Direct Expression of Mutual Information of Distant Regions

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1.Introduction

Entanglement Entropy (EE) is the quantity which measures the degree of entanglement.

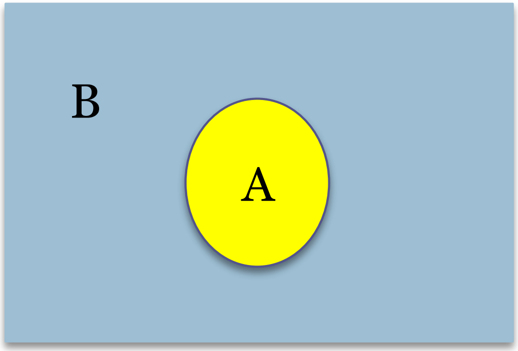
Entanglement entropy (EE) is generally defined as the von Neumann entropy

$$S_A := -tr_A \rho_A \log \rho_A$$

corresponding to the reduced density matrix of a subsystem A .

The Renyi entropy is the generalization of EE and defined as

$$S_A^{(n)} := \frac{1}{1-n} \log Tr(\rho_A{}^n) \qquad S_A = \lim_{n \to 1} S_A^{(n)}$$

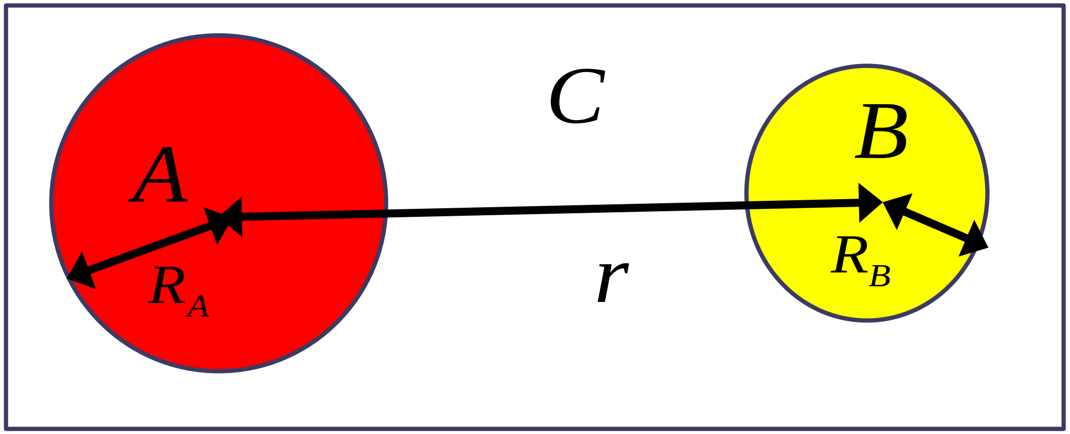


Mutual Information (MI)

Mutual information (MI) measures the correlation between two subsystems.
The mutual Renyi information is defined as,

$$I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)}$$

$$I(A,B) = \lim_{n\to 1} I^{(n)}(A,B)$$

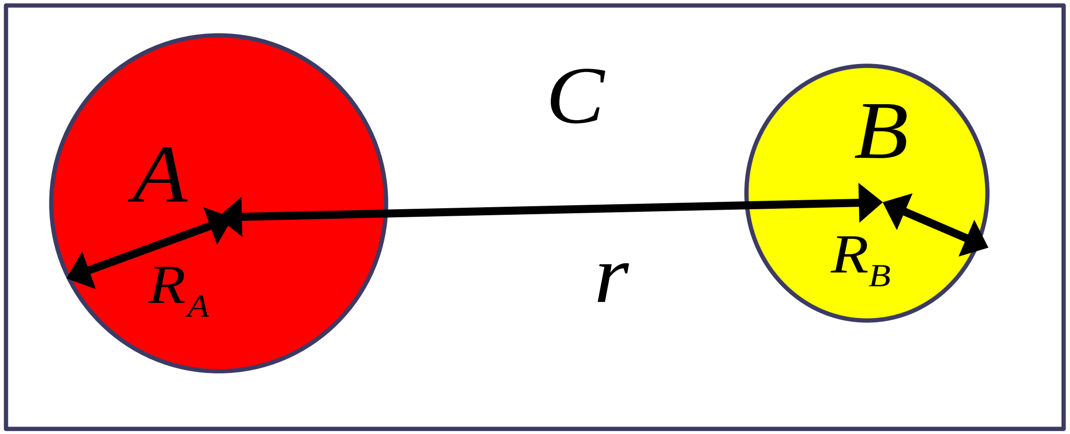


Today's talk

We consider the mutual (Renyi) information of distant compact spatial regions A and B in the vacuum state of a free scalar field. The distance r between A and B is much greater than their sizes $R_{A,B}$.

It is known that the mutual information is proportional to the square of the correlation function,

$$I^{(n)}(A,B) \approx C_{AB}^{(n)} \langle 0|\phi(r)\phi(0)|0\rangle^2 \qquad r \gg R_{A,B}$$



$$I^{(n)}(A,B) \approx C_{AB}^{(n)} \langle 0|\phi(r)\phi(0)|0\rangle^2 \qquad r \gg R_{A,B}$$

When both A and B are the spheres and the scalar field is massless, the coefficient $C_{AB}^{(n)}$ was calculated analytically.

Cardy, 2013

However, when both A and B are not the spheres or the dispersion relation of the scalar field is general, it is difficult to calculate the coefficient $C_{AB}^{(n)}$ analytically.

In this work, we obtain the direct expression of $C_{AB}^{(n)}$ for arbitrary regions A and B in free scalar fields which have general dispersion relations.

Comparison with the real time formalism

EE in free scalar fields can be calculated numerically by the real time formalism.

e.g. Bombelli et al. 1986, Srednicki 1993

In order to calculate the coefficient by the real time formalism, we have to plot the mutual information as a function of r and extract the coefficient.

So we have to calculate numerically $S_{A\cup B}$ many times to plot the mutual information as a function of r.

On the other hand, in our method, we separate the r dependence of the mutual information analytically and obtain the direct expression of the coefficient.

So, it reduces significantly the amount of computation.

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2.An operator method in EE

Shiba 2014

We consider the general scalar field in (d+1) dimensional spacetime and do not specify its Hamiltonian.

We consider n copies of the scalar fields and the j-th copy of the scalar field is denoted by $\{\phi^{(j)}\}$.
Thus the total Hilbert space, $H^{(n)}$, is the tensor product of the n copies of the Hilbert space,

$$H^{(n)} = H \otimes H \cdots \otimes H$$

where H is the Hilbert space of one scalar field.
We define the density matrix $\rho^{(n)}$ in $H^{(n)}$ as

$$\rho^{(n)} = \rho \otimes \rho \cdots \otimes \rho$$

where ρ is an arbitrary density matrix in H.
We can express $Tr\rho_{\Omega}{}^{n}$ as

$$Tr\rho_{\Omega}{}^{n} = Tr(\rho^{(n)} E_{\Omega})$$

$$|\psi^{(n)}\rangle = |\psi\rangle|\psi\rangle \ldots |\psi\rangle$$

for a pure state $\rho = |\psi\rangle\langle\psi|$

$$Tr\rho_{\Omega}{}^{n} = \langle\psi^{(n)}|E_{\Omega}|\psi^{(n)}\rangle$$

$$Tr\rho_{\Omega}{}^{n} = Tr(\rho^{(n)} E_{\Omega})$$

$$E_{\Omega} = \int \prod_{j=1}^{n} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) \exp\left[i \int d^d x \sum_{l=1}^{n} J^{(l+1)}(x)\phi^{(l)}(x) \right]$$
$$\times \exp\left[i \int d^d x \sum_{l=1}^{n} K^{(l)}(x)\pi^{(l)}(x) \right] \times \exp\left[-i \int d^d x \sum_{l=1}^{n} J^{(l)}\phi^{(l)} \right]$$

Shiba 2014

where $\pi(x)$ is a conjugate momenta of $\phi(x)$,
$[\phi(x), \pi(y)] = i\delta^d(x-y)$ and

$J^{(j)}(x)$ and $K^{(j)}(x)$ exist only in Ω and $J^{(n+1)} = J^{(1)}$
$J^{(j)}(x)$ and $K^{(j)}(x)$ are auxiliary fields .

Thus E_{Ω} is an operator at Ω .

We call E_{Ω} as a glueing operator.

General properties of E_Ω

(1) Symmetry:

$$E_\Omega(\phi^{(1)},\ldots,\phi^{(n)},\pi^{(1)},\ldots,\pi^{(n)}) = E_\Omega(-\phi^{(1)},\ldots,-\phi^{(n)},-\pi^{(1)},\ldots,-\pi^{(n)}).$$

(2) Locality: when $\Omega = A \cup B$ and $A \cap B = 0$

$$E_{A\cup B} = E_A E_B$$

From the locality, the mutual Renyi information in the vacuum state can be expressed as the correlation function of E,

$$\frac{Tr\rho_{A\cup B}{}^n}{Tr\rho_A{}^n Tr\rho_B{}^n} = \frac{\langle 0^{(n)}|E_\mathrm{A}E_\mathrm{B}|0^{(n)}\rangle}{\langle 0^{(n)}|E_\mathrm{A}|0^{(n)}\rangle\langle 0^{(n)}|E_\mathrm{B}|0^{(n)}\rangle}$$

$$I^{(n)}(A,B) = \frac{1}{n-1}\ln\frac{Tr\rho_{A\cup B}{}^n}{Tr\rho_A{}^n Tr\rho_B{}^n}$$

Normal ordering and expansion of E_Ω

For free scalar fields, it is useful to represent the operator E_Ω as the normal ordered operator.

$$E_\Omega = \int \prod_{j=0}^{n-1} \prod_{x\in\Omega} DJ^{(j)}(x) DK^{(j)}(x) : \exp[i \sum_{l=0}^{n-1} \int d^d x ((J^{(l+1)} - J^{(l)})\phi^{(l)*} + K^{(l)}\pi^{(l)*}$$

$$+ (J^{(l+1)*} - J^{(l)*})\phi^{(l)} + K^{(l)*}\pi^{(l)})] : \exp[-\tilde{S}],$$

$$\tilde{S} \equiv \sum_{l=1}^{n} [\int d^d x d^d y [\frac{1}{2} K^{(l)}(x) A(x,y) K^{(l)*}(y) + \frac{1}{2}(J^{(l+1)} - J^{(l)})(x) D(x,y) (J^{(l+1)*} - J^{(l)*})(y)]$$

$$+ \frac{i}{2} \int d^d x (K^{(l)}(x)(J^{(l+1)*} + J^{(l)*})(x) + K^{(l)*}(x)(J^{(l+1)} + J^{(l)})(x))].$$

$$\langle 0| \phi(x)\phi(y) |0\rangle = \int \frac{d^d p}{(2\pi)^d} \frac{1}{2E_p} e^{ip(x-y)} \equiv \frac{1}{2} D(x,y),$$

$$\langle 0| \pi(x)\pi(y) |0\rangle = \int \frac{d^d p}{(2\pi)^d} \frac{E_p}{2} e^{ip(x-y)} \equiv \frac{1}{2} A(x,y),$$

Here, $\phi^{(l)}(x)$ is a free complex scalar field.

We consider a complex scalar field because it is useful for later calculation.

We use the following Fourier transformation,

$$f^{(l)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i k l / n} \tilde{f}^{(k)}$$

Then, we rewrite the operator as,

$$E_\Omega = \prod_{k=0}^{n-1} E_\Omega^{(k)}$$

$$E_\Omega^{(k)} = \int \prod_{x \in \Omega} D\tilde{J}^{(k)}(x) D\tilde{K}^{(k)}(x) : \exp[iQ^{(k)}] : \exp[-\tilde{S}^{(k)}]$$

$$Q^{(k)} \equiv \int d^d x [(e^{2\pi i k/n} - 1)\tilde{J}^{(k)}\tilde{\phi}^{(k)*} + (e^{-2\pi i k/n} - 1)\tilde{J}^{(k)*}\tilde{\phi}^{(k)} + \tilde{K}^{(k)}\tilde{\pi}^{(k)*} + \tilde{K}^{(k)*}\tilde{\pi}^{(k)}]$$

$$\tilde{S}^{(k)} \equiv \int d^d x d^d y [\frac{1}{2}\tilde{K}^{(k)}(x) A(x,y) \tilde{K}^{(k)*}(y) + \frac{1}{2}(1 - \cos(\frac{2\pi k}{n}))\tilde{J}^{(k)}(x) D(x,y) \tilde{J}^{(k)*}(y)]$$
$$+ \frac{i}{2} \int d^d x ((e^{-2\pi i k/n} + 1)\tilde{K}^{(k)}(x)\tilde{J}^{(k)*}(x) + (e^{2\pi i k/n} + 1)\tilde{K}^{(k)*}(x)\tilde{J}^{(k)}(x)).$$

By expanding the exponential in the normal ordered product and performing the Gauss integral of J and K, we can rewrite the E_Ω as a series of operators.

$$\frac{E_\Omega^{(k)}}{\langle 0| E_\Omega^{(k)} |0\rangle} = 1- :\tilde{\phi}^{(k)*}(x_0)\tilde{\phi}^{(k)}(x_0): C_\Omega^{(k)} + \cdots$$

$$C_\Omega^{(k)} \equiv \left(2 - 2\cos\left(\frac{2\pi k}{n}\right)\right) \int d^d x d^d y \langle \tilde{J}^{(k)}(x)\tilde{J}^{(k)*}(y)\rangle$$

$$\langle \tilde{J}^{(k)}(x)\tilde{J}^{(k)*}(y)\rangle \equiv \frac{\int \prod_{x\in\Omega} D\tilde{J}^{(k)}(x) D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}]\tilde{J}^{(k)}(x)\tilde{J}^{(k)*}(y)}{\int \prod_{x\in\Omega} D\tilde{J}^{(k)}(x) D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}]}$$

$$= \left(A^{-1} + D + \cos\left(\frac{2\pi k}{n}\right)(A^{-1} - D)\right)^{-1}(x,y)$$

$$\frac{E_\Omega^{(k)}}{\langle 0|\, E_\Omega^{(k)}\, |0\rangle} = 1- :\tilde{\phi}^{(k)*}(x_0)\tilde{\phi}^{(k)}(x_0): C_\Omega^{(k)} + \cdots$$

In order to separate the n dependence of $C_\Omega^{(k)}$,

we use the following matrices,

$$X \equiv (A^{-1}+D)^{-1/2}, \quad Y \equiv X(D-A^{-1})X$$

$$Y = O^T \Lambda O, \quad \Lambda = \mathrm{diag}(\lambda_i)$$

Then, we can rewrite $C_\Omega^{(k)}$ as,

$$C_\Omega^{(k)} = \left(2-2\cos\left(\frac{2\pi k}{n}\right)\right)\sum_i\sum_j\sum_l Z_{li}\frac{1}{1-\lambda_l\cos\left(\frac{2\pi k}{n}\right)}Z_{lj}$$

$$Z = OX$$

3. Application to the mutual (Renyi) information of distant compact spatial regions

$$I^{(n)}(A,B) = \frac{1}{n-1} \ln \frac{Tr\rho_{A\cup B}{}^{n}}{Tr\rho_{A}{}^{n} Tr\rho_{B}{}^{n}}$$

$$\simeq \frac{f(r)}{n-1} \sum_{k=0}^{n-1} C_A^{(k)} C_B^{(k)}$$

$$f(r) \equiv \langle 0| :\tilde{\phi}^{(k)*}(x_A)\tilde{\phi}^{(k)}(x_A) :: \tilde{\phi}^{(k)*}(x_B)\tilde{\phi}^{(k)}(x_B) :) |0\rangle$$

$$= \langle 0|\phi(r)\phi(0)|0\rangle^2$$

$$I^{(n)}(A,B) \approx C_{AB}^{(n)} \langle 0|\phi(r)\phi(0)|0\rangle^2 \qquad r \gg R_{A,B}$$

$$C_{AB}^{(n)} \simeq \frac{4}{n-1} \sum_{i_A} \sum_{j_A} \sum_{l_A} \sum_{i_B} \sum_{j_B} \sum_{l_B} Z_{l_A i_A}^{(A)} Z_{l_A j_A}^{(A)} Z_{l_B i_B}^{(B)} Z_{l_B j_B}^{(B)} F(n, \lambda_{l_A}^{(A)}, \lambda_{l_B}^{(B)})$$

$$F(n,a,b) \equiv \sum_{k=0}^{n-1} \left(1 - \cos\left(\frac{2\pi k}{n}\right)\right)^2 \frac{1}{1 - a\cos\left(\frac{2\pi k}{n}\right)} \frac{1}{1 - b\cos\left(\frac{2\pi k}{n}\right)}$$

$$\left(\frac{\partial}{\partial n} F(n,a,b)\right)\bigg|_{n=1} = \frac{1}{2} \frac{(1+p^2)(1+q^2)}{(1+p)(1+q)(p-q)(1-pq)} \left[(1-p)(1+q)\ln p - (1+p)(1-q)\ln q\right]$$

$$p \equiv \rho(a) = \frac{1}{a}(1 - \sqrt{1-a^2}), \quad q \equiv \rho(b) = \frac{1}{b}(1 - \sqrt{1-b^2})$$

4.Conclusion

$$I^{(n)}(A,B) \approx C_{AB}^{(n)} \langle 0|\phi(r)\phi(0)|0\rangle^2 \qquad r \gg R_{A,B}$$

In this work, we obtain the direct expression of $C_{AB}^{(n)}$ for arbitrary regions A and B in free scalar fields which have general dispersion relations.

The direct expression is useful for the numerical computation.

It reduces significantly the amount of computation in comparison with the computation by the real time formalism.

Application to fermionic fields, negativity, and so on.