Constructing k-uniform states of non-minimal support

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ICFO - The Institute of Photonic Sciences

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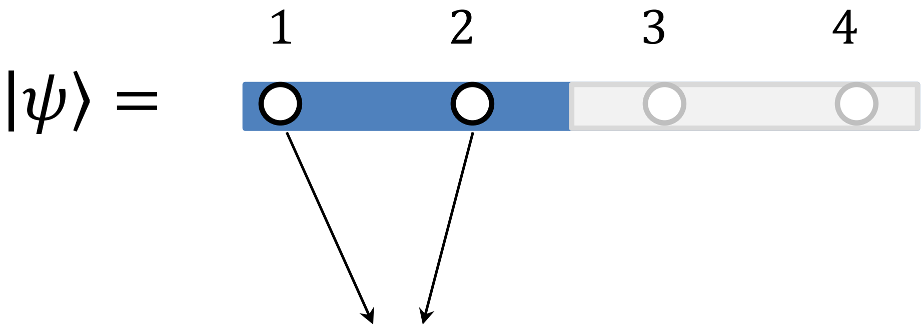
k-uniform states
and Absolutely Maximally Entangled (AME) states

There is a fundamental question to ask, which states are useful for quantum information applications?

What are AME states?

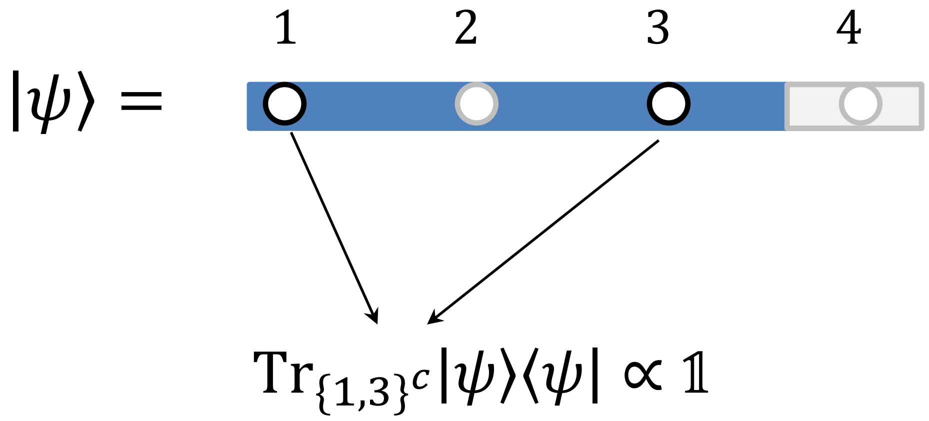
$$|\psi\rangle =$$

What are AME states?



$$\mathrm{Tr}_{\{1,2\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

What are AME states?



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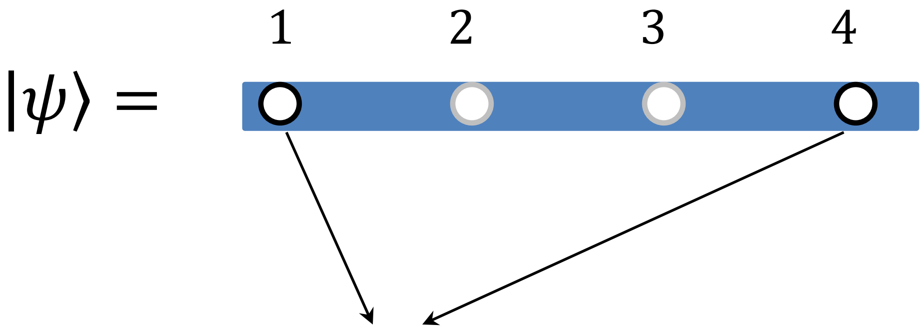
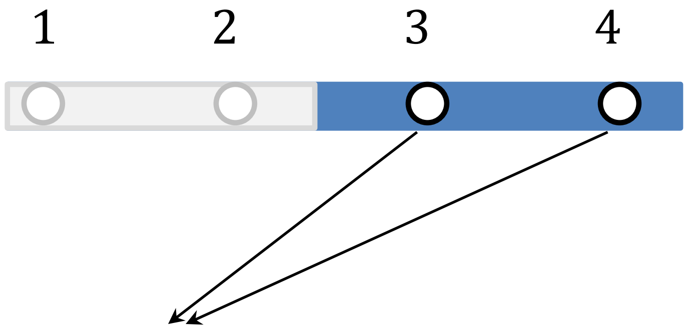


$$\mathrm{Tr}_{\{1,4\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

What are AME states?

$|\psi\rangle =$



$$\mathrm{Tr}_{\{3,4\}^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

$AME(n,q)$:

A pure state of n parties with local dimension q is AME if for all $S \subset \{1,\ldots,n\}$

$$|S| \leq \lfloor {}^{n}/_{2} \rfloor \quad \Longrightarrow \quad \rho_S = \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

Existence of AME states

- Still fundamental questions open.

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[1] A. Higuchi, A. and Sudbery, Phys. Lett. A 273,213 (2000).

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[3] F. Huber, O. Gühne, and J. Siewert, Phys. Rev. Lett. 118, 200502 (2017).

Existence of AME states

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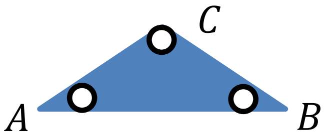
  - For qubits, ($q = 2$): $n = 2, 3$

$$|\phi^+\rangle = |00\rangle + |11\rangle \longrightarrow \rho_i = \mathbb{1} \qquad \forall i$$



$$|GHZ\rangle = |000\rangle + |111\rangle \longrightarrow \rho_i = \mathbb{1} \qquad \forall i$$

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Existence of AME states

- Still fundamental questions open.
  - For qubits, ($q = 2$): $n = 2, 3, 4, 5, 6, 7, 8, 9, \dots$ [1,2,3]

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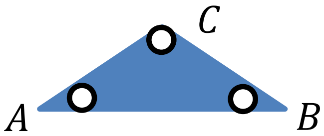

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Existence of AME states

- Still fundamental questions open.
  - For qubits, ($q = 2$): $n = 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ [1,2,3]

- By increasing the local dimension q, we can find AME state
  - For qutrits, ($q = 3$):

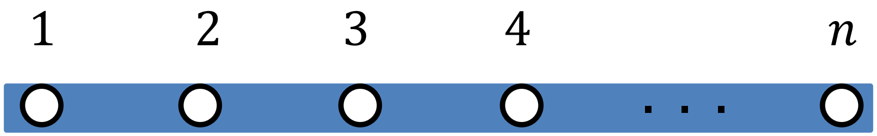
$$|AME(4,3)\rangle = \sum_{i,j=0}^{2} |i, j, i+j, i+2j\rangle \longrightarrow \rho_{ij} = \mathbb{1} \qquad \forall i, j$$

$\text{modulo}(3)$

k-uniform states

- Since AME states may not always exist, one can loosen the criteria for maximal mixedness,

$$|\psi\rangle =$$



$AME(n,q)$ states:

A pure state $|\psi\rangle$ of n parties with local dimension q is AME if for all $S \subset \{1,2,\ldots,n\}$,

$$|S| \leq \lfloor {}^{n}/_{2} \rfloor \quad \Longrightarrow \quad \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

k-UNI(n,q) states:

A pure state $|\psi\rangle$ of n parties with local dimension q is k-uniform if for all $S \subset \{1,2,\ldots,n\}$,

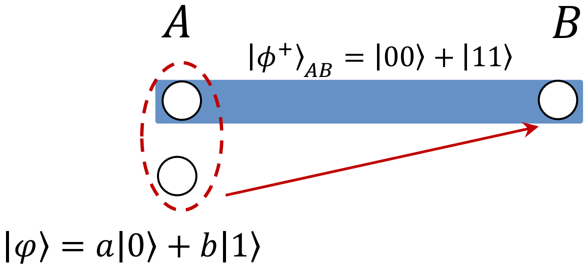
$$|S| \leq k \quad \Longrightarrow \quad \mathrm{Tr}_{S^c}|\psi\rangle\langle\psi| \propto \mathbb{1}$$

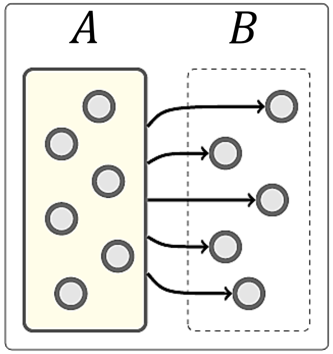
- Obviously, an AME state is a $k = \lfloor \frac{n}{2} \rfloor$-uniform state.

Why are k-uniform states interesting?

- Natural generalization of EPR and GHZ states

- Resource for multipartite parallel teleportation and quantum secret sharing [1]





- k-uniform states are a type of quantum error correcting codes having the maximal distance allowed by the Singleton bound (optimal codes) [2,3]

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[1] W. Helwig, W. Cui, J. I. Latorre, A. Riera, and H.K. Lo, Phys. Rev. A, 86, 052335 (2012).

[2] A.J. Scott, Phys. Rev. A 69, 052330 (2004).

[3] M. Grassl and M Rötteler, IEEE Int. Symp. Inf. Teory (ISTT), 1108 (2015) .

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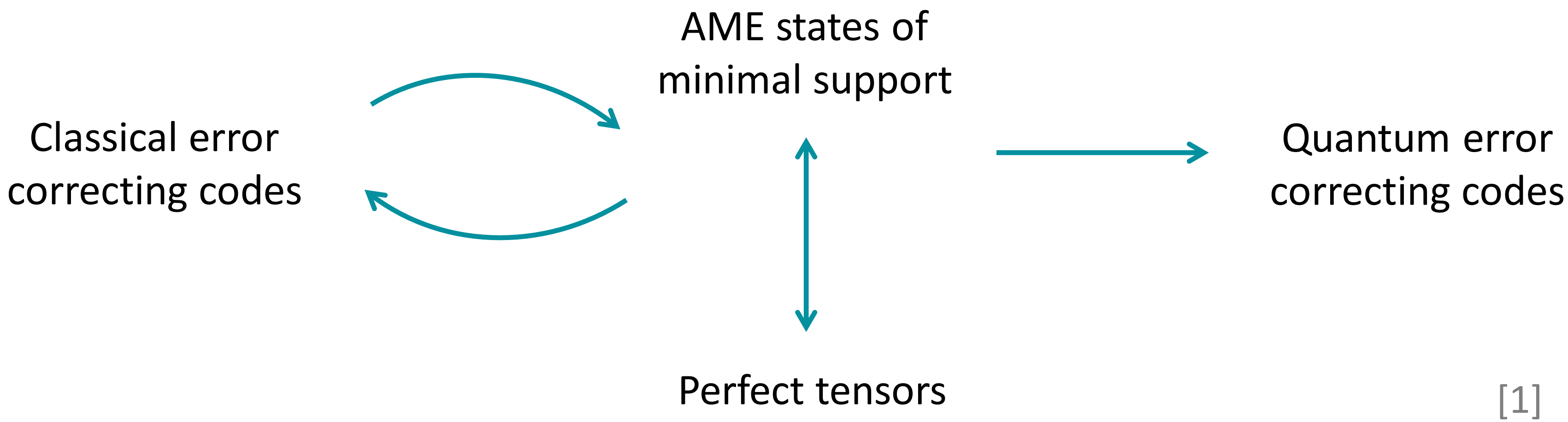

AME states of minimal support

Classical error correcting codes

Quantum error correcting codes

Perfect tensors

[1]

- Holographic models implementing the AdS/CFT correspondence [2]

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[1] Z. R., C. Gogolin, A. Riera, A. Acín, J. Phys. A, 51, 7 (2018)

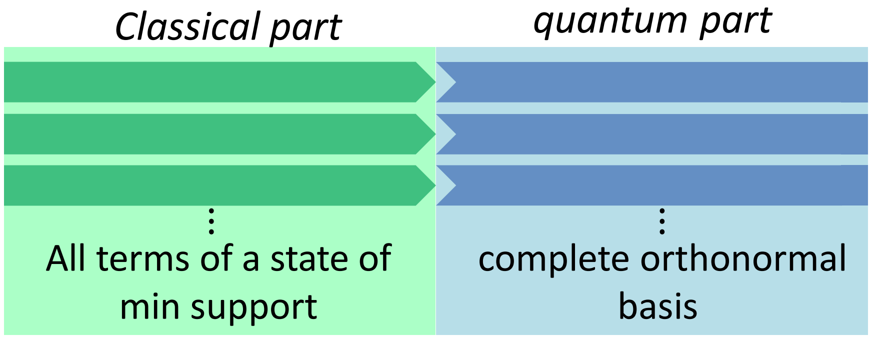
[2] F. Patawski, B. Yoshida, D. Harlow, and J. Preskill, HEP, 06, 149 (2015).

Content of this talk

1. Classical error correcting codes $\longrightarrow$ k-uniform states of minimal support $|\psi\rangle$

2. Constructing k-uniform state of non-minimal support:

$|\phi\rangle$



Classical part

quantum part

⋮

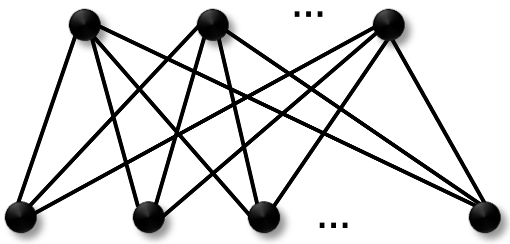
All terms of a state of min support

⋮

complete orthonormal basis

$$|\psi\rangle \xrightarrow{\text{LU } \cancel{\text{equivalent}}} |\phi\rangle$$

3. Graph states:



k-uniform states of minimal support

Classical error correcting codes $\longrightarrow$ k-uniform states of minimal support

k-uniform states of minimal support

- Classifying the k-uniform states according to the number of their terms ⟶ they are expanded in product basis.

$$|\psi\rangle = \sum_{j_1,\ldots,j_n=0}^{q-1} c_{j_1,\ldots,j_n} |j_1, \ldots, j_n\rangle$$

$$\downarrow$$

$$q^k \leq \#\text{ terms} \leq q^n$$

- $\#$ **terms** $= q^k$: states with this number of terms or local unitary equivalent to this state are called k-uniform of minimal support.
- $\#$ **terms** $> q^k$: states with this number of terms are k-uniform of non-minimal support.

Classical error correcting codes

Channel

Alice Bob

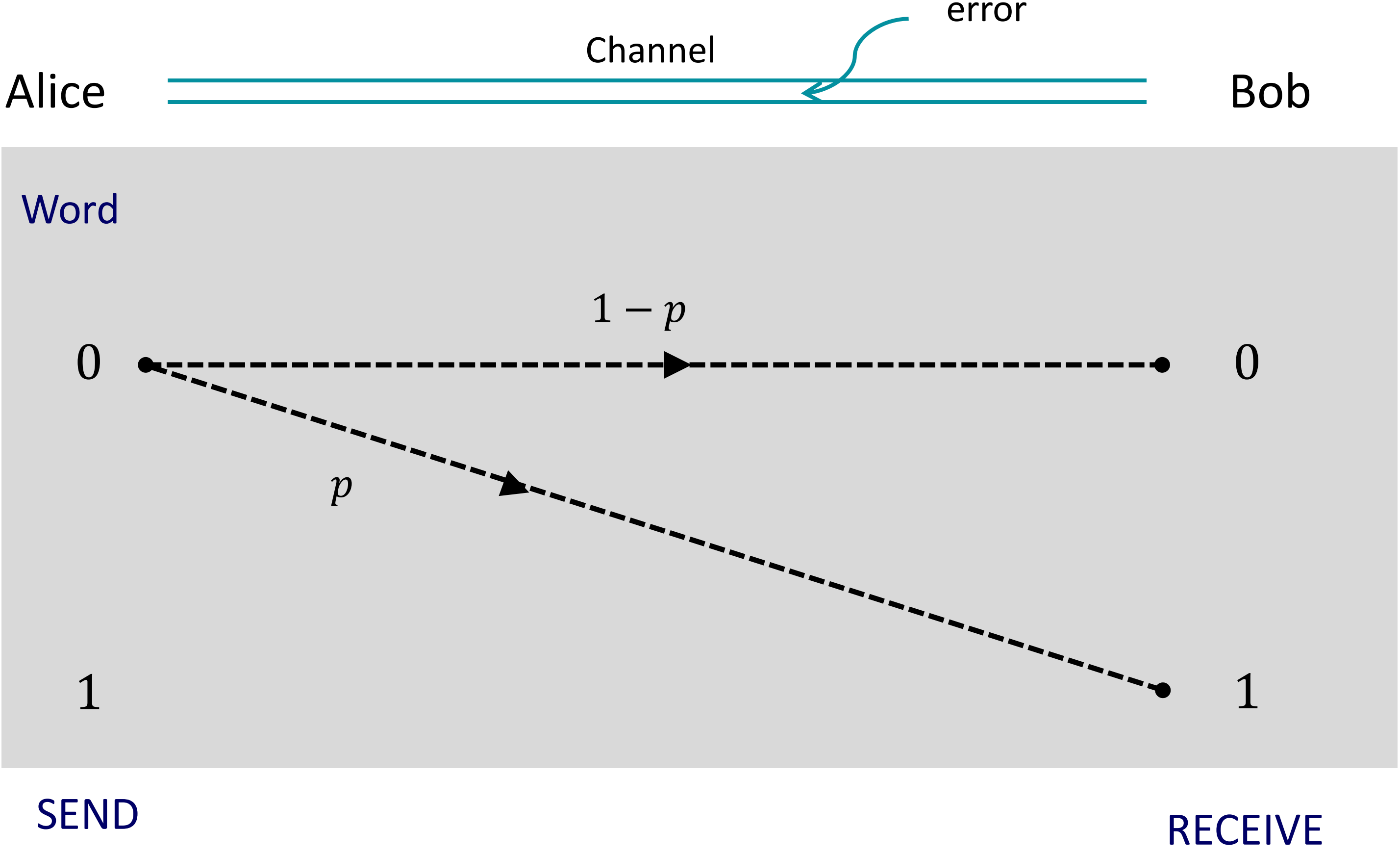
Word

0

1

F.J. MacWilliams and N.J.A. Sloane, The theory of error-correction codes (1977) - chapter 1.

Classical error correcting codes



F.J. MacWilliams and N.J.A. Sloane, The theory of error-correction codes (1977) - chapter 1.

Classical error correcting codes

Word	Encoding (Codewords)
0	000
1	111

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Classical error correcting codes

Word	Encoding (Codewords)		Error
0	000	$\longrightarrow$	$0\textcolor{red}{1}0$
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Classical error correcting codes

Word	Encoding (Codewords)		Error		Correction	Decoding
0	000	→	010	→	000	0
1	111	→	101	→	111	1

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Classical error correcting codes

Word	Encoding (Codewords)		Error		Correction	Decoding
0	000	⟶	010	⟶	000	0
$\underbrace{1}_{k=1}$	111	⟶	101	⟶	111	1

$$\vec{m} = (x_1, \dots, x_k)$$

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Classical error correcting codes

Word	Encoding (Codewords)		Error		Correction	Decoding
0	000	$\longrightarrow$	$0\textcolor{red}{1}0$	$\longrightarrow$	000	0
$\underbrace{1}_{k=1}$	$\underbrace{111}_{n=3}$	$\longrightarrow$	$1\textcolor{red}{0}1$	$\longrightarrow$	111	1

$$\vec{m} = (x_1, \dots, x_k) \longrightarrow \vec{c} = (\underbrace{x_1, \dots, x_k}_{\text{Message symbols}} \quad , \quad \underbrace{x_{k+1}, \dots, x_n}_{\text{Check symbols}})$$

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Classical error correcting codes

Word	Encoding (Codewords)	Error	Correction	Decoding
0	000	010	000	0
1	111	101	111	1

d_H

$k = 1$ $\quad$ $n = 3$

$$d_H = 2t + 1$$

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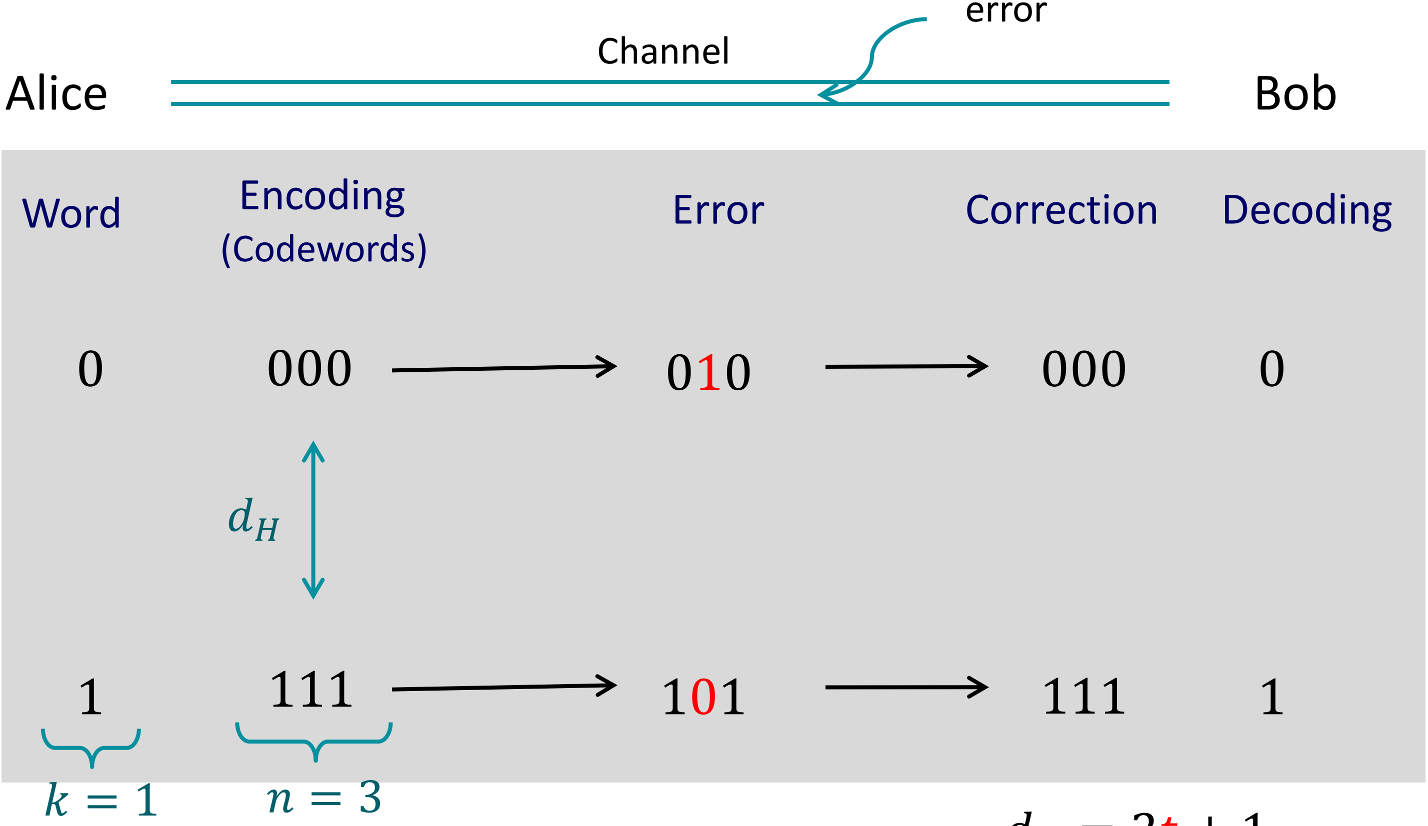

Word	Encoding (Codewords)		Error		Correction	Decoding
0	000	$\longrightarrow$	010	$\longrightarrow$	000	0
	$\updownarrow d_H$					
$\underbrace{1}_{k=1}$	$\underbrace{111}_{n=3}$	$\longrightarrow$	101	$\longrightarrow$	111	1

$$d_H = 2t + 1$$

$$[n = 3,\ \ k = 1,\ d_H = 3]_{q=2}$$

F.J. MacWilliams and N.J.A. Sloane, The theory of error-correction codes (1977) - chapter 1.

MDS codes

- Constructing linear codes $[n, k, d_H]_q$ [1]

Word

$$\underbrace{\vec{m}}_{k}\ G_{k\times n} = \underbrace{\vec{c}}_{n}$$

Codeword

Generator Matrix

$$d_H = 2t + 1$$

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[1] F.J. MacWilliams, N.J.A. Sloane, The theory of error-correction codes (1977).

[2] R. Singleton, IEEE Trans. Inf. Theor., 10, 116 (2006).

MDS codes

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$$d_H = 2t + 1$$

- Singleton bound for any linear code: $d_H \leq n - k + 1$ [2]

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MDS codes

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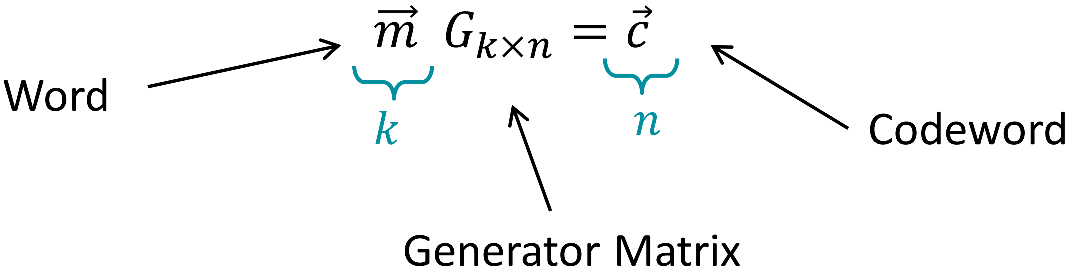

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Optimal Code ⟹ Maximum Distance Separable (MDS) code

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[1] F.J. MacWilliams, N.J.A. Sloane, The theory of error-correction codes (1977).

[2] R. Singleton, IEEE Trans. Inf. Theor., 10, 116 (2006).

Constructing k-uniform state from MDS codes

- From an MDS code a k-uniform state can be constructed $\longrightarrow$ taking the equally weighted superposition of all the codewords

Classical MDS codes → k-uniform states of minimal support

$$|\psi\rangle = \sum |\text{all codewords}\rangle$$

- The existence of the MDS codes and hence a set of k-uniform states of minimal support

$$\begin{cases} n \leq q+1 & k\text{-uniform states} \\ n \leq q+2 & \text{If } q \text{ is even, 3-uniform states} \end{cases}$$

q is a power of prime

An example of k-uniform state

- Generator matrix of an MDS code $[4,2,3]_3$

$$m = (i,j)\,, \qquad G_{2\times 4} = \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \qquad \longrightarrow \qquad c = (i, j, i+j, i+2j)\,,$$

- $\mathrm{AME}(4,3)$:

$$|\psi\rangle = \sum |\vec{m}\, G_{2\times 4}\rangle = \sum_{i,j=0}^{2} |i, j, i+j, i+2j\,\rangle \begin{array}{cl} = & |0 \quad 0 \quad 0 \quad 0\rangle \\ + & |0 \quad 1 \quad 1 \quad 2\rangle \\ + & |0 \quad 2 \quad 2 \quad 1\rangle \\ + & |1 \quad 0 \quad 1 \quad 1\rangle \\ + & |1 \quad 1 \quad 2 \quad 0\rangle \\ + & |1 \quad 2 \quad 0 \quad 2\rangle \\ + & |2 \quad 0 \quad 2 \quad 2\rangle \\ + & |2 \quad 1 \quad 0 \quad 1\rangle \\ + & |2 \quad 2 \quad 1 \quad 0\rangle \end{array}$$

(All additions and multiplications modulo $q = 3$.)

Basis

- Given a k-uniform state of minimal support $|\psi\rangle = \sum |\vec{m}G_{k\times n}\rangle$

$$|\psi_{\vec{a}}\rangle \coloneqq M(\vec{a})\,|\psi\rangle$$

#states= q^n

form a complete orthonormal basis

$$M(\vec{a}) \coloneqq M(\vec{a}_Z) \otimes M(\vec{a}_X)$$

$$= \underbrace{Z^{a_1} \otimes Z^{a_{k+1}} \otimes \cdots \otimes Z^{a_k}}_{k} \otimes \underbrace{X^{a_{v_{k+1}}} \otimes X^{a_{k+2}} \otimes \cdots \otimes X^{a_n}}_{n-k}\ , \qquad \forall a_i \in \{0, \dots, q-1\}$$

Generalized Pauli operators

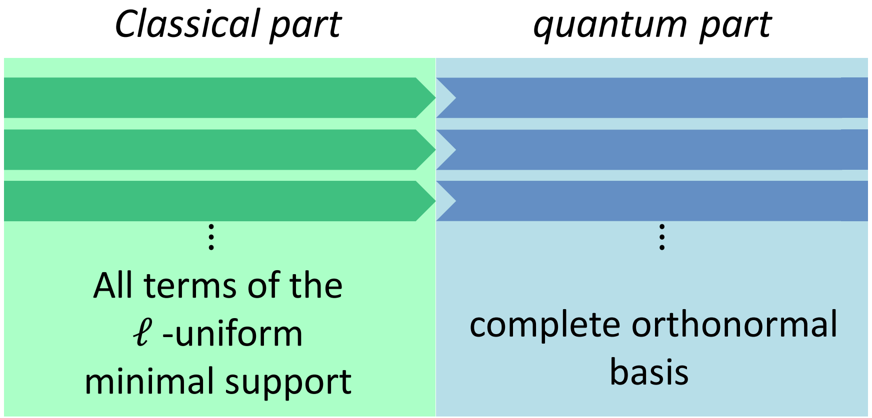
$$\longrightarrow \qquad \langle\psi|M(\vec{a})^{\dagger}M(\vec{b})|\psi\rangle = \prod_i \delta_{a_i,b_i}$$

$$\begin{cases} X\,|j\rangle = |j+1 \mod q\rangle \\ Z\,|j\rangle = \omega^j|j\rangle \qquad \omega \coloneqq \mathrm{e}^{\frac{2\pi i}{q}} \end{cases}$$

$$X^q = Z^q = \mathbb{1}$$

Constructing k-uniform states of non-minimal support

A systematic method to construct a set of non-minimal support states.



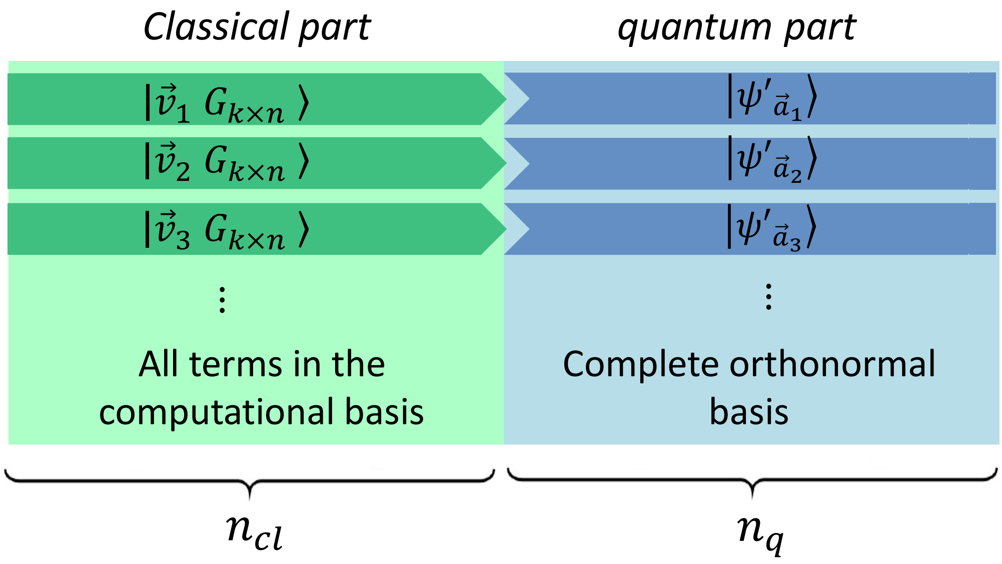
Classical part

quantum part

All terms of the ℓ -uniform minimal support

complete orthonormal basis

k-uniform states of non-minimal support



Classical part

quantum part

$|\vec{v}_1\, G_{k\times n}\,\rangle$ $|\psi'_{\vec{a}_1}\rangle$

$|\vec{v}_2\, G_{k\times n}\,\rangle$ $|\psi'_{\vec{a}_2}\rangle$

$|\vec{v}_3\, G_{k\times n}\,\rangle$ $|\psi'_{\vec{a}_3}\rangle$

⋮ ⋮

All terms in the computational basis

Complete orthonormal basis

n_{cl}

n_q

$\ell - \mathrm{UNI}\ (n_{\mathrm{cl}}, q)$ state

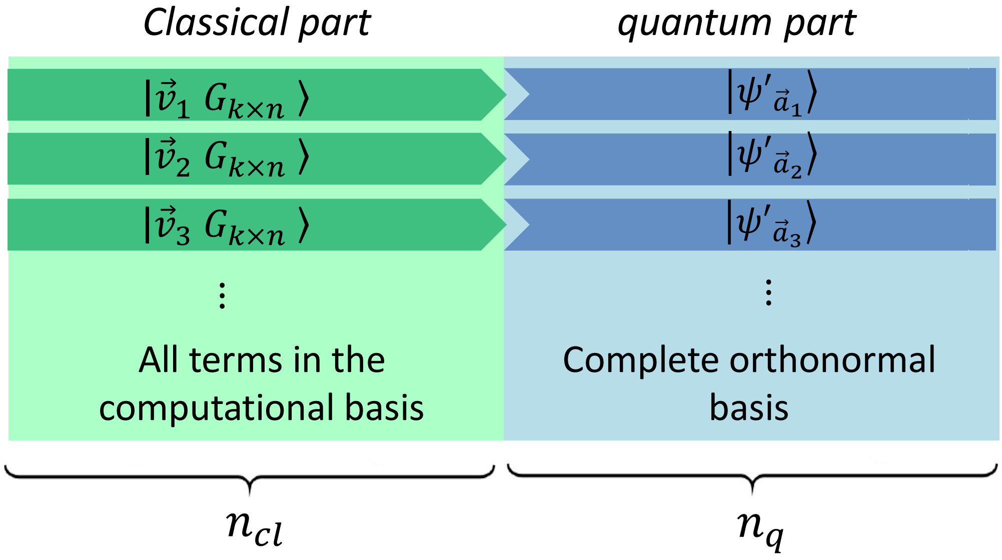
$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ basis

$$|\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle$$

$$|\psi'_{\vec{a}}\rangle \coloneqq M(\vec{a})\ |\psi'\rangle$$

k-uniform states of non-minimal support

$|\phi\rangle \equiv$



Classical part

quantum part

$|\vec{v}_1\, G_{k\times n}\rangle$

$|\vec{v}_2\, G_{k\times n}\rangle$

$|\vec{v}_3\, G_{k\times n}\rangle$

$\vdots$

All terms in the computational basis

$|\psi'_{\vec{a}_1}\rangle$

$|\psi'_{\vec{a}_2}\rangle$

$|\psi'_{\vec{a}_3}\rangle$

$\vdots$

Complete orthonormal basis

n_{cl}

n_q

$\longrightarrow \quad n = n_{cl} + n_q$

$\ell - \mathrm{UNI}\ (n_{\mathrm{cl}}, q)$ state

$$|\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle$$

$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ basis

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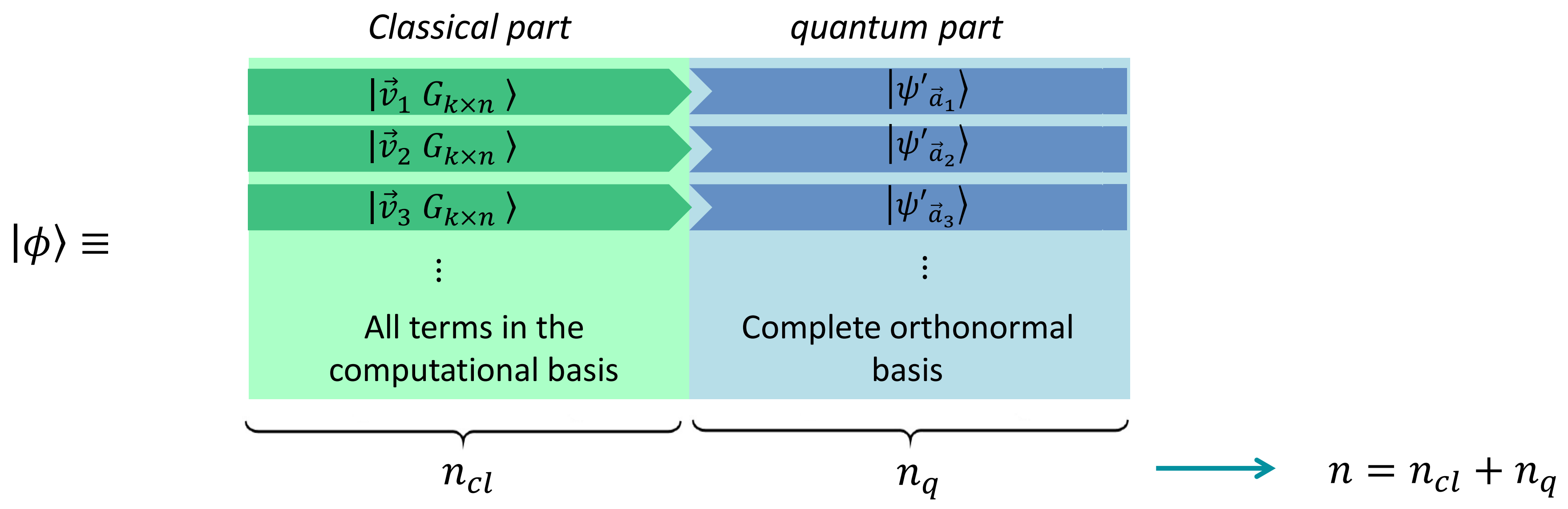
k-uniform states of non-minimal support



$|\phi\rangle \equiv$

Classical part	*quantum part*
$\lvert\vec{v}_1\, G_{k\times n}\,\rangle$	$\lvert\psi'_{\vec{a}_1}\rangle$
$\lvert\vec{v}_2\, G_{k\times n}\,\rangle$	$\lvert\psi'_{\vec{a}_2}\rangle$
$\lvert\vec{v}_3\, G_{k\times n}\,\rangle$	$\lvert\psi'_{\vec{a}_3}\rangle$
$\vdots$	$\vdots$
All terms in the computational basis	Complete orthonormal basis
n_{cl}	n_q

$\longrightarrow \quad n = n_{cl} + n_q$

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$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ basis

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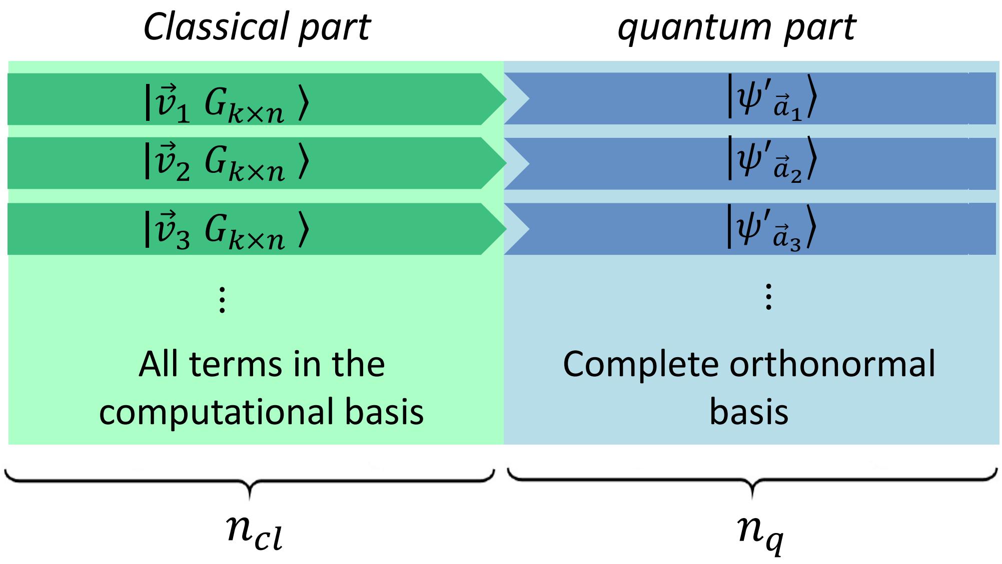
e.g. # q^{ℓ} - term # q^{n_q} - states $\longrightarrow$ $\ell = n_q$

k-uniform states of non-minimal support



Classical part *quantum part*

$|\phi\rangle \equiv$

$|\vec{v}_1\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_1}\rangle$

$|\vec{v}_2\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_2}\rangle$

$|\vec{v}_3\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_3}\rangle$

$\vdots$ $\vdots$

All terms in the computational basis

Complete orthonormal basis

n_{cl} n_q $\longrightarrow$ $n = n_{cl} + n_q$

$\ell -$ UNI (n_{cl}, q) state

$$|\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle$$

$\ell' -$ UNI (n_{q}, q) basis

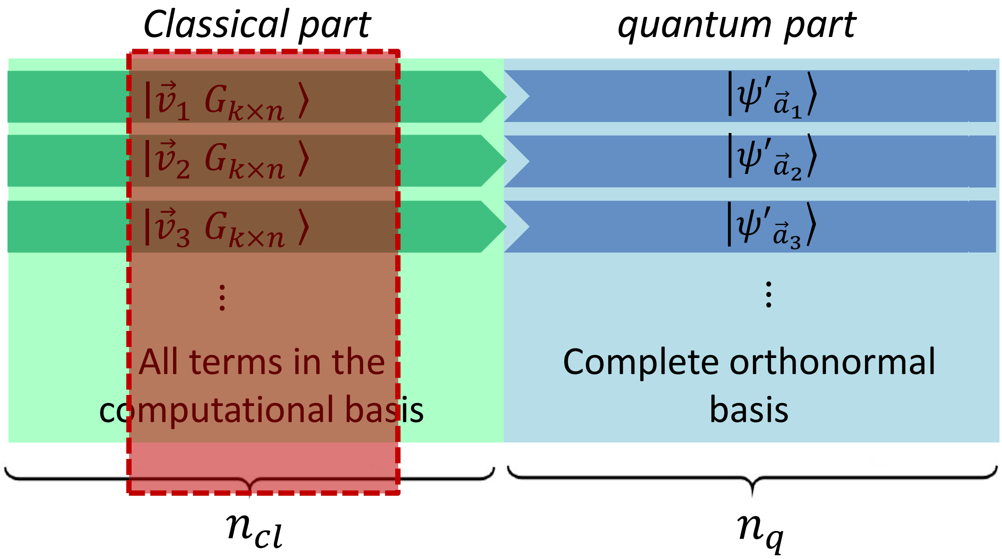
$$|\psi'_{\vec{a}}\rangle := M(\vec{a})\, |\psi'\rangle$$

e.g. # q^{ℓ} - term # q^{n_q} - states $\longrightarrow$ $\ell = n_q$

- $|\phi\rangle$ which is k-uniform state in $k -$ UNI (n, q) and $k = \min\{\ell + 1, \ell' + 1\}$

k-uniform states of non-minimal support

$|\phi\rangle \equiv$



Classical part

quantum part

$|\vec{v}_1\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_1}\rangle$

$|\vec{v}_2\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_2}\rangle$

$|\vec{v}_3\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_3}\rangle$

$\vdots$ $\vdots$

All terms in the computational basis

Complete orthonormal basis

n_{cl} n_q

Case I.

$\longrightarrow$ $n = n_{cl} + n_q$

$\ell - \mathrm{UNI}\ (n_{\mathrm{cl}}, q)$ state

$$|\psi\rangle = \sum_{\vec{v}} |\vec{v} G_{k\times n}\rangle$$

$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ basis

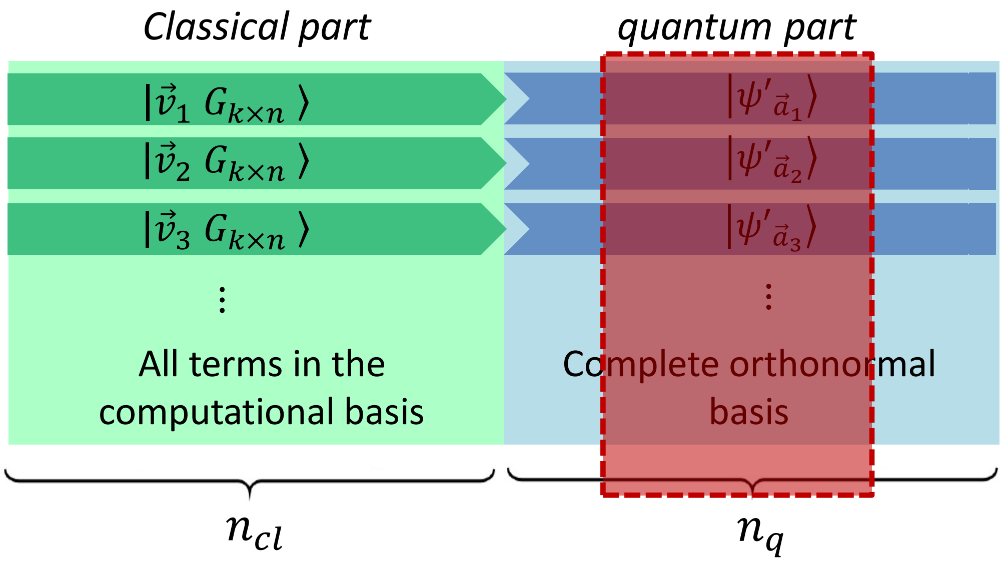
$$|\psi'_{\vec{a}}\rangle := M(\vec{a})\, |\psi'\rangle$$

e.g. # q^{ℓ} - term # q^{n_q} - states $\longrightarrow$ $\ell = n_q$

- $|\phi\rangle$ which is k-uniform state in $k - \mathrm{UNI}\ (n, q)$ and $k = \min\{\ell + 1, \ell' + 1\}$

k-uniform states of non-minimal support

$|\phi\rangle \equiv$



Classical part

quantum part

$|\vec{v}_1\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_1}\rangle$

$|\vec{v}_2\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_2}\rangle$

$|\vec{v}_3\, G_{k\times n}\rangle$ $|\psi'_{\vec{a}_3}\rangle$

$\vdots$ $\vdots$

All terms in the computational basis

Complete orthonormal basis

n_{cl} n_q

Case I.

Case II.

$\longrightarrow$ $n = n_{cl} + n_q$

$\ell - \mathrm{UNI}\ (n_{\mathrm{cl}}, q)$ state

$$|\psi\rangle = \sum_{\vec{v}} |\vec{v} G_{k\times n}\rangle$$

$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ basis

$$|\psi'_{\vec{a}}\rangle \coloneqq M(\vec{a})\, |\psi'\rangle$$

e.g. # q^{ℓ} - term # q^{n_q} - states $\longrightarrow$ $\ell = n_q$

- $|\phi\rangle$ which is k-uniform state in $k - \mathrm{UNI}\ (n, q)$ and $k = \min\{\ell + 1, \ell' + 1\}$

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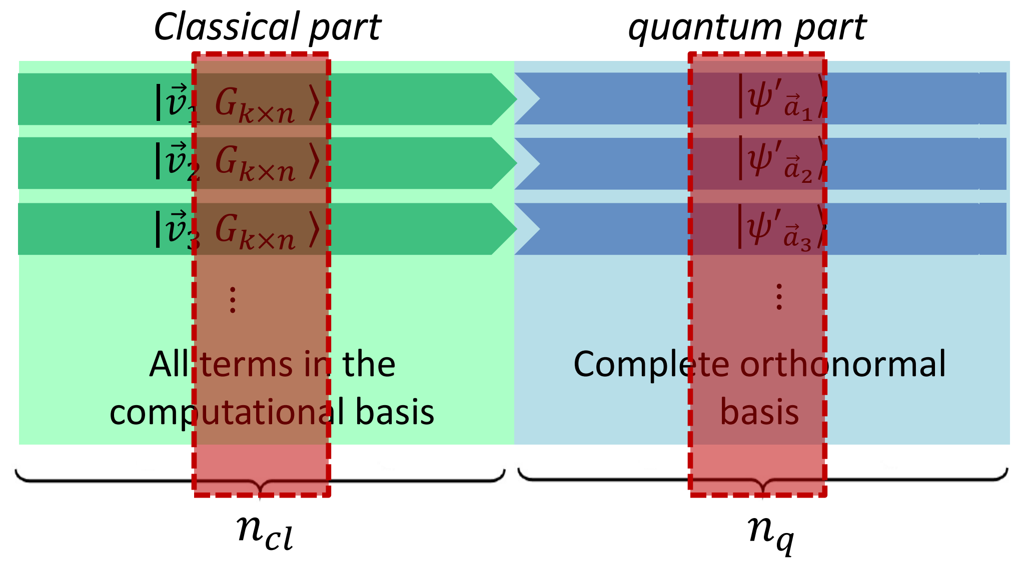

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Examples of k-uniform of non-minimal support

- $\mathrm{AME}(n = 5, q = 2)$:

$$|\psi\rangle = \begin{array}{cccc|c} & & \textit{Classical part} & & \textit{quantum part} \\ & 1 & 2 & 3 & 4\&5 \\ & |0\rangle & |0\rangle & |0\rangle & |\phi^+\rangle \\ + & |0\rangle & |1\rangle & |1\rangle & |\psi^+\rangle \\ + & |1\rangle & |0\rangle & |1\rangle & |\psi^-\rangle \\ + & |1\rangle & |1\rangle & |0\rangle & |\phi^-\rangle \end{array} \qquad q = 2$$

$$|\psi\rangle = \underbrace{\sum_{i,j} |i, j, i + j\rangle}_{n_{cl} = 3} \otimes \underbrace{Z^i \otimes X^j \sum_m |m, m\rangle}_{n_q = 2} \qquad q \geq 2$$

D. Goyeneche, Z. R., S. DiMartino, and K. Życzkowski, Phys. Rev. A, 97, 062326 (2018).

New states

- $AME(7,4)$
- $AME(11,8)$

↓

quantum error correcting codes

Table of AME states/perfect tensors / multi-unitary matrices, http://www.tp.nt.unisiegen.de/+fhuber/ame.html

	2	3	4	5	6	7	8
2	Stab CMDS	Stab CMDS	Stab CMDS	Stab CMDS	Stab	Stab CMDS	Stab CMDS
3	Stab	Stab CMDS	Stab CMDS	Stab CMDS	Stab	CMDS	Stab CMDS
4		Stab OA	Stab CMDS	Stab CMDS		CMDS	CMDS
5	Stab OA QOA	Stab OA QOA	Stab OA QOA	Stab CMDS OA QOA	OA QOA	CMDS OA QOA	CMDS OA QOA
6	Stab OA QOA	Stab	Stab CMDS	Stab		CMDS	CMDS
7		Stab	✓	Stab CMDS		CMDS	CMDS
8				Stab CMDS			CMDS
9		Stab	Stab CMDS	Stab CMDS		Stab CMDS	Stab CMDS
10		Stab	Stab CMDS	Stab CMDS		Stab CMDS	CMDS
11		No stab				Stab CMDS	✓

k-uniform of non-minimal support vs k-uniform of minimal support

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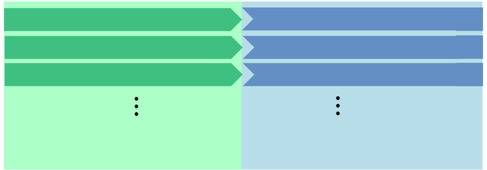
from an MDS code

$$|\psi\rangle = \sum |\text{all codewords}\rangle$$

$$\xrightarrow[S \subset \{1, \dots, n\}]{\cancel{\text{LU}}}$$

$|\phi\rangle$



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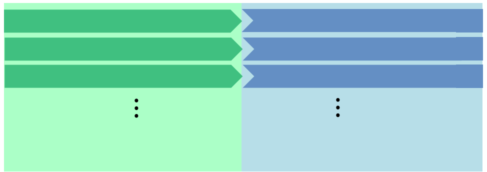


$\forall S$ s.t. $|S| \leq k \implies \rho_S \propto \mathbb{1}$

$\forall S$ s.t. $|S| = k + 1 \implies \rho_S \not\propto \mathbb{1}$

$\forall S$ s.t. $|S| \leq k \implies \rho_S \propto \mathbb{1}$

$\exists S$ s.t. $|S| = k + 1 \implies \rho_S \propto \mathbb{1}$

- Which state is better for teleportation and … ?

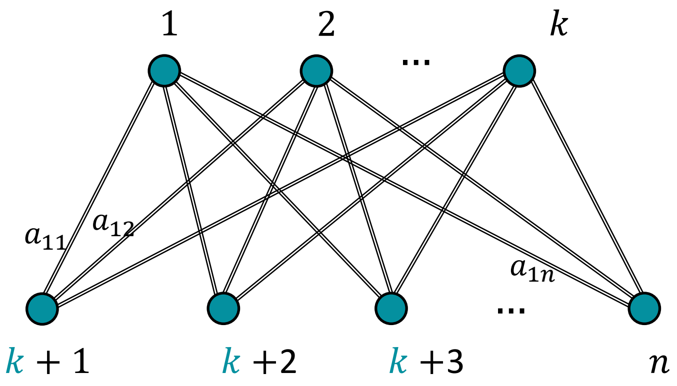
Graph state

Description of the k-uniform states within the graph state formalism.

Stabilizers formalism within the graph states

- State $|\psi\rangle$ constructed from an MDS code

$$|\psi\rangle = \sum |\text{all codewords}\rangle$$



Complete bipartite graph

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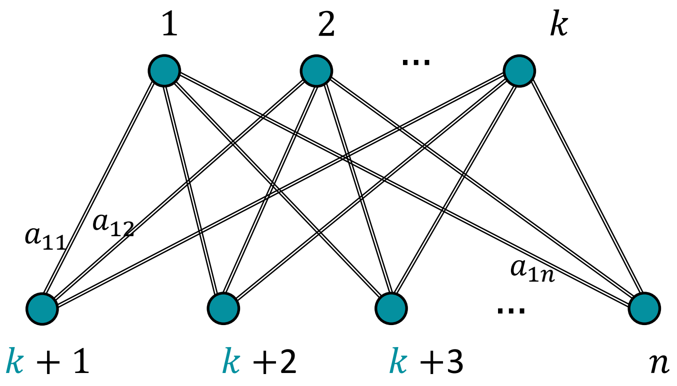
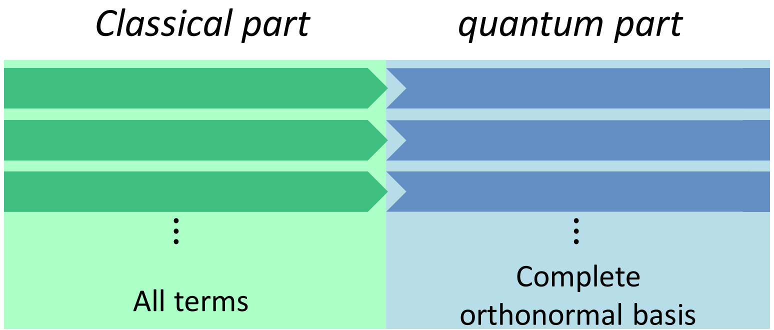


Complete bipartite graph

- State $|\phi\rangle$ constructed from

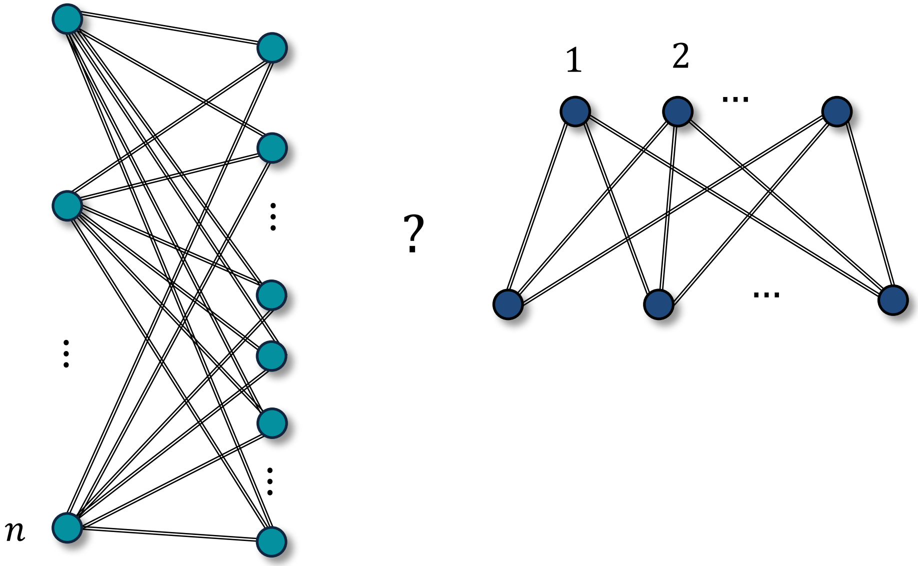


Classical part

quantum part

All terms

Complete orthonormal basis



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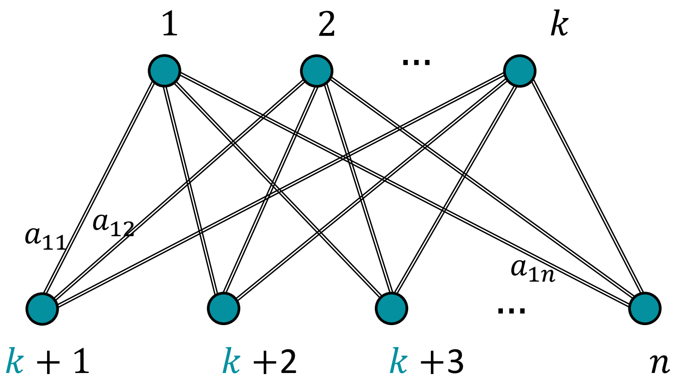

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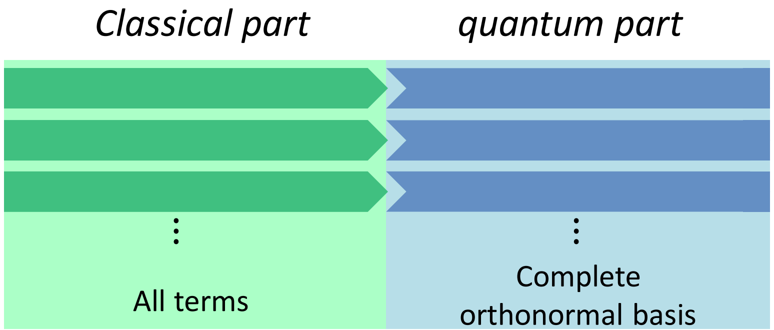

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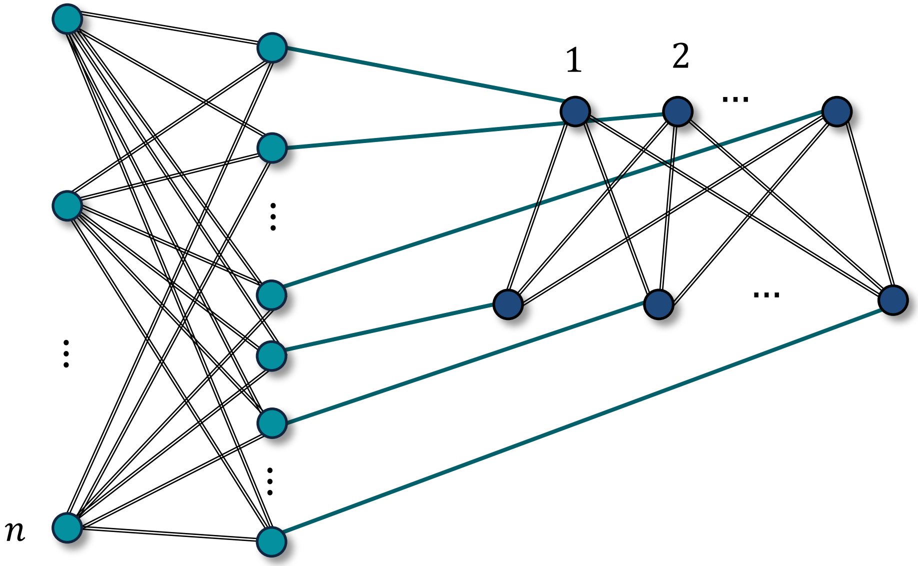
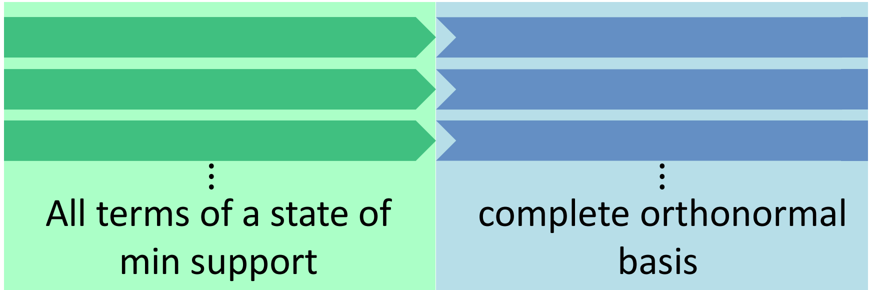


Summary

1. Classical error correcting codes → k-uniform states of minimal support $|\psi\rangle$

2. Constructing k-uniform state of non-minimal support:

$|\phi\rangle$

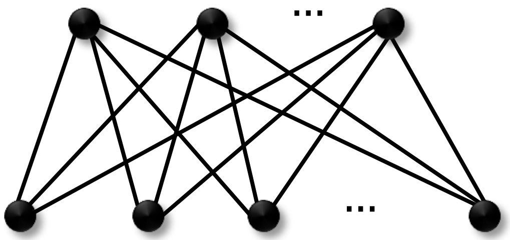


All terms of a state of min support

complete orthonormal basis

$|\psi\rangle$ LU ~~equivalent~~ → $|\phi\rangle$

3. Graph states:



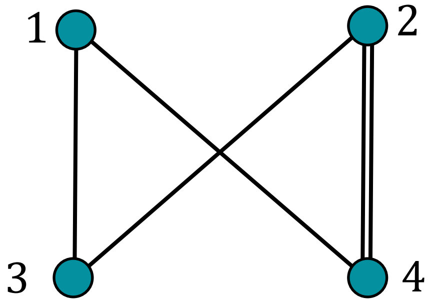
Thank you for your attention!

Graph states: Introduction

- X and Z that generalize the Pauli operators to Hilbert spaces of dimension $q \geq 2$

$$\begin{cases} X\,|j\rangle = |j+1 \mod q\rangle \\ Z\,|j\rangle = \omega^{j}|j\rangle \qquad \omega := \mathrm{e}^{\frac{2\pi i}{q}} \end{cases} \qquad X^{q} = Z^{q} = \mathbb{1}$$

- initialize each qudit as the state, $|+\rangle = |0\rangle + \cdots + |q-1\rangle$, perform $CZ_{\alpha\beta} = \sum_{l=0}^{q-1} |l\rangle\langle l|_{\alpha} \otimes Z^{l}{}_{\beta}$



$$\begin{cases} s_1 = X \quad \mathbb{1} \quad Z \quad Z \\ s_2 = \mathbb{1} \quad X \quad Z \quad Z^{2} \\ s_3 = Z \quad Z \quad X \quad \mathbb{1} \\ s_4 = Z \quad Z^{2} \quad \mathbb{1} \quad X \end{cases}$$

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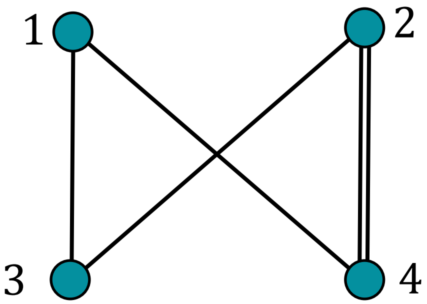
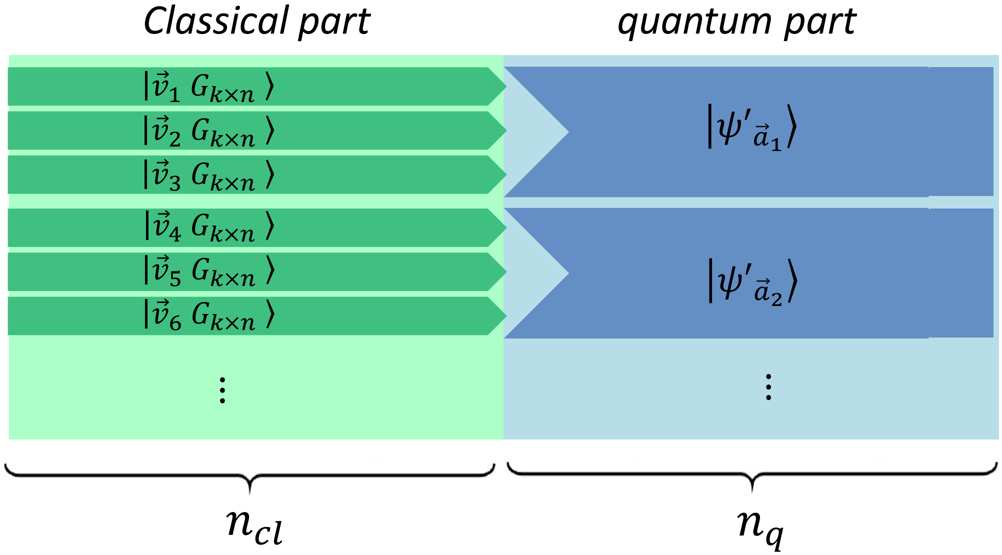


$$\begin{cases} s_1 = X & \mathbb{1} & Z & Z \\ s_2 = \mathbb{1} & X & Z & Z^2 \\ s_3 = Z & Z & X & \mathbb{1} \\ s_4 = Z & Z^2 & \mathbb{1} & X \end{cases}$$

$$\xrightarrow{\text{LU equivalent}} \qquad |AME(4,3)\rangle = \sum_{i,j=0}^{2} |i, j, i+j, i+2j\,\rangle$$

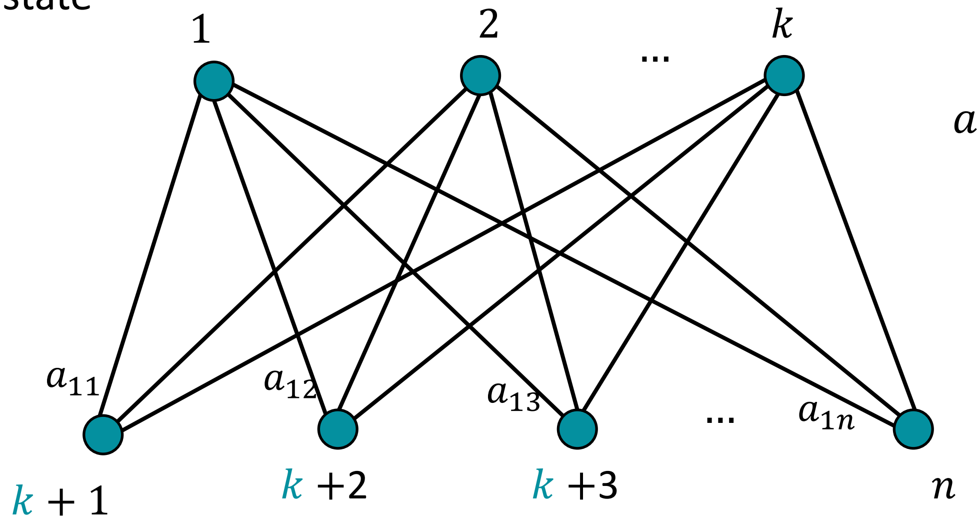
Repetition in the basis

- $AME(7,4)$
- $AME(11,8)$



Stabilizers formalism within the graph states

- The minimal support k-uniform state constructed from the MDS codes
  - Generator matrix of an MDS code $[n, k, d_H]_q$ $\longrightarrow$ $G_{k\times n} = [\mathbb{1}_k | A_{k\times n-k}]$
  - k-uniform state, $k - \mathrm{UNI}(n, q)$ $\longrightarrow$ $|\psi\rangle = \sum_{\vec{v}} |\vec{v}G_{k\times n}\rangle$
  - Graph state



a_{ij} are matrix elements of $A_{k\times(n-k)}$

$$\overset{a_{ij}}{\overline{}} \equiv \mathrm{CZ}^{a_{ij}}$$

- A complete bipartite graph shows the structure of the graph

Graph state

$\ell - \mathrm{UNI}\ (n_{\mathrm{cl}}, q)$ state

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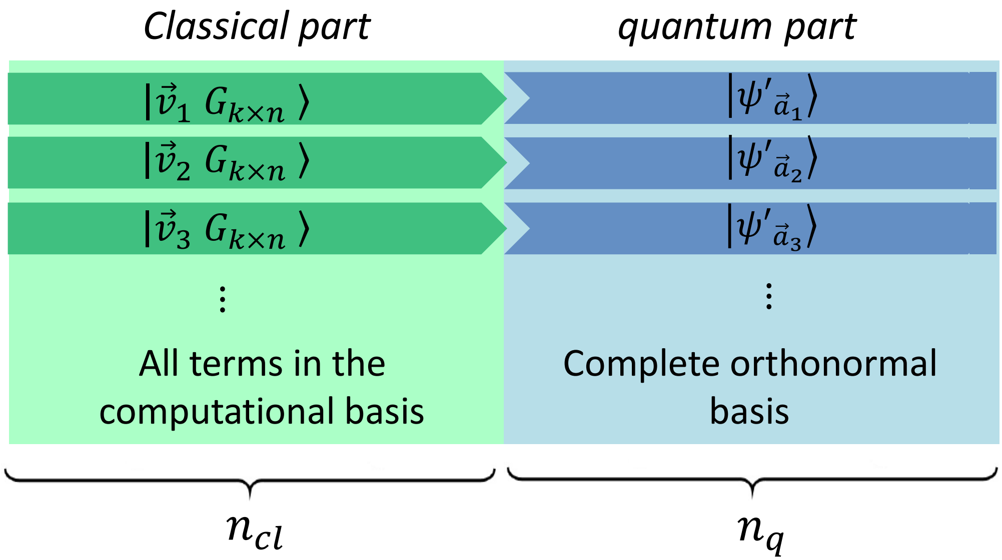

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⋮

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quantum part

$|\psi'_{\vec{a}_1}\rangle$

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$|\psi'_{\vec{a}_3}\rangle$

⋮

Complete orthonormal basis

n_{cl}

n_q

$\ell' - \mathrm{UNI}\ (n_{\mathrm{q}}, q)$ states

$$|\psi'_{\vec{a}}\rangle \coloneqq M(\vec{a})\ |\psi'\rangle$$