A computationally universal phase of quantum matter

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joint work with D.-S. Wang, D.T. Stephen, C. Okay, and H.P. Nautrup
The liquid phase of water
A quantum phase of spins

... which supports universal quantum computation

We consider:

- Phases of unique ground states of spin Hamiltonians, at $T = 0$
- In the presence of symmetry
Computational phases of quantum matter

physical characterization

mathematical characterization

group cohomology

computational characterization

MBQC power

phase

phase boundary
A quantum phase of spins in 2D

... which supports universal quantum computation

We show:

- There exists a quantum phase of matter which is universal for quantum computation
- The computational power is *uniform* across the phase.
- Employ measurement-based quantum computation
Outline

1. “Computational phases of quantum matter”:
   - Our motivation
   - A short history of the question (1D)

2. A computationally universal phase of matter in 2D
Part I:

A short history of

“computational phases of quantum matter”
Measurement-based quantum computation

Unitary transformation

deterministic, reversible

Projective measurement

probabilistic, irreversible
Information written onto the resource state, processed and read out by one-qubit measurements only.

Universal computational resources exist: cluster state, AKLT state.

Motivation #1: MBQC and symmetry

Can MBQC schemes be classified by symmetry, in a similar way as, say, elementary particles can?
An observation in quantum error-correction

There's good and bad entanglement. Good entanglement often comes with a symmetry.
Motivation #2

How rare are MBQC resource states?
1. MBQC resource states are rare

1. MBQC resource states are rare

\[ \text{Fraction of useful states smaller than } \exp(-n) \]

\[ n: \text{number of qubits} \]

What about systems with symmetry?

In the presence of symmetry

- Computational power is uniform across physical phases (known in 1D, conjectured beyond).
- Computationally useful quantum states are no longer rare.
Symmetry-protected topological order

Definition of SPT phases:

We consider ground states of Hamiltonians that are invariant under a symmetry group $G$. 
Two points in parameter space lie in the same SPT phase iff they can be connected by a path of Hamiltonians such that

1. At every point on the path, the corresponding Hamiltonian is invariant under $G$.

2. Along the path the energy gap never closes.
2. Symmetry protects computation

we observe low-maintenance features of the ground-code MQC in that this computation is doable without an exact (classical) description of the resource ground state as well as without an initialization to a pure state. It turns out these features are deeply intertwined with the physics of the 1D Haldane phase (cf. Fig. 1), that is well characterized as the symmetry-protected topological order in a modern perspective [6, 7]. We believe our approach must bring the study of MQC, conventionally based on the analysis of the model entangled states (e.g., [1, 8, 9]), much closer to the condensed matter physics, which is aimed to describe characteristic physics based on the Hamiltonian.

3. Symmetry-protected wire in MBQC

- Computational wire persists throughout symmetry-protected phases in 1D.
- Imports group cohomology from the classification of SPT phases.


4. The SPT $\Rightarrow$ MBQC meat grinder

Hints at the classification of MBQC schemes by symmetry.

Symmetry’s work and asymmetry’s contribution

In 1D (at least):

- MBQC schemes classified by symmetry
- MBQC schemes operated using symmetry breaking
The above waypoints 2 - 4 are about 1D systems.

1D is not sufficient for universal MBQC

here is why:

- MBQC in spatial dimension $D$ maps to the circuit model in dimension $D - 1$

$\Rightarrow$ Require $D \geq 2$ for universality.
Are there computationally universal quantum phases in two dimensions?

This talk describes one.
Part II:

A computationally universal SPT phase in 2D
Description of the 2D phase & result

• The symmetries of the phase are

• The 2D cluster state is inside the phase

Result. For a spin-1/2 lattice on a torus with circumferences $n$ and $Nn$, with $n$ even, all ground states in the 2D cluster phase, except a possible set of measure zero, are universal resources for measurement-based quantum computation on $n/2$ logical qubits.
Consider MBQC resource states as tensor networks
Cluster-like states

... have PEPS tensors with the following symmetries

\[
\begin{align*}
X &\otimes I = X \\
Z &\otimes I = Z \\
Z &\otimes I = Z \\
X &\otimes I = X
\end{align*}
\]

The cluster states have the additional symmetry

\[
\begin{align*}
Z &\otimes I = Z \\
X &\otimes I = X
\end{align*}
\]

(We do not require the latter symmetry for cluster-like states)
Splitting the problem into halves

Part A:

**Lemma 1.** All states in the 2D cluster phase are cluster-like.

Part B:

**Lemma 2.** All cluster-like states, except a set of measure zero, are universal for MBQC.
Part A: PEPS tensor symmetries

The physical symmetries in the 2D cluster phase imply the local PEPS tensor symmetries,

\[ X \times X = X, \quad Z = X = Z, \quad Z = Z \times X \]
Lemma 3. [*] Symmetric gapped ground states in the same SPT phase are connected by symmetric local quantum circuits of constant depth.

For any state $|\Phi\rangle$ in the phase,

$$|\Phi\rangle = U_k U_{k-1} \ldots U_1 |2D\text{ cluster}\rangle.$$ 

Look at an individual symmetry-respecting gate in the circuit,

$$U = \sum_j c_j T_j, \text{ with } T_j \in \mathcal{P}.$$ 

Which Pauli observables $T_j$ can be admitted in the expansion?

A: In cluster phase $\Rightarrow$ cluster-like

Which Paulis $T_j$ can be admitted in the expansion $U = \sum_j c_j T_j$?
A: In cluster phase $\Rightarrow$ cluster-like

Which Paulis $T_j$ can be admitted in the expansion $U = \sum_j c_j T_j$?

Only $X$-type Pauli operators survive in the expansion.
Description of the local tensors:

\[ C = A \Phi B \Phi \]

With

\[ \text{equivalent operators} \]

Hence

\[ C = B \Phi X B \Phi = \Phi \]

- Local tensors \( A_\Phi \) describing \( |\Phi\rangle \) are invariant under the cluster-like symmetries.
• The “virtual” quantum register is located on the horizontal tensor legs

Part B: Symmetry Lego

Just shown:
PEPS tensor symmetries hold throughout the 2D cluster phase

- Now weave them into larger patterns.
B: Cluster-like $\Rightarrow$ universal

The clock cycle:

- Every logical operator is mapped back to itself after $n$ columns ($n =$ circumference).

$\Rightarrow$ This defines the clock cycle for gate operation.
B: Cluster-like $\Rightarrow$ universal

- Map 2D system to effective 1D system
B: Cluster-like $\Rightarrow$ universal

Universal gate set on $n/2$ qubits
B: Cluster-like $\Rightarrow$ universal

2D cluster state:

Throughout the phase:

$$e^{i|\nu|d\alpha Z_k} \quad e^{i|\nu|d\alpha X_{k-1}Z_kX_{k+1}} \quad e^{i|\nu|d\alpha X_k}$$

$$|\nu| \leq 1$$

($\nu$ depends on the location in the phase)

Result

- The symmetries of the phase are

  ![Symmetry Diagram](image)

- The 2D cluster state is inside the phase

**Result.** For a spin-1/2 lattice on a torus with circumferences $n$ and $Nn$, with $n$ even, all ground states in the 2D cluster phase, except a possible set of measure zero, are universal resources for measurement-based quantum computation on $n/2$ logical qubits.
Summary and outlook

- There exists a symmetry-protected phase in 2D with uniform universal computational power for MBQC.

- **Goal**: Classification of MBQC schemes by symmetry.

- **Symmetry Lego is fun—Try it!**


Also see: **Quantum** 3, 142 (2019)
The parameter $\nu$

There is a complex-valued parameter $\nu$, $|\nu| \leq 1$, that needs to be known about the location of the resource state within the phase.

For infinitesimal angles $d\beta$, this results in a logical rotation [\*]

$e^{id\beta |\nu| T}$

for some Pauli operator $T$. (E.g., $T = Z_k, X_k, X_{k-1}Z_kX_{k+1}$).

We require that $\nu \neq 0$.