Fundamental exchange rate between coherence and asymmetry

H. Tajima, N. Shiraishi and K. Saito
Phys. Rev. Lett. **121**, 110403 (2018)

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Hiroyasu Tajima
@Kyoto university (YITP)

Collaborators:

N. Shiraishi
@Gakushuin university

K. Saito
@Keio university

Topic:

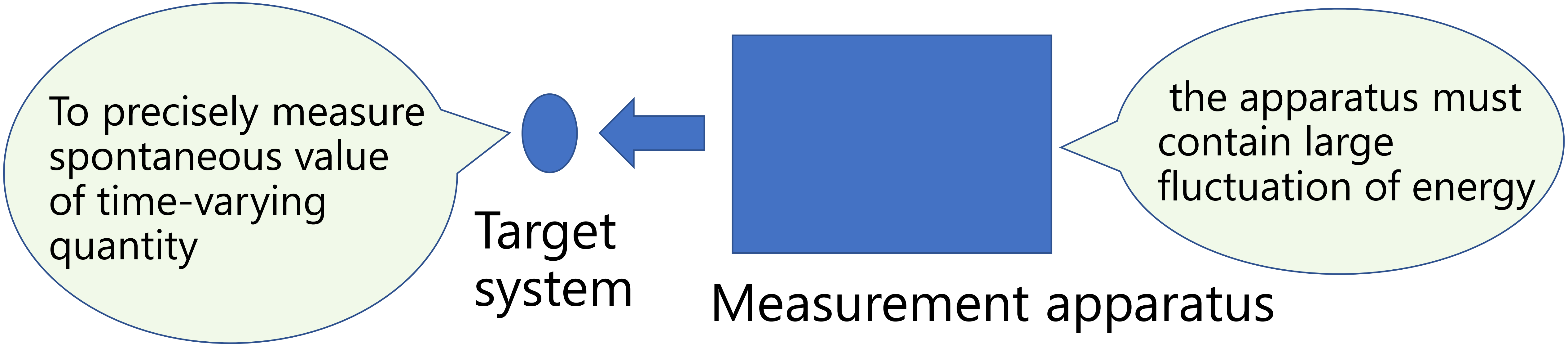
Resource cost for quantum operations under conservation laws

Restrictions imposed by conservation laws

conservation laws → restrict → operations

Some restrictions are about resource cost for operations

Example: Wigner-Araki-Yanase theorem



To perform precise measurement, we need large fluctuation of energy as a resource.

Restriction on unitary dynamics?

Is there any restriction similar to Wigner-Araki-Yanase theorem on implementing unitary dynamics under conservation laws?

initially proposed by Masanao Ozawa, about two decades ago:

M. Ozawa, Phys. Rev. Lett. **89**, 057902 (2002).

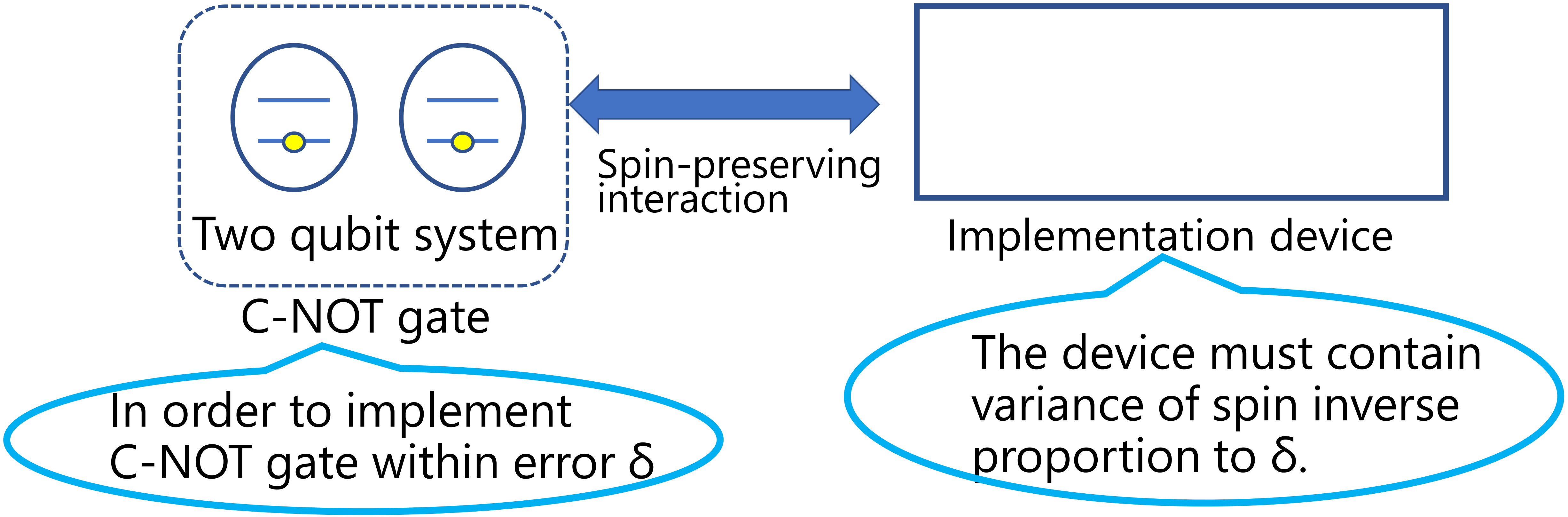
The motivation is to clarify the restrictions on quantum computing imposed by conservation laws.

Restriction on C-NOT gate: Ozawa's result

Ozawa considers the implementation of Controlled-NOT gate under spin-preserving interaction.

M. Ozawa, Phys. Rev. Lett. **89**, 057902 (2002).

Ozawa obtain a trade-off inequality between error and fluctuation for Controlled-NOT gate. (With using Wigner-Araki-Yanase theorem!)



Restriction on general unitary gate: A long standing open problem

After Ozawa's result, similar trade-off relations were given for various (but specific) unitary gates:

Not gate and Fredkin gate:

T. Karasawa and M. Ozawa, Phys. Rev. A **75**, 032324 (2007).

Hadamard gate:

M. Ozawa, Int. J. Quant. Inf. **1**, 569 (2003).

Question :

Is there any universal trade-off between fluctuation and error for general unitary, other than qubit gates ?

Although the above strong circumstantial evidence, the trade-off was never given.

Our result 1: An answer to the long standing open problem

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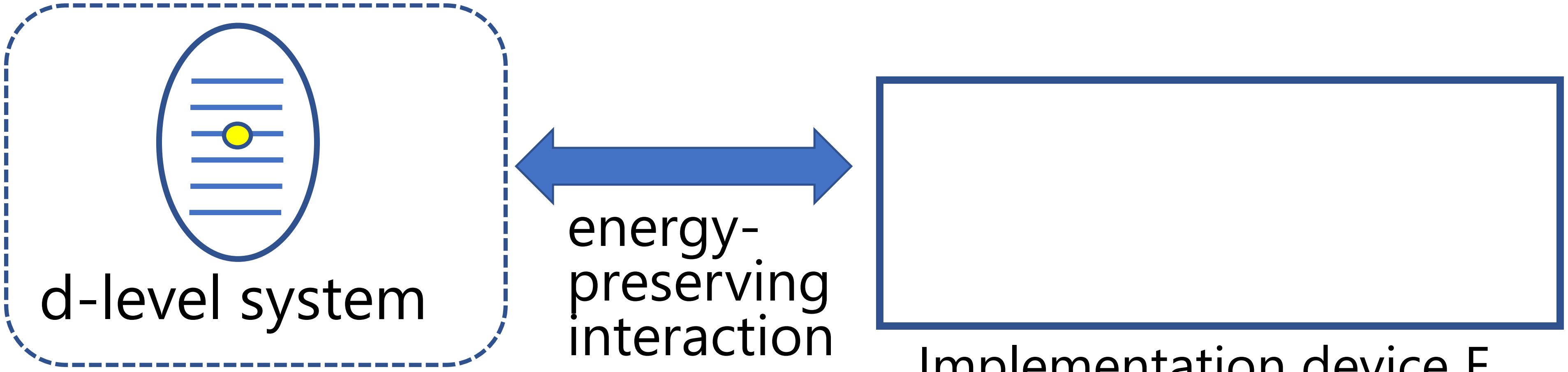
We consider the implementation of arbitrary unitary gate under conservation law of energy.

We derive a universal trade-off inequality between fluctuation of energy and implementation error of unitary operations. (without using Wigner-Araki-Yanase theorem)



Arbitrary unitary gate U_S

Implementation error δ

Variance of energy of E V

Trade-off:

$$\sqrt{V} \geq \frac{\|[U_S, H_S]\|}{40\delta}$$

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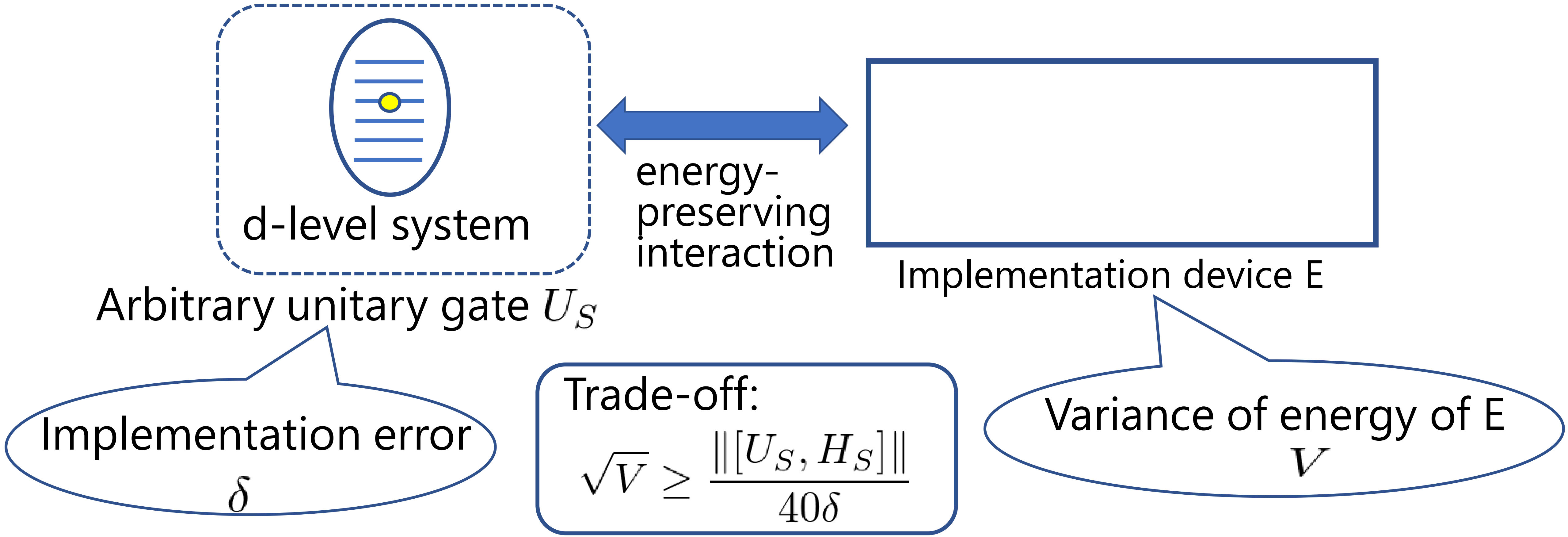

Our result 2: An answer to the long standing open problem

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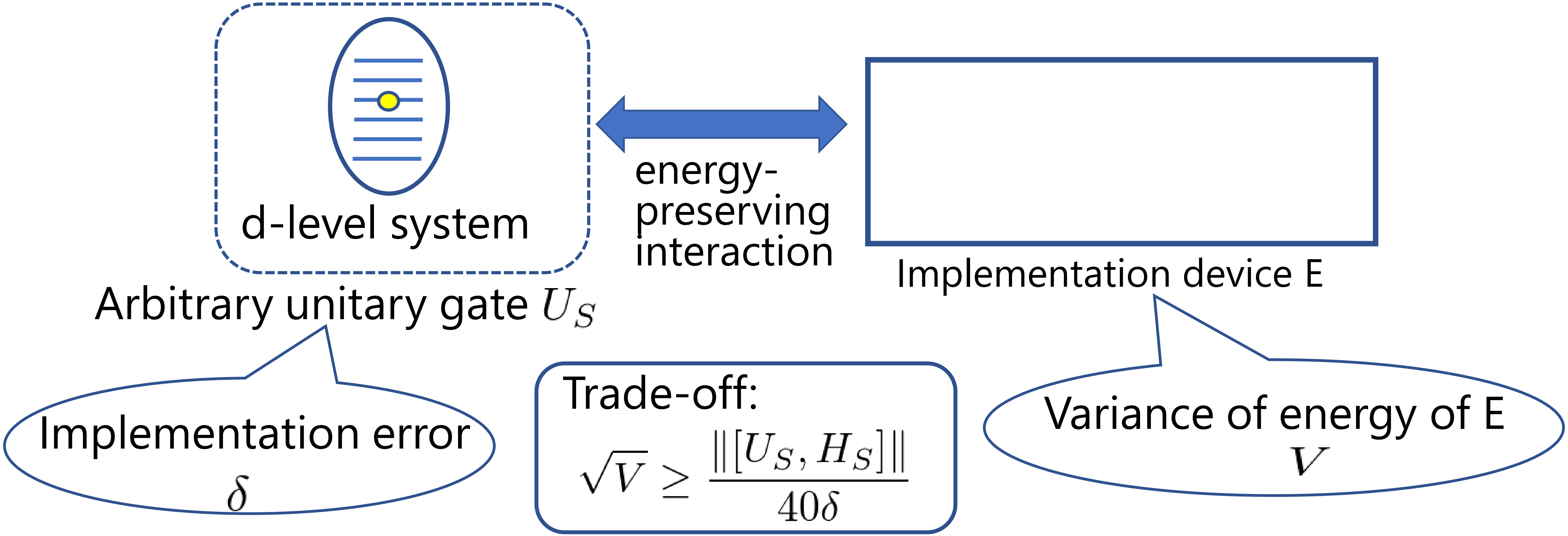
We also show that the required fluctuation must have quantum origin.



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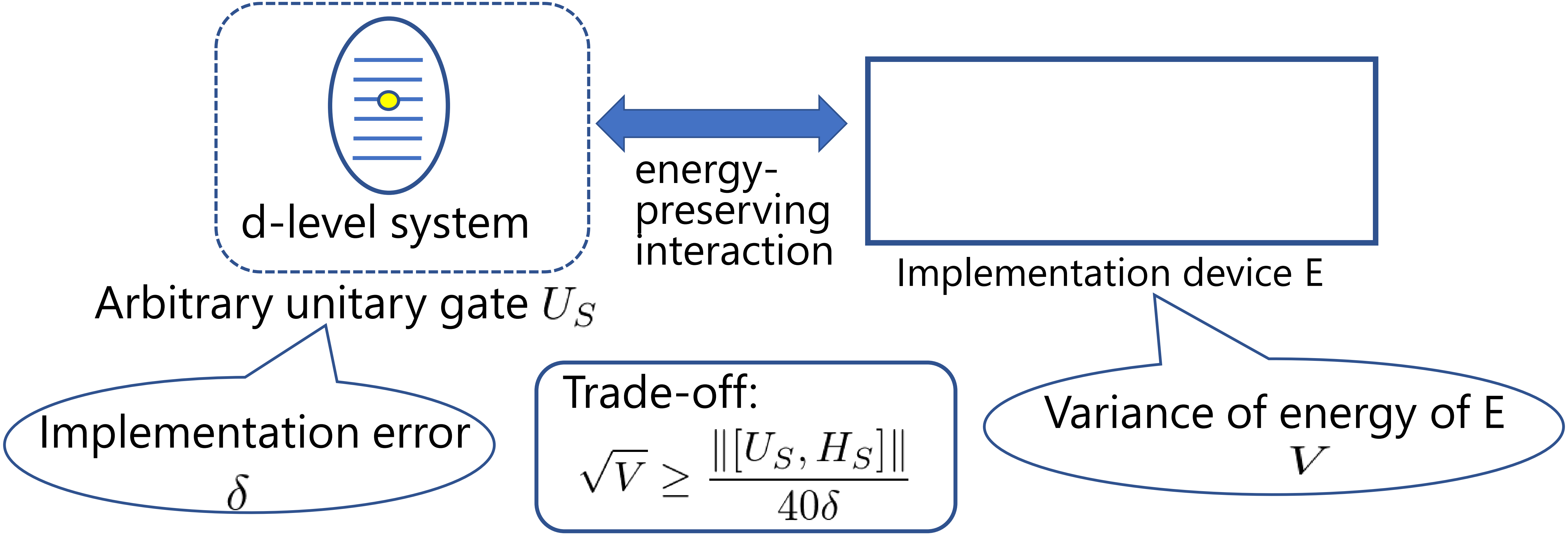
We derive another trade-off between implementation error and quantum Fisher information, which is a well-known measure of quantum fluctuation.



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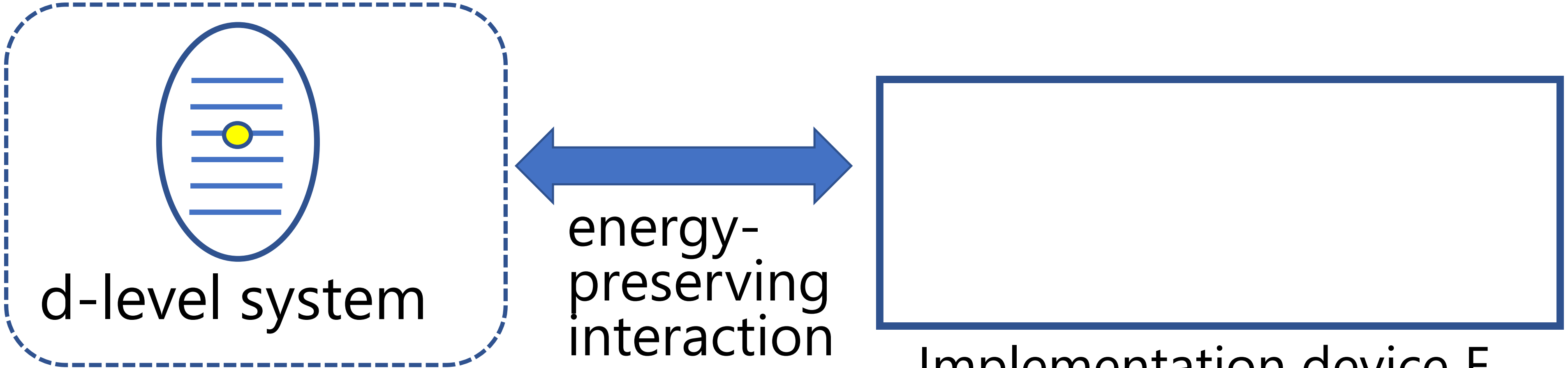

Arbitrary unitary gate U_S

Implementation error δ

Trade-off2:

$$\sqrt{\mathcal{F}} \geq \frac{\|[U_S, H_S]\|}{81\delta}$$

Quantum Fisher information of energy of E $\mathcal{F}$

Quantum Fisher information: A measure of coherence

$$\mathcal{F}_A(\rho) := 2\sum_{a,b}\frac{(p_a-p_b)^2}{p_a+p_b}|A_{ab}|^2$$

$$A_{ab} := \langle\psi_a|\,A\,|\psi_b\rangle$$

$\{p_a,\psi_a\}$ is eivenvalues and eivenvectors of ρ

Important feature：$\mathcal{F}_A(\rho) = 4 \min_{\{q_j,\phi_j\}:\rho=\sum_j q_j\phi_j} \sum_j q_j V_A(\phi_j).$

ρ is pure ➡ $\mathcal{F}_A(\rho) = 4V_A(\rho)$

$[\rho, A] = 0$ ➡ $\mathcal{F}_A(\rho) = 0$

Namely, QFI is “quantum part” of fluctuation of the physical quantity A.

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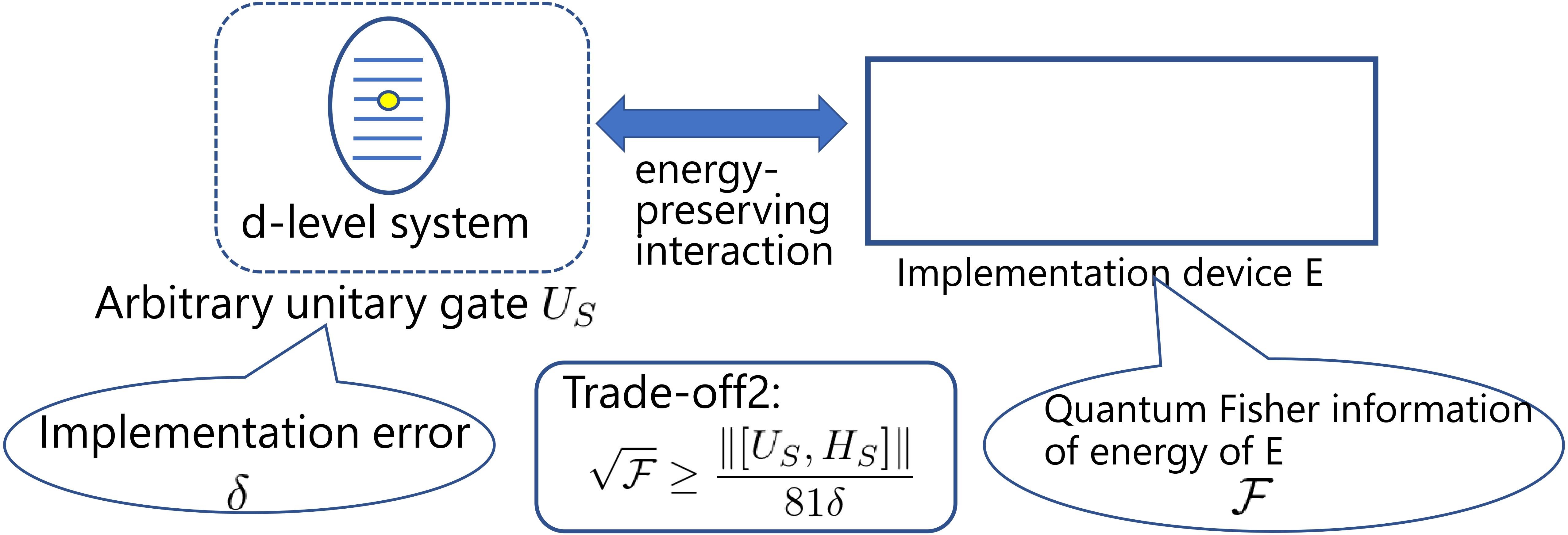

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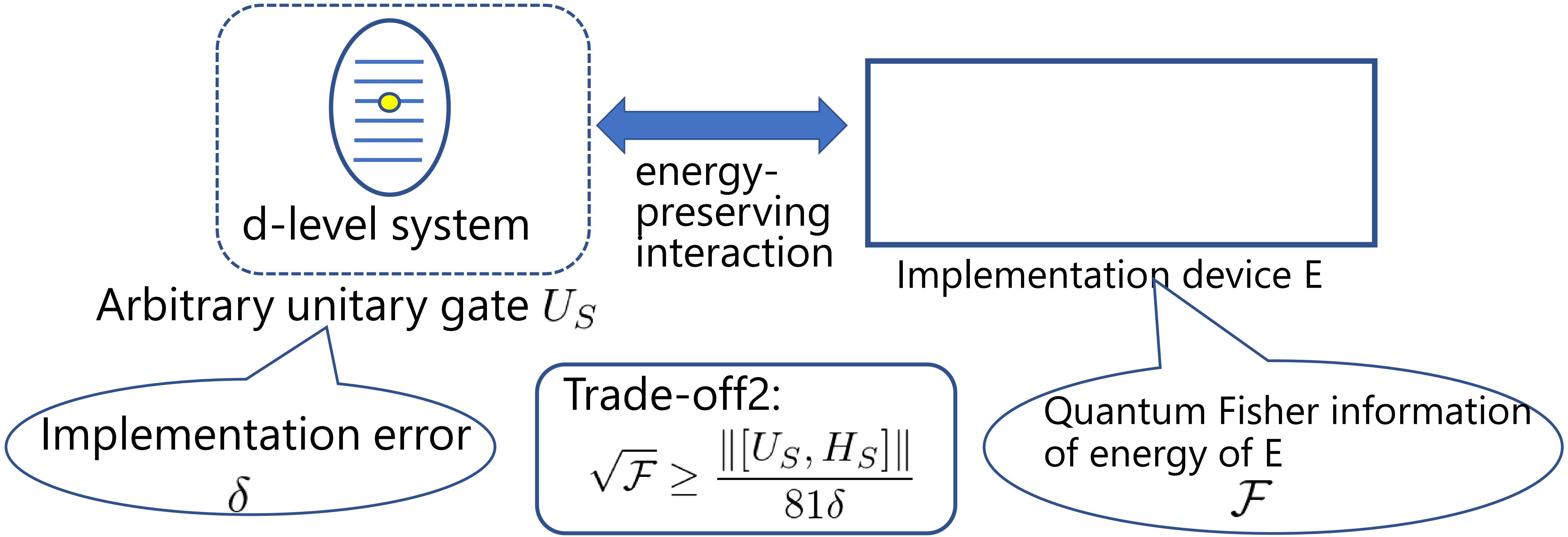

The remaining question: a generalization of Ozawa's question

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Quantum Fisher information $\mathcal{F}$ is a measure of coherence.

So, Trade-off 2 is a lower bound for coherence necessary to implement unitary dynamics under conservation law.

Question':
How much coherence is "necessary and sufficient" to implement unitary dynamics under conservation law?

This is a generalization of Ozawa's question.

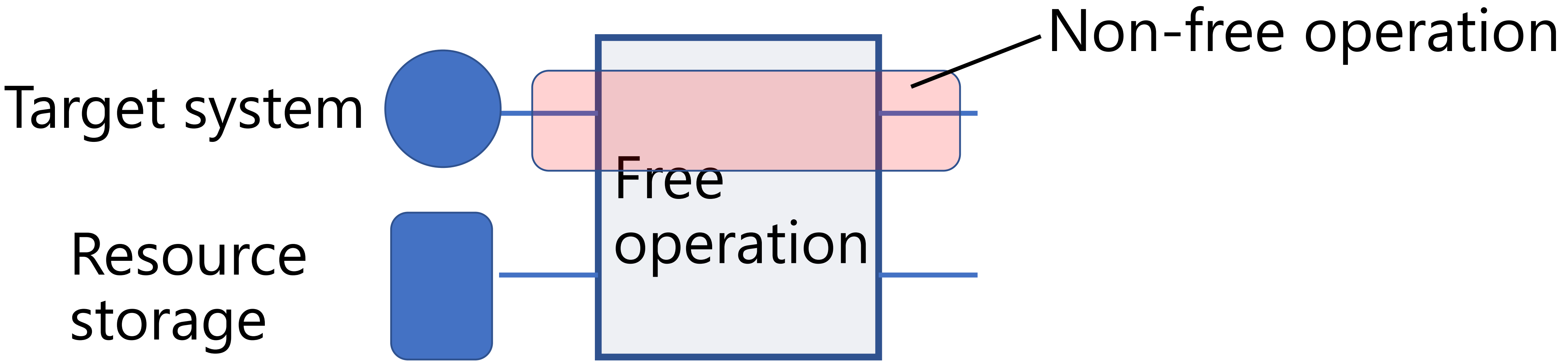
Trade-off2:

$$\sqrt{\mathcal{F}} \geq \frac{\|[U_S, H_S]\|}{81\delta}$$

Approach from quantum information
-resource theory of quantum channels

Free operations and free states: we can use freely

Resource states:
the states we cannot create from free operations and free states



Key question of resource theory of quantum channels:
How much resource do we need to implement the desired operations?

Approach from quantum information -resource theory of quantum channels

Partially solved in various cases

Quantum thermodynamics:

P. Faist and R. Renner. Phys. Rev. X, **8** 021011, (2018).
P. Faist, M. Berta and F. Brandao, Phys. Rev. Lett. **122**, 200601 (2019).

Resource erasure:

Z.-W. Liu and A. Winter, arXiv:1904.04201 (2019).

Incoherent operations:

M. G. Diaz, K. Fang, X. Wang, M. Rosati, M. Skotiniotis, J. Calsamiglia and A. Winter, Quantum **2**, 100 (2018).

Upper and lower bounds for "necessary and sufficient" resource to implement the desired operations

Key question of resource theory of quantum channels:
How much resource do we need to implement the desired operations?

Position of our question

Solved by us!
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Ozawa's Question :

Is there any universal trade-off between fluctuation and error for implementing unitary dynamics under conservation law?

generalization

unsolved

Our Question :

How much coherence is "necessary and sufficient" to implement unitary dynamics under conservation law?

we solve here!
arXiv:1906.04076 (2019)

Special case

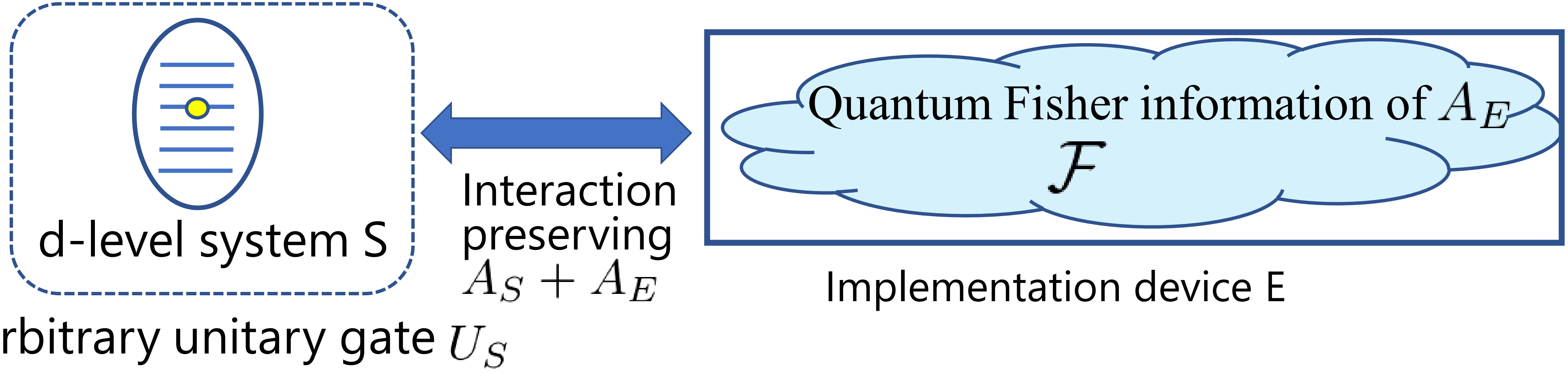
unsolved

Key question of resource theory of quantum channels:
How much resource do we need to implement the desired operations?

Situation that we treat (detail is shown later)

arXiv:1906.04076 (2019)

Situation:

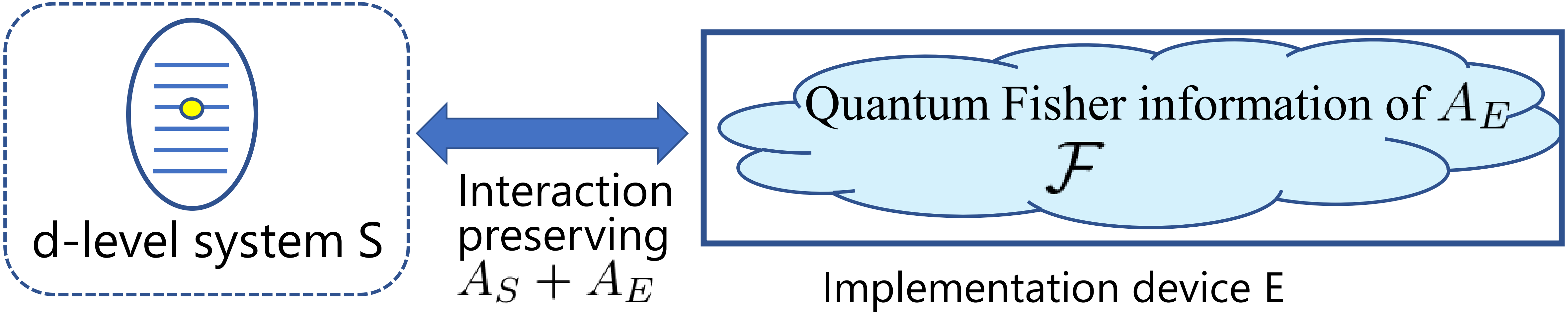


We derive “necessary and sufficient” $\mathcal{F}$ to implement U_S within error δ.

Our result

arXiv:1906.04076 (2019)

Situation:



Arbitrary unitary gate U_S

$$\mathcal{F}_{U_S,\delta} := \left(\text{"necessary and sufficient"}\ \mathcal{F} \text{ to implement } U_S \text{ within error } \delta. \right)$$

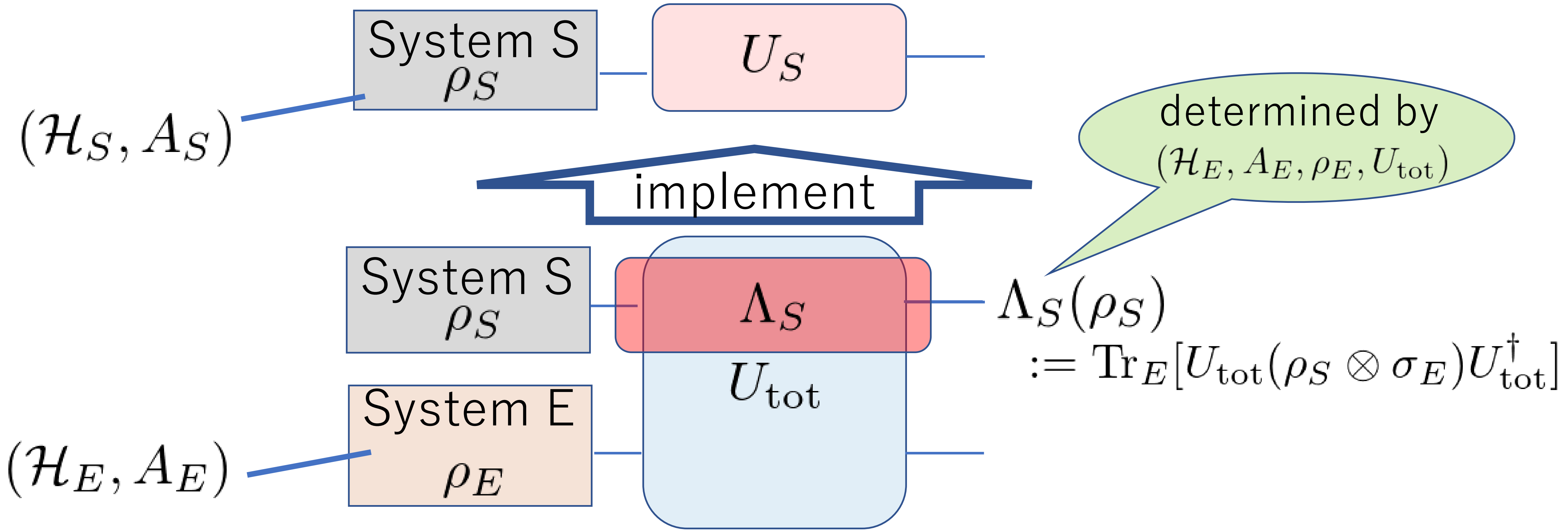
Result:

$$\sqrt{\mathcal{F}_{U_S,\delta}} = \frac{\mathcal{A}_{U_S}}{\delta} + O(\|A_S\|)$$

$\mathcal{A}_{U_S}$: degree of how $[U_S, A_S]$ is far from 0.

Situation that we treat (details)

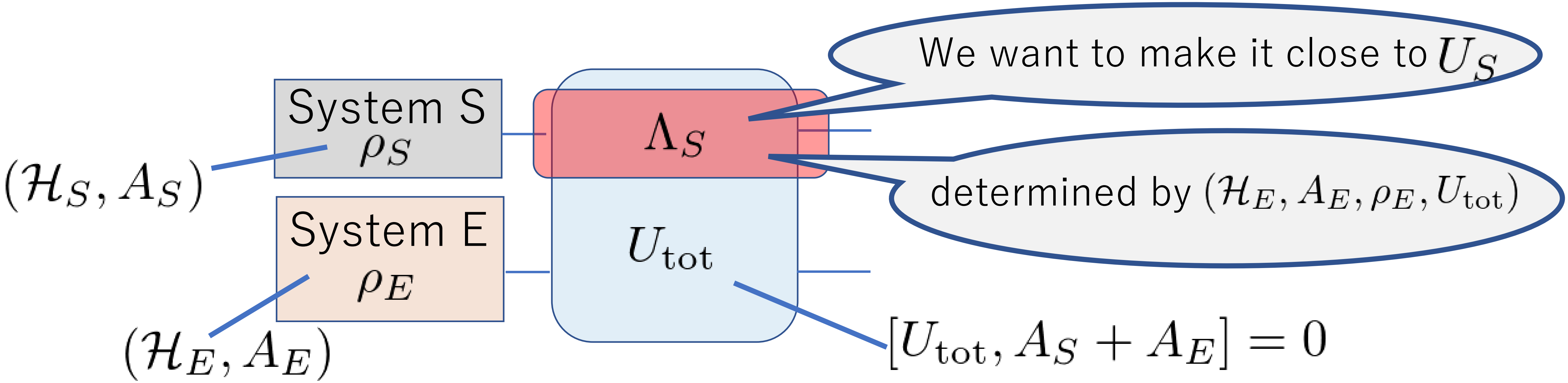
We approximately implement U_S on the target system S by the interaction with an external system E.



Under the restriction $[U_{\rm tot}, A_S + A_E] = 0$,

we take $(\mathcal{H}_E, A_E, \rho_E, U_{\rm tot})$ freely, and try to make Λ_S close to U_S .

Situation that we treat (details)



Under this setup, we define the following three quantities:

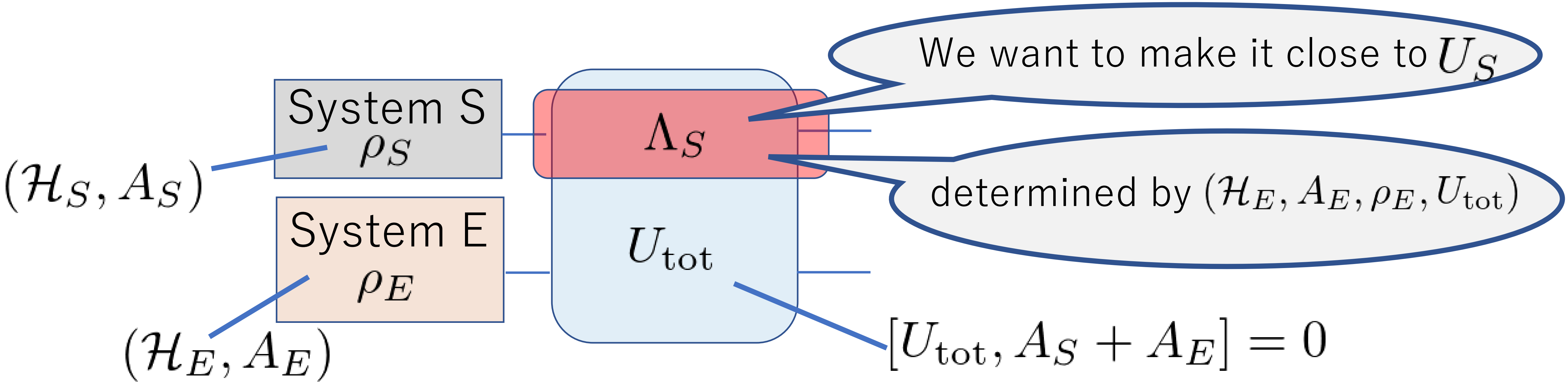
degree of how $[U_S, A_S]$ is far from 0

$$\sqrt{\mathcal{F}_{U_S,\delta}} = \frac{\mathcal{A}_{U_S}}{\delta} + O(\|A_S\|)$$

“necessary and sufficient” amount of Coherence to implement U_S

implementation error

Situation that we treat (details)



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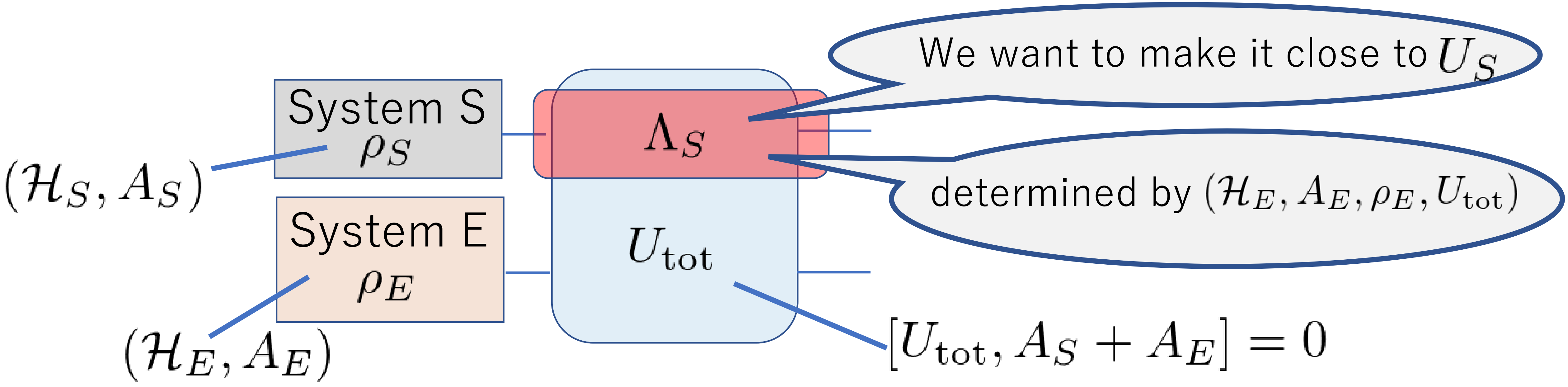


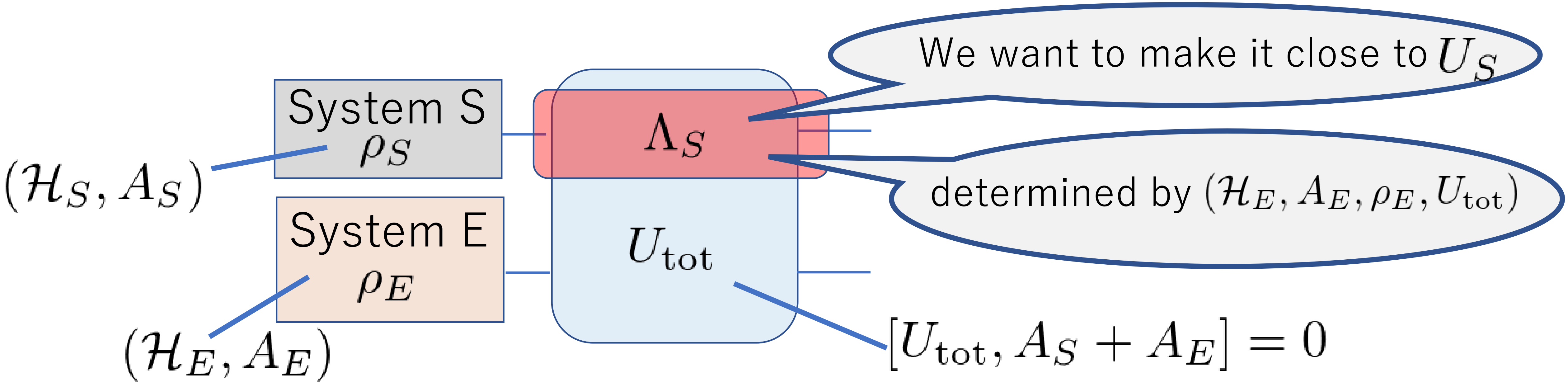
We define δ as maximal entanglement Bures distance between $U_S \rho_S U_S^\dagger$ and $\Lambda_S(\rho_S)$:

$\mathcal{I} = (\mathcal{H}_E, A_E, \rho_E, U_{\text{tot}})$ implements U_S within error δ

$\overset{\text{def}}{\Longleftrightarrow}$ $\delta \geq \max_{\rho_S} L_e(\rho_S, \Lambda_{U_S^\dagger} \circ \Lambda_S)$

$$\Lambda_{U_S^\dagger}(\rho) := U_S^\dagger \rho_S U_S \quad L_e(\rho_S, \Lambda_S) := \sqrt{2(1 - F_e(\rho, \Lambda_S))}$$

Situation that we treat (details)

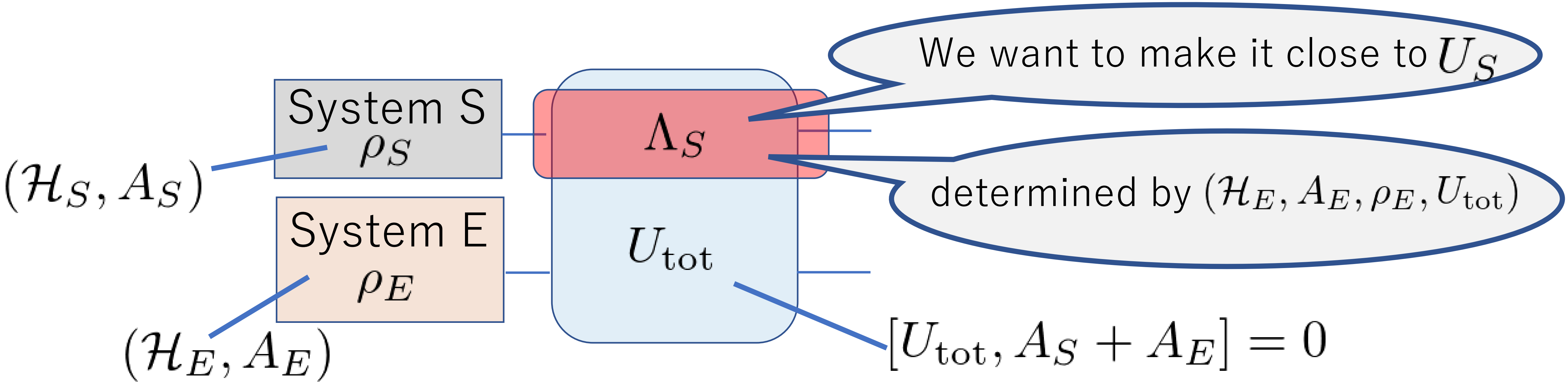


We define $\mathcal{F}_{U_S,\delta}$ as the minimal sufficient amount of QFI to implement U_S within error δ.

$$\mathcal{F}_{U_S,\delta} := \min_{\mathcal{I} \models_\delta U_S} \mathcal{F}_{A_E}(\rho_E)$$

$\mathcal{I} \models_\delta U_S$ means "$\mathcal{I}$ implements U_S within error δ"

Situation that we treat (details)



We define $\mathcal{A}_{U_S}$ as degree of how U_S changes the conserved quantity A_S

Maximum and minimum eigenvalues

$$\mathcal{A}_{U_S} := \frac{\lambda_{\max}(U_S^\dagger A_S U_S - A_S) - \lambda_{\min}(U_S^\dagger A_S U_S - A_S)}{2},$$

Results

The following two bounds hold：

$$\frac{\mathcal{A}_{U_S}}{\delta} - 4\|A_S\| \leq \sqrt{\mathcal{F}_{U_S,\delta}} \leq \frac{\mathcal{A}_{U_S}}{\delta} + \sqrt{2}\|A_S\|$$

Lower bound for
necessary coherence

Upper bound for
sufficient coherence

Results

Combining two bounds, we obtain an asymptotic equality:

$$\sqrt{\mathcal{F}_{U_S,\delta}} = \frac{\mathcal{A}_{U_S}}{\delta} + O(\|A_S\|) \qquad \delta \to 0$$

Simple equality between degree of asymmetry
(degree of violation of conservation law) and amount of coherence!

Summary

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We derived two inequalities and one equality.

Trade-off:

$$\sqrt{V} \geq \frac{\|[U_S, H_S]\|}{40\delta}$$

Fundamental trade-off between error and fluctuation

➡ Answer to Ozawa's question

Trade-off2:

$$\sqrt{\mathcal{F}} \geq \frac{\|[U_S, H_S]\|}{81\delta}$$

A lower bound for necessary coherence for implementing unitary dynamics under energy-conservation law

Trade-off3:

$$\sqrt{\mathcal{F}_{U_S,\delta}} = \frac{\mathcal{A}_{U_S}}{\delta} + O(\|A_S\|)$$

Asymptotic equality for "necessary and sufficient" coherence for implementing unitary dynamics under conservation laws

➡ Answer to the key question of resource theory of quantum channels in a special case.