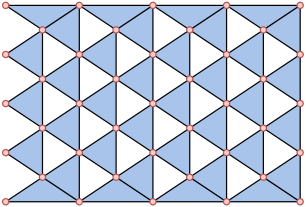
Lattice mappings for RQCs

An additional simplification occurs when we sum over all the blue nodes, defining an effective plaquette term



where $J^{\sigma_1}_{\sigma_2\sigma_3} \equiv \sum_{\tau \in S_k}$ 

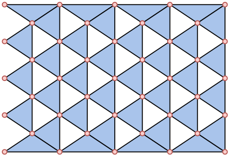
The frame potential is then a partition function on a triangular lattice

$\mathcal{F}^{(k)}_{\mathrm{RQC}} = \sum_{\{\sigma\}}$ 

Frame potential as a partition function

The result is then that we can write the k-th frame potential as

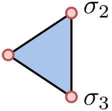
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \sum_{\{\sigma\}} \prod_{\triangleleft} J_{\sigma_2\sigma_3}^{\sigma_1} = \sum_{\{\sigma\}}$$



of width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, with pbc in time.

The plaquettes are functions of three $\sigma \in S_k$, written explicitly as

$$J_{\sigma_2\sigma_3}^{\sigma_1} = \sigma_1$$



$$= \sum_{\tau \in S_k} \mathcal{W}g(\sigma_1^{-1}\tau, q^2) q^{\ell(\tau^{-1}\sigma_2)} q^{\ell(\tau^{-1}\sigma_3)} .$$

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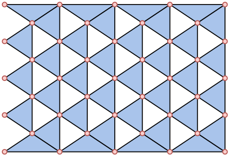


of width $n_g = \lfloor n/2 \rfloor$, depth $2(t-1)$, with pbc in time.

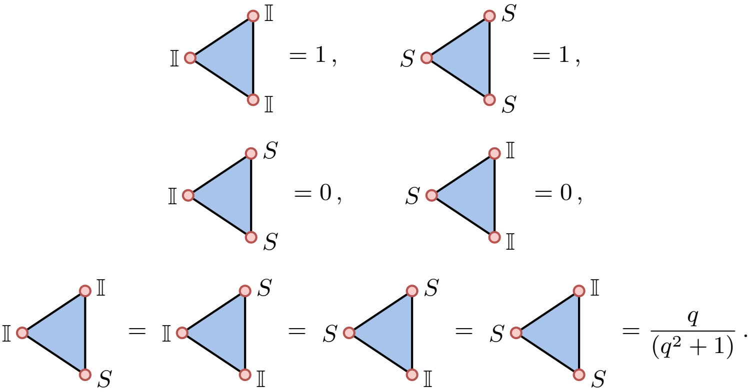
We can show that $J^{\sigma}_{\sigma\sigma} = 1$, and thus the minimal Haar value of the frame potential comes from the $k!$ ground states of the lattice model

$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = k! + \ldots$$

Also, for $k = 1$ we have $\mathcal{F}^{(1)}_{\mathrm{RQC}} = 1$, RQCs form exact 1-designs.

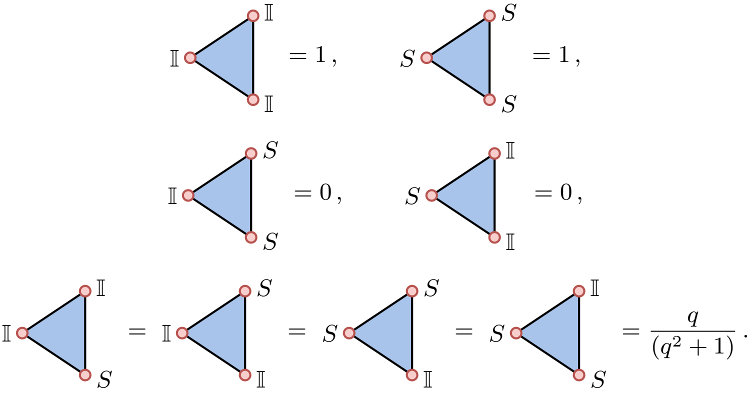
$k = 2$ plaquette terms

For $k = 2$, where the local degrees of freedom are $\sigma \in S_2 = \{\mathbb{I}, S\}$, the plaquettes terms $J^{\sigma_1}_{\sigma_2\sigma_3}$ are simple to compute



$k = 2$ plaquette terms

we can interpret these in terms of domain walls separating regions of $\mathbb{I}$ and S spins



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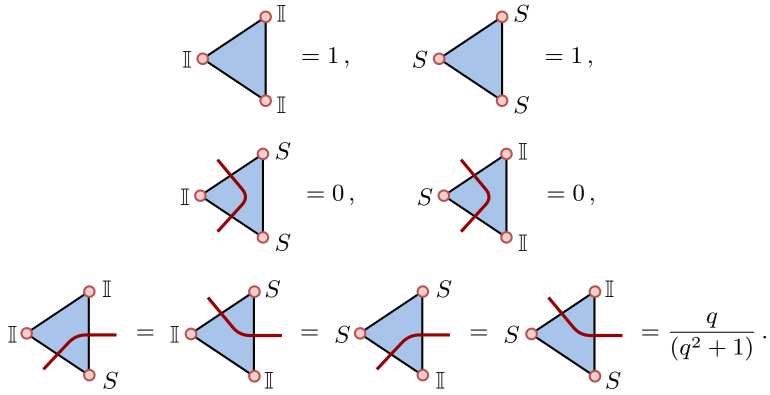
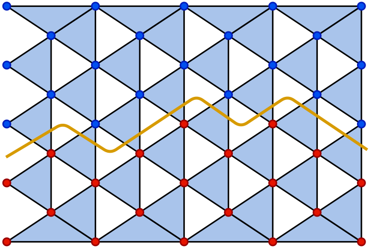


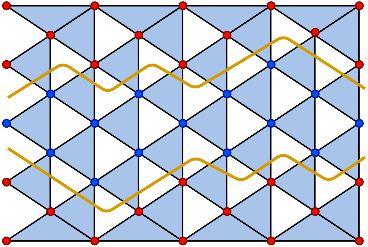
$k = 2$ domain walls

all non-zero contributions to $\mathcal{F}^{(2)}_{\mathrm{RQC}}$ are domain walls (which must wrap the circuit)

a single domain wall configuration:



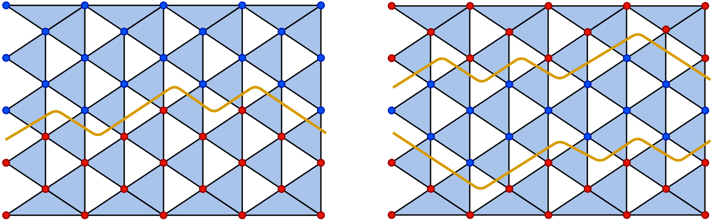
a double domain wall configuration:



2-designs from domain walls

To compute the 2-design time, we simply need to count the domain wall configurations

$$\mathcal{F}^{(2)}_{\mathrm{RQC}} = 2\left(1 + \sum_{1\ \mathrm{dw}} wt(q,t) + \sum_{2\ \mathrm{dw}} wt(q,t) + \dots\right)$$



2-designs from domain walls

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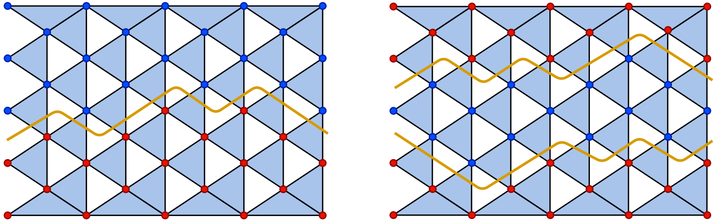
$$\mathcal{F}^{(2)}_{\mathrm{RQC}} = 2\left(1+c_1(n,t)\left(\frac{q}{q^2+1}\right)^{2(t-1)}+c_2(n,t)\left(\frac{q}{q^2+1}\right)^{4(t-1)}+\dots\right)$$



2-designs from domain walls

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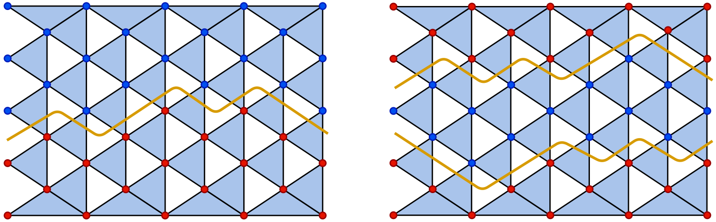
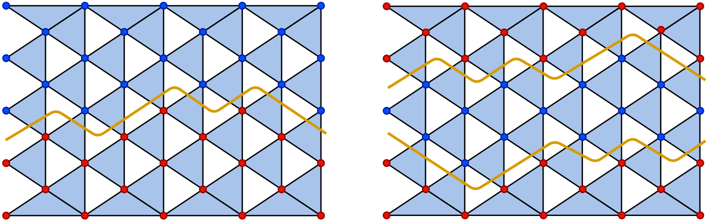
$$\mathcal{F}^{(2)}_{\mathrm{RQC}} \leq 2\left(1+\left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right)^{n_g-1}$$



2-designs from domain walls

To compute the 2-design time, we simply need to count the domain wall configurations

$$\mathcal{F}_{\mathrm{RQC}}^{(2)} = 2\left(1 + \sum_{p} c_p(n,t)\left(\frac{q}{q^2+1}\right)^{2p(t-1)}\right)$$



We can actually compute the $c_p(n,t)$ coefficients exactly by solving the problem of p nonintersecting random walks in the presence of boundaries

[Fisher], [Huse, Fisher].

RQC 2-design time

We have the $k=2$ frame potential for random circuits

$$\mathcal{F}^{(2)}_{\mathrm{RQC}} \leq 2\left(1+\left(\frac{2q}{q^2+1}\right)^{2(t-1)}\right)^{n_g-1}$$

and recalling that $\left\|\Phi^{(2)}_{\mathrm{RQC}}-\Phi^{(2)}_{\mathrm{Haar}}\right\|^2_\diamond \leq d^4\big(\mathcal{F}^{(2)}_{\mathrm{RQC}}-\mathcal{F}^{(2)}_{\mathrm{Haar}}\big)$,

the circuit depth at which we form an ϵ-approximate 2-design is then

$$t_2 \geq C\big(2n\log q+\log n+\log 1/\epsilon\big) \quad \text{with} \quad C=\left(\log\frac{q^2+1}{2q}\right)^{-1}$$

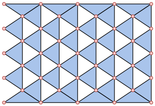
and where for $q=2$ we have $t_2\approx 6.2n$, and in the limit $q\to\infty$ we find $t_2\approx 2n$

(reproducing the known result that t_2 is $O(n+\log(1/\epsilon))$ [Harrow, Low])

k-designs in RQCs

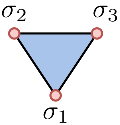
We wrote the k-th FP as a lattice partition function of $\sigma \in S_k$ spins

$$\mathcal{F}^{(k)}_{\mathrm{RQC}} = \sum_{\{\sigma\}} \prod_{\triangleleft} J^{\sigma_1}_{\sigma_2\sigma_3} = \sum_{\{\sigma\}}$$



and had plaquette terms

$$J^{\sigma_1}_{\sigma_2\sigma_3} =$$



$$= \sum_{\tau \in S_t} \mathcal{W}g(\sigma_1^{-1}\tau, q^2) q^{\ell(\tau^{-1}\sigma_2)} q^{\ell(\tau^{-1}\sigma_3)}$$

k-designs in RQCs

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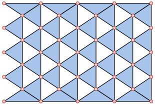
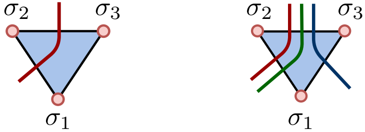
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} = \sum_{\{\sigma\}} \prod_{\triangleleft} J_{\sigma_2\sigma_3}^{\sigma_1} = \sum_{\{\sigma\}}$$ 

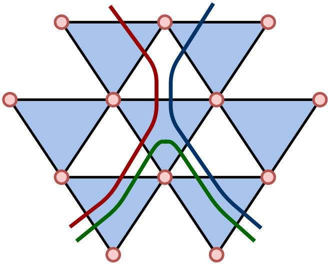
with domain walls representing transpositions between permutations



i.e. denoting the generating set of transpositions for S_k, of which there are $\binom{k}{2}$

A panoply of domain walls
(and ominous combinatorics)

For general k, domain walls are now allowed to interact, pair create, and annihilate

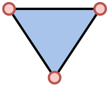


this means we can have closed loops in the circuit

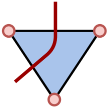
so there is no longer a nice division into multidomain walls sectors

Domain walls - a tractable sector

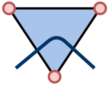
But there are a few facts about $J^{\sigma_1}_{\sigma_2\sigma_3}$'s that we can prove for any k, which guarantee the independence of the single domain wall sector



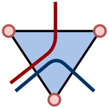
$= 1\,,$



$= \dfrac{q}{(q^2+1)}$



$= 0\,,$



$= 0$

for any domain wall in the k-th moment (i.e. any transpositions in S_k)

Domain walls - a tractable sector

For general k, we then have the contribution from the ground states and single domain wall sector, plus higher order contributions

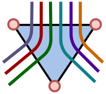
$$\mathcal{F}_{\mathrm{RQC}}^{(k)} \leq k! \left(1 + (n_g - 1) \binom{k}{2} \binom{2(t-1)}{t-1} \left(\frac{q}{q^2+1} \right)^{2(t-1)} + \dots \right)$$

Domain walls - a tractable sector

For general k, we then have the contribution from the ground states and single domain wall sector, plus higher order contributions

$$\mathcal{F}_{\mathrm{RQC}}^{(k)} \leq k! \left(1 + (n_g - 1) \binom{k}{2} \binom{2(t-1)}{t-1} \left(\frac{q}{q^2+1} \right)^{2(t-1)} + \dots \right)$$

Moreover, the multi-domain wall terms are heavily suppressed and higher order interactions are subleading in $1/q$ as



$$\sim \frac{1}{q^p}$$

In the large q limit, the single domain wall sector gives the ϵ-approximate k-design time: $t_k \geq C(2nk \log q + k \log k + \log(1/\epsilon))$, which is

$$t_k = O(nk)$$

k-designs from stat-mech

RQCs form k-designs in $O(nk)$ depth

we showed this in the large q limit, but this limit is likely not necessary

- the multi-domain walls terms with no intersections are bounded by the single domain wall terms
- for interacting domain wall configurations, the more complicated the interaction term the stronger the suppression
- many of the interaction terms have negative weight

Conjecture: *The single domain wall sector of the lattice partition function dominates the multi-domain wall sectors for higher moments k and any local dimension q.*

As the lower bound on the design depth is $O(nk)$, RQCs are then **optimal implementations of randomness**

Future science

- Can we rigorously bound the higher order terms in $\mathcal{F}^{(k)}_{\mathrm{RQC}}$ at small q?
- These stat-mech approaches are powerful, can we use them for other RQCs?
  e.g. RQCs with different geometries, higher dimensions, Floquet RQCs, RQCs with symmetry/conservation laws
  - show that orthogonal circuits [NHJ] form k-designs for $O(d)$
  - do z-spin conserving RQCs [Khemani, Vishwanath, Huse], [Rakovszky, Pollmann, von Keyserlingk] form k-designs in fixed charge sectors?
- A linear growth in design also has implications for the growth of complexity
- Apply these techniques to the RQCs in the Google experiments?

Thanks!

(ご清聴ありがとうございました)