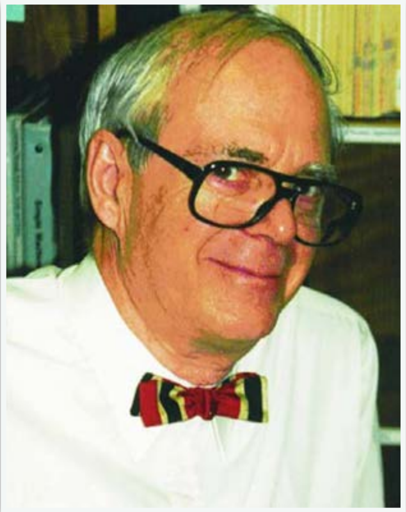
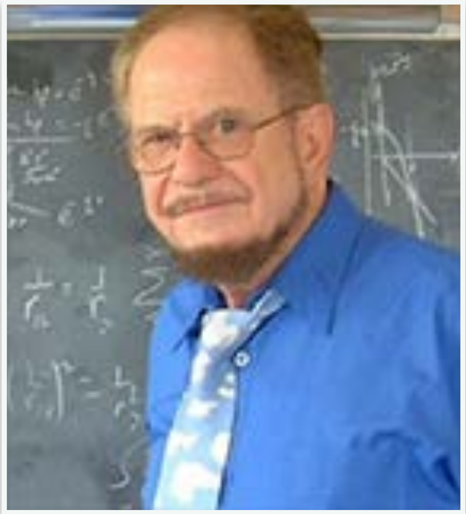




Quantum Spin Chains and von Neumann Algebra

Lieb-Schultz-Mattis type theorem without continuous symmetry

Hal Tasaki

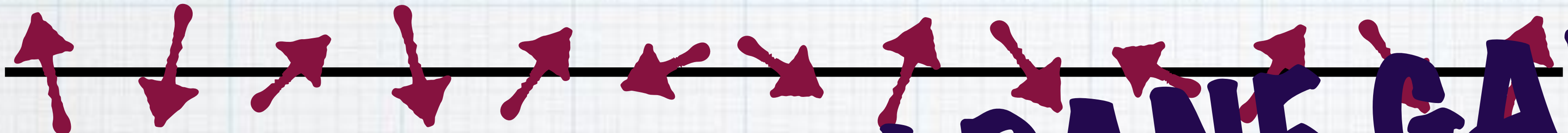
Quantum Information and String Theory 2019 (YITP, June 10)

Yoshiko Ogata and Hal Tasaki, "Lieb–Schultz–Mattis Type Theorems for Quantum Spin Chains Without Continuous Symmetry"
arXiv:1808.08740, Commnun. Math. Phys.

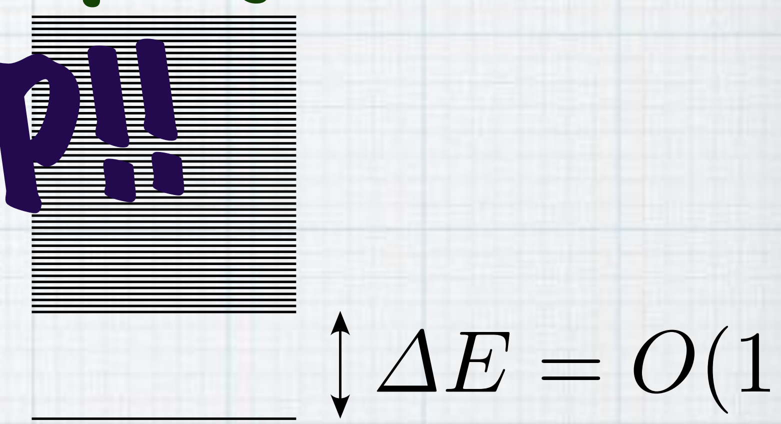
Lieb-Schultz-Mattis Theorem and its Generalizations

Lieb-Schultz-Mattis (LSM) type theorem

No-go theorem which states that certain quantum many-body systems **CANNNOT** have a gapped unique ground state



NO HALDANE GAP!!



the original theorem Lieb, Schultz, Mattis 1961, Affleck, Lieb 1986

antiferromagnetic Heisenberg chain

$$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1} \text{ with } S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$$

for any $\ell < L$, there exists an energy eigenvalue E such that $E_{\mathrm{GS}} < E \leq E_{\mathrm{GS}} + \frac{\text{const.}}{\ell}$

there are gapless excitations in the limit $L \uparrow \infty$

Proof of the original theorem

Lieb, Schultz, Mattis 1961, Affleck, Lieb 1986

$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1}$ unique ground state $|\mathrm{GS}\rangle$

(1) variational estimate

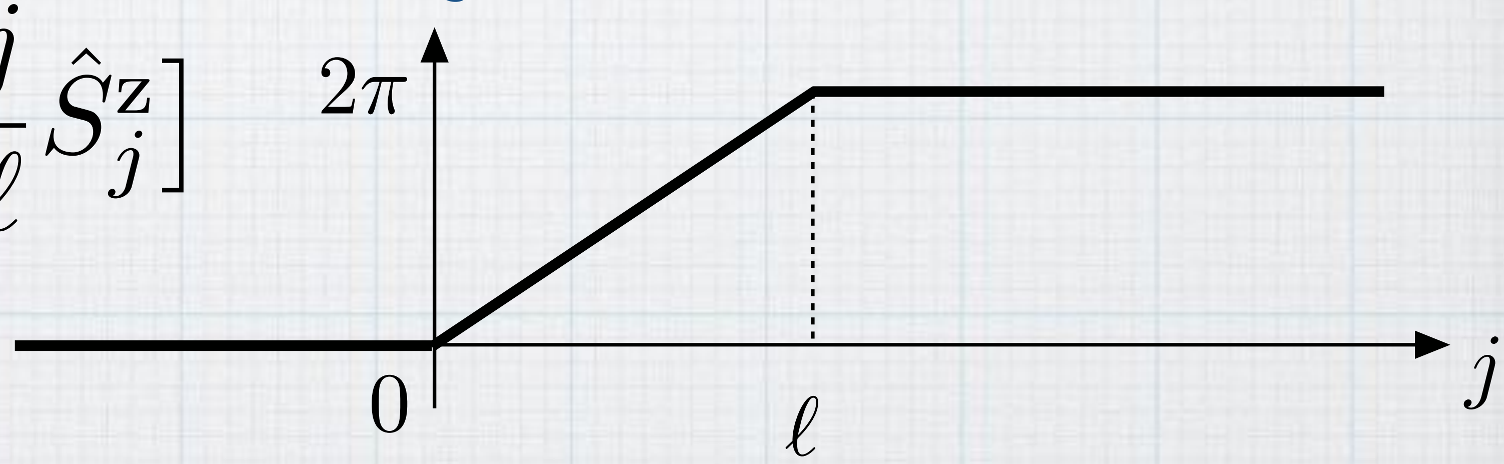
g.s. is rotation invariant $\exp\left[i\sum_{j=1}^{L}\theta\, S_j^{\mathrm{z}}\right]|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

uniform rotation about z

gradual non-uniform rotation to g.s.

$$\hat{V}_\ell = \exp\left[i\sum_{j=0}^{\ell} 2\pi\frac{j}{\ell}\hat{S}_j^{\mathrm{z}}\right]$$

$$|\Psi_\ell\rangle = \hat{V}_\ell|\mathrm{GS}\rangle$$



from an elementary estimate

$$\langle\Psi_\ell|\hat{H}|\Psi_\ell\rangle - E_{\mathrm{GS}} \le \frac{\mathrm{const.}}{\ell}$$

Proof of the original theorem

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

$\hat{H} = \sum_{j=1}^{L} \hat{\boldsymbol{S}}_j \cdot \hat{\boldsymbol{S}}_{j+1}$ unique ground state $|\mathrm{GS}\rangle$

(1) variational estimate

$$\langle \Psi_\ell | \hat{H} | \Psi_\ell \rangle - E_{\mathrm{GS}} \leq \frac{\text{const.}}{\ell} \qquad |\Psi_\ell\rangle = \hat{V}_\ell |\mathrm{GS}\rangle$$

(2) orthogonality

it can be shown (by symmetry) that $\langle \Psi_\ell | \mathrm{GS} \rangle = 0$

for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

for any $\ell < L$, there exists an energy eigenvalue E such that $E_{\mathrm{GS}} < E \leq E_{\mathrm{GS}} + \frac{\text{const.}}{\ell}$

there cannot be a unique gapped ground state!

Lieb-Schultz-Mattis (LSM) type theorem

No-go theorem which states that certain quantum many-body systems **CANNNOT** have a gapped unique ground state

the original theorem and its extensions

Lieb, Shultz, Mattis 1961, Affleck, Lieb 1986

Oshikawa, Yamanaka, Affleck 1997

Oshikawa 2000, Hastings 2004, Nachtergaele, Sims 2007

U(1) invariance is essential

recent "extensions"

Chen, Gu, Wen 2011   Watanabe, Po, Vishwanath, Zaletel 2013

similar no-go statements for models without continuous symmetry, but with some discrete symmetry

projective representation of the symmetry is inconsistent with the existence of unique disordered state

the argument appears already in Matsui 2001

A Typical Theorem

THEOREM 1: Consider a quantum spin chain with $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2\times\mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

$\mathbb{Z}_2\times\mathbb{Z}_2$ transformation

π-rotations about the three axes

$\mathcal{R}_{\rm x}$	$\hat{S}^{\rm x}_j\to\hat{S}^{\rm x}_j$	$\hat{S}^{\rm y}_j\to-\hat{S}^{\rm y}_j$	$\hat{S}^{\rm z}_j\to-\hat{S}^{\rm z}_j$
$\mathcal{R}_{\rm y}$	$\hat{S}^{\rm x}_j\to-\hat{S}^{\rm x}_j$	$\hat{S}^{\rm y}_j\to\hat{S}^{\rm y}_j$	$\hat{S}^{\rm z}_j\to-\hat{S}^{\rm z}_j$
$\mathcal{R}_{\rm z}$	$\hat{S}^{\rm x}_j\to-\hat{S}^{\rm x}_j$	$\hat{S}^{\rm y}_j\to-\hat{S}^{\rm y}_j$	$\hat{S}^{\rm z}_j\to\hat{S}^{\rm z}_j$

invariant Hamiltonian (an example)

$$\hat{H}=\sum_j\{J_{\rm x}\hat{S}^{\rm x}_j\hat{S}^{\rm x}_{j+1}+J_{\rm y}\hat{S}^{\rm y}_j\hat{S}^{\rm y}_{j+1}+J_{\rm z}\hat{S}^{\rm z}_j\hat{S}^{\rm z}_{j+1}+K\hat{S}^{\rm x}_j\hat{S}^{\rm y}_j\hat{S}^{\rm z}_j\}$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation of a single spin

spin operator $\hat{\boldsymbol{S}} = (\hat{S}^{\mathrm{x}}, \hat{S}^{\mathrm{y}}, \hat{S}^{\mathrm{z}})$ $\quad \hat{\boldsymbol{S}}^2 = S(S+1)$ $\quad S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

$\hat{S}^{\mathrm{x}}, \hat{S}^{\mathrm{y}}, \hat{S}^{\mathrm{z}}$ (representation of the) generators of $\mathfrak{su}(2)$

$(2S+1) \times (2S+1)$ matrices

the simplest (but an important) case with $S = 1/2$

$$\hat{S}^{\mathrm{x}} = \frac{X}{2} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad \hat{S}^{\mathrm{y}} = \frac{Y}{2} = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix} \quad \hat{S}^{\mathrm{z}} = \frac{Z}{2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}$$

π-rotation about the α-axis $\quad \hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha] \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\hat{u}_{\mathrm{x}} \hat{u}_{\mathrm{y}} = \hat{u}_{\mathrm{z}} \qquad \hat{u}_{\mathrm{y}} \hat{u}_{\mathrm{z}} = \hat{u}_{\mathrm{x}} \qquad \hat{u}_{\mathrm{z}} \hat{u}_{\mathrm{x}} = \hat{u}_{\mathrm{y}} \qquad \hat{u}_{\mathrm{z}} \hat{u}_{\mathrm{y}} \hat{u}_{\mathrm{x}} = \hat{1}$$

for $S = 1/2$ we have

$$\hat{u}^{\mathrm{x}} = -iX \qquad \hat{u}^{\mathrm{y}} = -iY \qquad \hat{u}^{\mathrm{z}} = -iZ$$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation of a single spin

spin operator $\hat{\boldsymbol{S}} = (\hat{S}^{\mathrm{x}}, \hat{S}^{\mathrm{y}}, \hat{S}^{\mathrm{z}})$ $\quad \hat{\boldsymbol{S}}^2 = S(S+1) \quad S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$

π-rotation about the α-axis $\quad \hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha] \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\hat{u}_{\mathrm{x}}\hat{u}_{\mathrm{y}} = \hat{u}_{\mathrm{z}} \qquad \hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{z}} = \hat{u}_{\mathrm{x}} \qquad \hat{u}_{\mathrm{z}}\hat{u}_{\mathrm{x}} = \hat{u}_{\mathrm{y}} \qquad \hat{u}_{\mathrm{z}}\hat{u}_{\mathrm{y}}\hat{u}_{\mathrm{x}} = \hat{1}$$

integer S $\quad (S = 1, 2, \ldots)$

$$(\hat{u}_\alpha)^2 = \hat{1} \qquad \hat{u}_\alpha \hat{u}_\beta = \hat{u}_\beta \hat{u}_\alpha$$

$\hat{1}, \hat{u}_{\mathrm{x}}, \hat{u}_{\mathrm{y}}, \hat{u}_{\mathrm{z}}$ give a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

half-odd-integer S $\quad (S = \frac{1}{2}, \frac{3}{2}, \ldots)$

$$(\hat{u}_\alpha)^2 = -\hat{1} \qquad \hat{u}_\alpha \hat{u}_\beta = -\hat{u}_\beta \hat{u}_\alpha \qquad \alpha \neq \beta$$

$\hat{1}, \hat{u}_{\mathrm{x}}, \hat{u}_{\mathrm{y}}, \hat{u}_{\mathrm{z}}$ give a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

for $S = 1/2$ we have $\hat{u}^{\mathrm{x}} = -iX \qquad \hat{u}^{\mathrm{y}} = -iY \qquad \hat{u}^{\mathrm{z}} = -iZ$

Theorem for Matrix Product States (MPS)

Watanabe, Po, Vishwanath, Zaletel 2013

Matrix Product States (MPS)

Fannes, Nachtergaele, Werner 1991, 1992

quantum spin system with spin S on $\{1, 2, \ldots, L\}$

standard basis states $|\sigma_1, \ldots, \sigma_L\rangle = \bigotimes_{j=1}^{L} |\sigma_j\rangle_j$

$\hat{S}^{\mathrm{z}}|\sigma\rangle = \sigma|\sigma\rangle$

$D \times D$ matrices M^σ with $\sigma = -S, \ldots, S$

translation invariant state (MPS)

$$|\Phi\rangle = \sum_{\sigma_1, \ldots, \sigma_L = -S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}] |\sigma_1, \ldots, \sigma_L\rangle$$

it is known that disordered states (area-law states) can be approximated by MPS

$|\Phi\rangle$ is said to be injective if $\sum_{\sigma=-S}^{S} \mathsf{M}^\sigma (\mathsf{M}^\sigma)^\dagger = \mathsf{I}$, and there is ℓ such that $\mathsf{M}^{\sigma_1}\mathsf{M}^{\sigma_2} \cdots \mathsf{M}^{\sigma_\ell}$ with all possible $\sigma_1, \ldots, \sigma_\ell$ span the whole space of $D \times D$ matrices

heuristic

$|\Phi\rangle$ is injective if it is disordered, and not a "cat"

Theorem for MPS

translation invariant state (MPS)

$$|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}] |\sigma_1,\ldots,\sigma_L\rangle$$

$|\Phi\rangle$ is said to be injective if $\sum_{\sigma=-S}^{S} \mathsf{M}^{\sigma}(\mathsf{M}^{\sigma})^{\dagger} = \mathsf{I}$, and there is ℓ such that $\mathsf{M}^{\sigma_1}\mathsf{M}^{\sigma_2}\cdots\mathsf{M}^{\sigma_\ell}$ with all possible $\sigma_1,\ldots,\sigma_\ell$ span the whole space of $D \times D$ matrices

heuristic

$|\Phi\rangle$ is injective if it is disordered, and not a "cat"

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Proof of Theorem 1'

Watanabe, Po, Vishwanath, Zaletel 2013 (arranged by H.T.)

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

assume that $|\Phi\rangle = \sum_{\sigma_1,\ldots,\sigma_L=-S}^{S} \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}] \, |\sigma_1, \ldots, \sigma_L\rangle$

is injective, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ **invariant, i.e.,**

$\exp[-i\pi \sum_j \hat{S}_j^\alpha] |\Phi\rangle = \mathrm{const}\, |\Phi\rangle$ **for** $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\sum \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1} \ldots \tilde{\mathsf{M}}^{\sigma_L}] \, |\sigma_1, \ldots, \sigma_L\rangle = \mathrm{const} \sum \mathrm{Tr}[\mathsf{M}^{\sigma_1} \ldots \mathsf{M}^{\sigma_L}] \, |\sigma_1, \ldots, \sigma_L\rangle$$

with $\tilde{\mathsf{M}}^\sigma = \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha | \sigma' \rangle \mathsf{M}^{\sigma'}$ $\qquad \hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$

Proof of Theorem 1'

$$\sum \mathrm{Tr}[\tilde{\mathsf{M}}^{\sigma_1} \dots \tilde{\mathsf{M}}^{\sigma_L}] \, |\sigma_1, \dots, \sigma_L\rangle = \mathrm{const} \sum \mathrm{Tr}[\mathsf{M}^{\sigma_1} \dots \mathsf{M}^{\sigma_L}] \, |\sigma_1, \dots, \sigma_L\rangle$$

with $\tilde{\mathsf{M}}^{\sigma} = \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha | \sigma' \rangle \mathsf{M}^{\sigma'}$

uniqueness of injective MPS

Fannes, Nachtergaele, Werner 1992
Perez-Garcia, Wolf, Sanz, Verstraete, and Cirac 2008
Pollmann, Turner, Berg, Oshikawa 2010

there are $D \times D$ unitary matrices $\mathsf{U}_\mathrm{x}, \mathsf{U}_\mathrm{y}, \mathsf{U}_\mathrm{z}$ which form a projective representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and constants $\zeta_\alpha \in \mathbb{C}$ with $|\zeta_\alpha| = 1$ for $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$, such that

$$\tilde{\mathsf{M}}^{\sigma} = \zeta_\alpha \, \mathsf{U}_\alpha^\dagger \mathsf{M}^{\sigma} \mathsf{U}_\alpha$$

thus the matrices satisfy nontrivial constraints

$$\mathsf{M}^{\sigma} = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \mathsf{U}_\alpha^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\alpha \quad \text{for } \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

Proof of Theorem 1'

thus the matrices satisfy nontrivial constraints

$$\mathsf{M}^{\sigma} = \zeta_\alpha \sum_{\sigma'} \langle\sigma|\hat{u}_\alpha^\dagger|\sigma'\rangle \mathsf{U}_\alpha^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\alpha \quad \text{for } \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi\hat{S}^\alpha]$$

we then find

S is a half-odd integer

$$\mathsf{M}^{\sigma} = (\zeta_\alpha)^2 \sum_{\sigma'} \langle\sigma|\underbrace{(\hat{u}_\alpha^\dagger)^2}_{=-\hat{1}}|\sigma'\rangle \underbrace{(\mathsf{U}_\alpha^\dagger)^2 \mathsf{M}^{\sigma'} (\mathsf{U}_\alpha)^2}_{=\mathsf{M}^{\sigma'}} = -(\zeta_\alpha)^2\, \mathsf{M}^{\sigma}$$

and

$$\mathsf{M}^{\sigma} = \zeta_\mathrm{x} \sum_{\sigma'} \langle\sigma|\hat{u}_\mathrm{x}^\dagger|\sigma'\rangle \mathsf{U}_\mathrm{x}^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\mathrm{x}$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y} \sum_{\sigma'} \langle\sigma|(\hat{u}_\mathrm{y}\hat{u}_\mathrm{x})^\dagger|\sigma'\rangle (\mathsf{U}_\mathrm{y}\mathsf{U}_\mathrm{x})^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\mathrm{y}\mathsf{U}_\mathrm{x}$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z} \sum_{\sigma'} \langle\sigma|\underbrace{(\hat{u}_\mathrm{z}\hat{u}_\mathrm{y}\hat{u}_\mathrm{x})^\dagger}_{=\hat{1}}|\sigma'\rangle \underbrace{(\mathsf{U}_\mathrm{z}\mathsf{U}_\mathrm{y}\mathsf{U}_\mathrm{x})^\dagger \mathsf{M}^{\sigma'} \mathsf{U}_\mathrm{z}\mathsf{U}_\mathrm{y}\mathsf{U}_\mathrm{x}}_{=\mathsf{M}^{\sigma'}}$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z}\, \mathsf{M}^{\sigma}$$

$$(\zeta_\alpha)^2 = -1$$

$$\zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z} = 1$$

contradiction!

Theorem for MPS

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$

nontrivial projective representation of the on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry is inconsistent with the existence of an injective MPS

Matsui 2001 Chen, Gu, Wen 2011 Watanabe, Po, Vishwanath, Zaletel 2013

$$\mathsf{M}^{\sigma} = \zeta_{\alpha} \sum_{\sigma'} \langle \sigma | \hat{u}^{\dagger}_{\alpha} | \sigma' \rangle \mathsf{U}^{\dagger}_{\alpha} \mathsf{M}^{\sigma'} \mathsf{U}_{\alpha}$$

projective representation genuine representation

contradiction!

if S is an integer, there are translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS, such as the AKLT state

Toward the Full Theorem

From Theorem 1' to Theorem 1

we have proved

THEOREM 1': There cannot be a translation invariant and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant injective MPS for $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

this seems to imply the desired

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

From Theorem 1' to Theorem 1

assume that the GS is unique and gapped

Hamiltonian has translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

the GS is disordered, and translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

disordered states can be approximated by MPS

there exists an injective MPS that is translationally and $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant

this contradicts Theorem 1'

this "proof" looks plausible, but does not work!!!

the approximation by MPS is not that precise

the proof of Theorem 1 makes an essential use of operator algebraic formulation

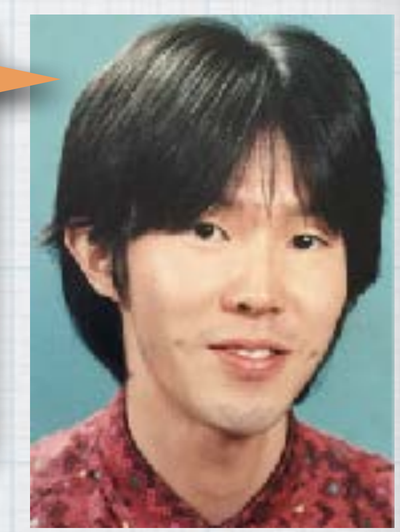
Opinions of a mathematical physicist on operator algebraic approaches to spin systems

early 20's (student)

Hey! Here's a formulation that allows us to treat infinite systems as they are! Probably we can solve phase transitions, renormalization, and everything!

In most cases physically interesting results are proved in finite systems without operator algebra...

It's useful for formulating various concepts of infinite systems, but not for proving concrete results. We can work within finite systems to prove important and interesting results!



mid 20's (posdoc)

Opinions of a mathematical physicist on operator algebraic approaches to spin systems

WOW!
IT'S USEFUL!!!!!

Ogata, Tasaki 2018
Ogata 2018, 2019

late 50's (old guy)

index theorems for SPT phases

the core of the proof of Theorem 1

if the g.s. is unique and accompanied by a gap, there is a representation of the Cuntz algebra $c^\sigma \in B(\tilde{\mathcal{H}}_\mathrm{R})$

$\sigma = -S, \ldots, S$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation of c^σ

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}^\dagger_\alpha | \sigma' \rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'}) \quad \zeta_\alpha \in \mathbb{C} \quad |\zeta_\alpha| = 1 \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

$\tilde{\mathcal{R}}_\mathrm{x}, \tilde{\mathcal{R}}_\mathrm{y}, \tilde{\mathcal{R}}_\mathrm{z}$ *-automorphisms on $B(\tilde{\mathcal{H}}_\mathrm{R})$

give a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

exactly the same transformation rule as in MPS!

$$\mathsf{M}^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}^\dagger_\alpha | \sigma' \rangle \mathsf{U}^\dagger_\alpha \mathsf{M}^{\sigma'} \mathsf{U}_\alpha$$

the same argument leads to contradiction Matsui 2001

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}^\dagger_\alpha | \sigma' \rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'})$$

projective representation | genuine representation

the core of the proof of Theorem 1

if the g.s. is unique and accompanied by a gap, there is a representation of the Cuntz algebra $c^\sigma \in B(\tilde{\mathcal{H}}_\mathrm{R})$

$\sigma = -S, \ldots, S$

$(c^\sigma)_{\sigma=-S,\ldots,S}$ "infinite dimension version" of matrices for MPS

$$(c^\sigma)^* c^{\sigma'} = \delta_{\sigma,\sigma'} \hat{1}$$

$$\pi_\mathrm{R}(|\sigma\rangle\langle\sigma'| \otimes \hat{1}_{[1,\infty)}) = c^\sigma (c^{\sigma'})^*$$

$$\sum_\sigma c^\sigma \pi_\mathrm{R}(\hat{A})(c^\sigma)^* = \pi_\mathrm{R}(\tau(\hat{A}))$$

related to the shift in a half-finite chain

$$c^\sigma |\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$

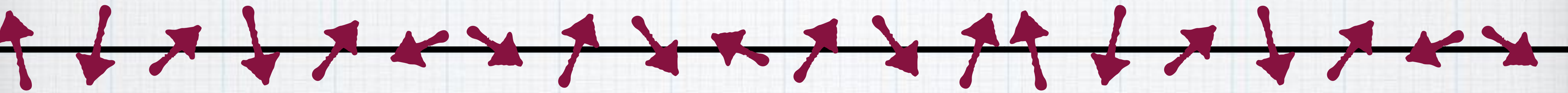
$$= |\sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$

non-rigorous picture!

reminds us of Hilbert's hotel with infinitely many rooms!

Outline of the Formulation and the Proof

Operator algebraic formulation of an infinite quantum spin chain 1/4



C^*-algebra $\mathfrak{A}$

$\mathfrak{A}_{\rm loc}$ the set of all polynomials of $\hat{S}_j^{(\alpha)}$ $\quad j \in \mathbb{Z} \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$\mathfrak{A} := \overline{\mathfrak{A}_{\rm loc}}$ (completion w.r.t. the operator norm)

the set of all local operators + a little bit more

states on $\mathfrak{A}$

a state is a linear function $\rho : \mathfrak{A} \to \mathbb{C}$

such that $\rho(\hat{1}) = 1$ and $\rho(\hat{A}^* \hat{A}) \geq 0$ for any $\hat{A} \in \mathfrak{A}$

$\rho(\hat{A})$ the expectation value of $\hat{A}$ in the state

(Rem: the set of all states is weak-* compact)

Operator algebraic formulation of an infinite quantum spin chain 2/4

Hamiltonian and commutator

formal Hamiltonian $\hat{H} = \sum_{j\in\mathbb{Z}} \hat{h}_j$ with $\hat{h}_j \in \mathfrak{A}_{\rm loc}$

commutator $[\hat{H}, \hat{A}] = [\sum_{j=-\ell}^{\ell} \hat{h}_j, \hat{A}]$ for sufficiently large ℓ

is well-defined for $\hat{A} \in \mathfrak{A}_{\rm loc}$

ground states

a state ω is a g.s. iff $\omega(\hat{A}^*[\hat{H}, \hat{A}]) \geq 0$ for any $\hat{A} \in \mathfrak{A}_{\rm loc}$

in a finite system

$$\langle \mathrm{GS}|\hat{A}^*[\hat{H}, \hat{A}]|\mathrm{GS}\rangle = \langle \mathrm{GS}|\hat{A}^*\hat{H}\hat{A}|\mathrm{GS}\rangle - E_{\rm GS}\langle \mathrm{GS}|\hat{A}^*\hat{A}|\mathrm{GS}\rangle \geq 0$$

unique gapped ground states

a unique g.s. ω is accompanied by a nonzero gap iff there exists $\gamma > 0$ such that $\omega(\hat{A}^*[\hat{H}, \hat{A}]) \geq \gamma\, \omega(\hat{A}^*\hat{A})$ for any $\hat{A} \in \mathfrak{A}_{\rm loc}$ with $\omega(\hat{A}) = 0$

Operator algebraic formulation of an infinite quantum spin chain 3/4

what is the Hilbert space of the model?

$\mathcal{H}_\infty := \bigotimes_{j\in\mathbb{Z}} \mathbb{C}^{2S+1}$ is too large (physically and mathematically)

GNS (Gelfand-Naimark-Segal) construction

given a state ρ on $\mathfrak{A}$, one can define

- a separable Hilbert space $\mathcal{H}$
- a representation π of $\mathfrak{A}$ on $\mathcal{H}$, i.e., $\pi : \mathfrak{A} \to B(\mathcal{H})$ s.t.

the set of all bounded operators

$$\pi(\alpha\hat{A} + \beta\hat{B}) = \alpha\,\pi(\hat{A}) + \beta\,\pi(\hat{B}) \qquad \pi(\hat{A}\hat{B}) = \pi(\hat{A})\,\pi(\hat{B})$$

$$\pi(\hat{A}^*) = \pi(\hat{A})^* \qquad \pi(\hat{1}) = \hat{1}$$

- a vector $\Omega \in \mathcal{H}$ s.t. $\rho(\hat{A}) = \langle \Omega, \pi(\hat{A})\Omega\rangle$ for any $\hat{A} \in \mathfrak{A}$

$(\mathcal{H}, \pi, \Omega)$ or, more precisely $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$, is the GNS triple

$\mathcal{H}$ is a physical Hilbert space that consists of the state ρ (which is now Ω) and other states "close" to it

Operator algebraic formulation of an infinite quantum spin chain 4/4

GNS (Gelfand-Naimark-Segal) construction

given a state ρ on $\mathfrak{A}$, one can define

- a separable Hilbert space $\mathcal{H}$
- a representation π of $\mathfrak{A}$ on $\mathcal{H}$, i.e., $\pi : \mathfrak{A} \to B(\mathcal{H})$ s.t.
- a vector $\Omega \in \mathcal{H}$ s.t. $\rho(\hat{A}) = \langle \Omega, \pi(\hat{A})\Omega \rangle$ for any $\hat{A} \in \mathfrak{A}$

the set of all bounded operators

the idea of the construction

we already have $\mathfrak{A}$, which is a vector space

define an inner product in $\mathfrak{A}$ by $\langle \hat{A}, \hat{B} \rangle := \rho(\hat{A}^* \hat{B})$

make $\mathfrak{A}$ into a Hilbert space by $\mathcal{H} := \overline{\mathfrak{A}/\sim}$

$$\hat{A} \sim \hat{B} \Leftrightarrow \langle \hat{A} - \hat{B}, \hat{A} - \hat{B} \rangle = 0$$

for $\psi_{\hat{A}} \in \mathcal{H}$ we define the representation by $\pi(\hat{B})\psi_{\hat{A}} = \psi_{\hat{B}\hat{A}}$

we set $\Omega := \psi_{\hat{1}}$

Setup for Theorem 1

spin operators $\hat{S}_j^{(\alpha)}$ $j \in \mathbb{Z}$ $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

$$\hat{\boldsymbol{S}}_j = (\hat{S}_j^{(\mathrm{x})}, \hat{S}_j^{(\mathrm{y})}, \hat{S}_j^{(\mathrm{z})}) \qquad (\hat{\boldsymbol{S}}_j)^2 = S(S+1)\,\hat{1}$$

we only consider models with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

translation automorphism $\tau(\hat{S}_j^{(\alpha)}) = \hat{S}_{j+1}^{(\alpha)}$ etc.
(linear *-automorphism)

$\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphism $\mathcal{R}_\alpha(\hat{S}_j^{(\beta)}) = \begin{cases} \hat{S}_j^{(\beta)} & \alpha = \beta \\ -\hat{S}_j^{(\beta)} & \alpha \neq \beta \end{cases}$
(linear *-automorphism)

Hamiltonian $\hat{H} = \sum_{j \in \mathbb{Z}} \hat{h}_j$

short ranged: $\hat{h}_j$ depends only on $\hat{S}_i^{(\alpha)}$ with $|i-j| \le r$

translation invariant: $\tau(\hat{h}_j) = \hat{h}_{j+1}$ for any $j \in \mathbb{Z}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant: $\mathcal{R}_\alpha(\hat{h}_j) = \hat{h}_j$ for any $j \in \mathbb{Z}$ $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$

we assume that the g.s. ω is unique and accompanied by a nonzero energy gap

Setup for Theorem 1

we assume that the g.s. ω is unique and accompanied by a nonzero energy gap

we shall derive the transformation rule

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle\sigma|\hat{u}_\alpha^\dagger|\sigma'\rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'}) \qquad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

projective representation — genuine representation

contradiction!

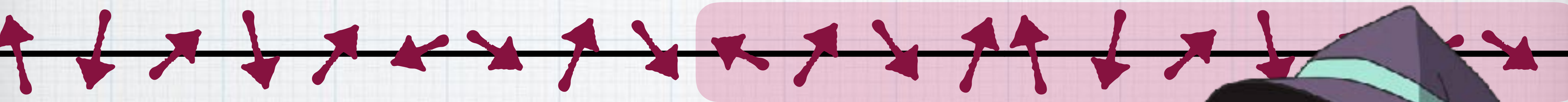
THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

algebras for the half-infinite chain

$\mathfrak{A}_{\mathrm{R}}$ C^*-algebra generated from $\hat{S}_j^{(\alpha)}$ with $j = 0, 1, 2, \ldots$

ω_{R} restriction of the unique g.s. ω onto $\mathfrak{A}_{\mathrm{R}}$

$(\mathcal{H}_{\mathrm{R}}, \pi_{\mathrm{R}}, \Omega_{\mathrm{R}})$ the corresponding GNS triple



$\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})$ —bicommutant→ $\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})'' =: \mathfrak{M}_{\mathrm{R}}$

representation of the C* algebra

von Neumann algebra

$$\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}}) \subset \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})'' = \mathfrak{M}_{\mathrm{R}} \subset B(\mathcal{H}_{\mathrm{R}})$$

closure of $\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})$ w.r.t. the weak topology

Def. of commutant

$\mathfrak{M} \subset B(\mathcal{H})$ $\quad \mathfrak{M}' := \{\hat{A} \in B(\mathcal{H}) \,|\, [\hat{A}, \hat{B}] = 0 \text{ for any } \hat{B} \in \mathfrak{M}\}$

shift on the von Neumann algebra

a unique gapped g.s. ω satisfies the split property

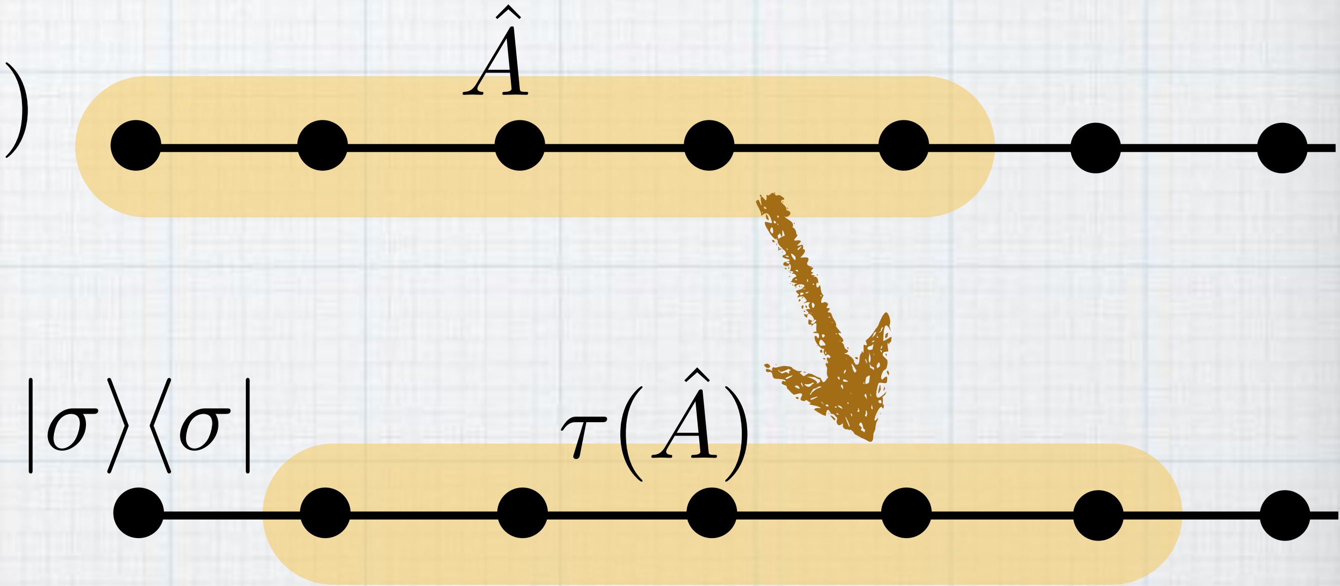
Hastings 2007, Matsui 2013

this means the von Neumann algebra $\mathfrak{M}_{\rm R}$ is a type-I factor

then there is a separable Hilbert space $\tilde{\mathcal{H}}_{\rm R}$ and $\mathfrak{M}_{\rm R} \cong B(\tilde{\mathcal{H}}_{\rm R})$

for $\sigma = -S, \ldots, S$, and $\hat{A} \in \mathfrak{A}_{\rm R}$ we define

$$\Theta^\sigma(\pi_{\rm R}(\hat{A})) = \pi_{\rm R}(|\sigma\rangle\langle\sigma| \otimes \tau(\hat{A}))$$



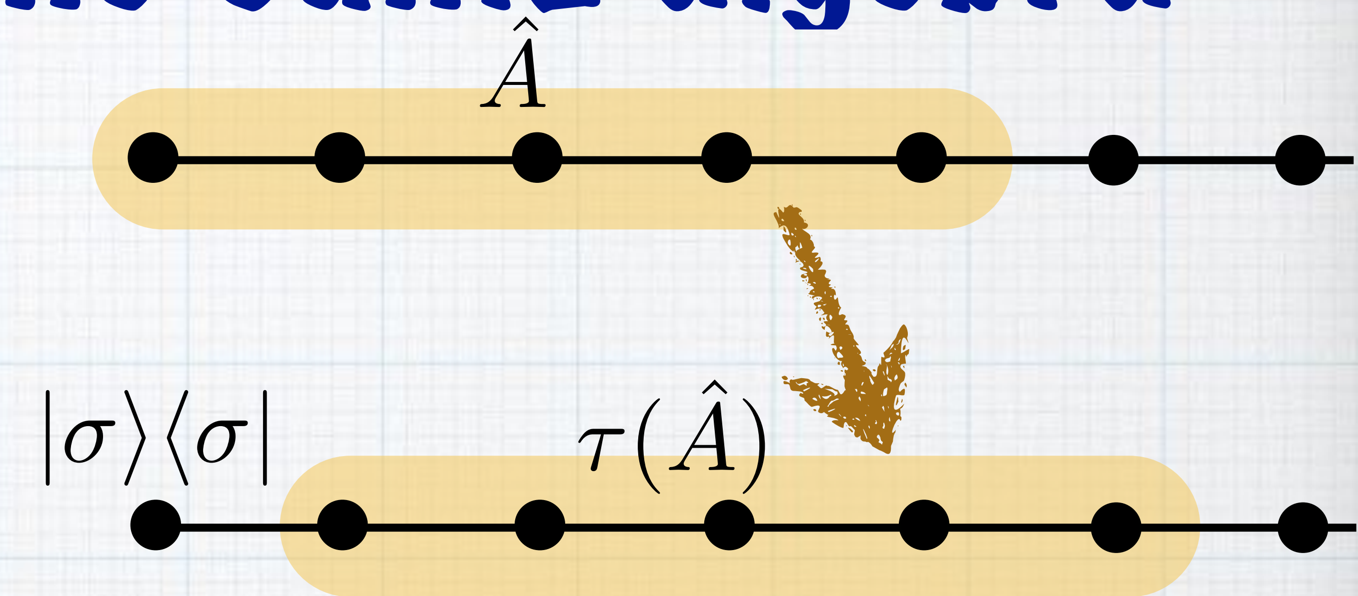
since $\pi_{\rm R}(\hat{A}) \in \pi_{\rm R}(\mathfrak{A}_{\rm R}) \subset \mathfrak{M}_{\rm R}$

Θ^σ can be extended to a unital endomorphism on $\mathfrak{M}_{\rm R}$

translation invariance is essential!

"Wigner's theorem" guarantees that there are operators $c^\sigma \in B(\tilde{\mathcal{H}}_{\rm R})$ such that $\Theta^\sigma X = c^\sigma X (c^\sigma)^*$ for any $X \in \mathfrak{M}_{\rm R}$

representation of the Cuntz algebra



"Wigner's theorem" guarantees that there are operators $c^\sigma \in B(\tilde{\mathcal{H}}_\mathrm{R})$ such that $\Theta^\sigma X = c^\sigma X (c^\sigma)^*$ for any $X \in \mathfrak{M}_\mathrm{R}$

they roughly correspond to

$$c^\sigma \, |\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$

$$= |\sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$

one can show that

$$(c^\sigma)^* c^{\sigma'} = \delta_{\sigma,\sigma'} \hat{1}$$

$$\pi_\mathrm{R}(|\sigma\rangle\langle\sigma'| \otimes \hat{1}_{[1,\infty)}) = c^\sigma (c^{\sigma'})^*$$

$$\sum_\sigma c^\sigma \pi_\mathrm{R}(\hat{A})(c^\sigma)^* = \pi_\mathrm{R}(\tau(\hat{A}))$$

$(c^\sigma)_{\sigma=-S,\ldots,S}$ gives a representation of the Cuntz algebra

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation on $\mathfrak{M}_{\mathrm{R}}$

$\mathfrak{A}_{\mathrm{R}}$ C^*-algebra generated from $\hat{S}_j^{(\alpha)}$ with $j = 0, 1, 2, \ldots$

ω_{R} restriction of the unique g.s. ω onto $\mathfrak{A}_{\mathrm{R}}$

$(\mathcal{H}_{\mathrm{R}}, \pi_{\mathrm{R}}, \Omega_{\mathrm{R}})$ the corresponding GNS triple

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance of the g.s.

$\omega_{\mathrm{R}}(\mathcal{R}_\alpha(\hat{A})) = \omega_{\mathrm{R}}(\hat{A})$ for any $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$ and $\hat{A} \in \mathfrak{A}_{\mathrm{R}}$

the invariance of the GNS inner product

$$\langle \psi_{\hat{A}}, \psi_{\hat{B}} \rangle = \omega_{\mathrm{R}}(\hat{A}^* \hat{B}) = \omega_{\mathrm{R}}(\mathcal{R}_\alpha(\hat{A}^*)\mathcal{R}^\alpha(\hat{B})) = \langle \psi_{\mathcal{R}_\alpha(\hat{A})}, \psi_{\mathcal{R}_\alpha(\hat{B})} \rangle$$

unitary $\hat{U}_\alpha$ on $\mathcal{H}_{\mathrm{R}}$ can be defined by $\hat{U}_\alpha \psi_{\hat{A}} = \psi_{\mathcal{R}_\alpha(\hat{A})}$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ invariance is essential!

$\hat{U}_{\mathrm{x}}, \hat{U}_{\mathrm{y}}, \hat{U}_{\mathrm{z}}$ form a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation on $\mathfrak{M}_{\mathrm{R}}$

unitary $\hat{U}_\alpha$ on $\mathcal{H}_{\mathrm{R}}$ can be defined by $\hat{U}_\alpha \psi_{\hat{A}} = \psi_{\mathcal{R}_\alpha(\hat{A})}$

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$\hat{U}_{\mathrm{x}}, \hat{U}_{\mathrm{y}}, \hat{U}_{\mathrm{z}}$ form a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

for $X \in \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})$, we define $\tilde{\mathcal{R}}_\alpha(X) := \hat{U}_\alpha X \hat{U}_\alpha^*$

which satisfies $\tilde{\mathcal{R}}_\alpha(\pi_{\mathrm{R}}(\hat{A})) = \pi_{\mathrm{R}}(\mathcal{R}_\alpha(\hat{A}))$

$\tilde{\mathcal{R}}_\alpha$ is then extended to $\pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})'' = \mathfrak{M}_{\mathrm{R}}$

$\tilde{\mathcal{R}}_{\mathrm{x}}, \tilde{\mathcal{R}}_{\mathrm{y}}, \tilde{\mathcal{R}}_{\mathrm{z}}$ form a genuine representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$\tilde{\mathcal{R}}_\alpha \circ \tilde{\mathcal{R}}_\beta = \tilde{\mathcal{R}}_\beta \circ \tilde{\mathcal{R}}_\alpha \qquad \tilde{\mathcal{R}}_\alpha \circ \tilde{\mathcal{R}}_\alpha = \mathrm{id}$$

$$\tilde{\mathcal{R}}_{\mathrm{x}} \circ \tilde{\mathcal{R}}_{\mathrm{y}} = \tilde{\mathcal{R}}_{\mathrm{z}} \qquad \tilde{\mathcal{R}}_{\mathrm{y}} \circ \tilde{\mathcal{R}}_{\mathrm{z}} = \tilde{\mathcal{R}}_{\mathrm{x}} \qquad \tilde{\mathcal{R}}_{\mathrm{z}} \circ \tilde{\mathcal{R}}_{\mathrm{x}} = \tilde{\mathcal{R}}_{\mathrm{y}}$$

transformation of the Cuntz algebra

$c^\sigma \in B(\tilde{\mathcal{H}}_{\mathrm{R}}) \cong \mathfrak{M}_{\mathrm{R}} = \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})''$

$$(c^\sigma)^* c^{\sigma'} = \delta_{\sigma,\sigma'} \hat{1}$$
$$\pi_{\mathrm{R}}(|\sigma\rangle\langle\sigma'| \otimes \hat{1}_{[1,\infty)}) = c^\sigma (c^{\sigma'})^*$$
$$\sum_\sigma c^\sigma \pi_{\mathrm{R}}(\hat{A})(c^\sigma)^* = \pi_{\mathrm{R}}(\tau(\hat{A}))$$

$$c^\sigma \, |\sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$
$$= |\sigma, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \ldots\rangle$$

fix $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$, **and let** $t^\sigma = \sum_{\sigma'} \langle\sigma|\hat{u}_\alpha^\dagger|\sigma'\rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'})$

-π rotation in the spin space

π rotation in the operator space

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

the we can show

$$(t^\sigma)^* t^{\sigma'} = \delta_{\sigma,\sigma'} \hat{1}$$
$$\pi_{\mathrm{R}}(|\sigma\rangle\langle\sigma'| \otimes \hat{1}_{[1,\infty)}) = t^\sigma (t^{\sigma'})^*$$
$$\sum_\sigma t^\sigma \pi_{\mathrm{R}}(\hat{A})(t^\sigma)^* = \pi_{\mathrm{R}}(\tau(\hat{A}))$$

$(t^\sigma)_{\sigma=-S,\ldots,S}$ **also gives a representation of the Cuntz algebra**

transformation of the Cuntz algebra

$c^\sigma \in B(\tilde{\mathcal{H}}_{\mathrm{R}}) \cong \mathfrak{M}_{\mathrm{R}} = \pi_{\mathrm{R}}(\mathfrak{A}_{\mathrm{R}})''$

fix $\alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$, and let $t^\sigma = \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'})$

$(t^\sigma)_{\sigma=-S,\ldots,S}$ also gives a representation of the Cuntz algebra

from the uniqueness of representation of the Cuntz algebra

$t^\sigma = \zeta_\alpha \, c^\sigma$ with $\zeta_\alpha \in \mathbb{C}$ $\quad |\zeta_\alpha| = 1$

we finally get the desired transformation rule

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'}) \quad \text{for } \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

projective representation

genuine representation

S is a half-odd integer

contradiction!

Proof of Theorem 1

we finally get the desired transformation rule

$$c^\sigma = \zeta_\alpha \sum_{\sigma'} \langle \sigma | \hat{u}_\alpha^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\alpha(c^{\sigma'}) \quad \text{for} \quad \alpha = \mathrm{x}, \mathrm{y}, \mathrm{z}$$

$$\hat{u}_\alpha = \exp[-i\pi \hat{S}^\alpha]$$

S is a half-odd integer

we then find

$$c^\sigma = (\zeta_\alpha)^2 \sum_{\sigma'} \langle \sigma | \underbrace{(\hat{u}_\alpha^\dagger)^2}_{= -\hat{1}} | \sigma' \rangle \underbrace{\tilde{\mathcal{R}}_\alpha \circ \tilde{\mathcal{R}}_\alpha(c^{\sigma'})}_{= c^{\sigma'}} = -(\zeta_\alpha)^2 \, c^\sigma$$

and

$$c^\sigma = \zeta_\mathrm{x} \sum_{\sigma'} \langle \sigma | \hat{u}_\mathrm{x}^\dagger | \sigma' \rangle \tilde{\mathcal{R}}_\mathrm{x}(c^{\sigma'})$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y} \sum_{\sigma'} \langle \sigma | (\hat{u}_\mathrm{y}\hat{u}_\mathrm{x})^\dagger | \sigma' \rangle \, \tilde{\mathcal{R}}_\mathrm{x} \circ \tilde{\mathcal{R}}_\mathrm{y}(c^{\sigma'})$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z} \sum_{\sigma'} \langle \sigma | \underbrace{(\hat{u}_\mathrm{z}\hat{u}_\mathrm{y}\hat{u}_\mathrm{x})^\dagger}_{= \hat{1}} | \sigma' \rangle \underbrace{\tilde{\mathcal{R}}_\mathrm{x} \circ \tilde{\mathcal{R}}_\mathrm{y} \circ \tilde{\mathcal{R}}_\mathrm{z}(c^{\sigma'})}_{= c^{\sigma'}}$$

$$= \zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z} \, c^\sigma$$

$$(\zeta_\alpha)^2 = -1$$

$$\zeta_\mathrm{x}\zeta_\mathrm{y}\zeta_\mathrm{z} = 1$$

contradiction!

Extensions

symmetry

THEOREM 1: Consider a quantum spin chain with $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2\times\mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

$\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry

↓

any on-site symmetry whose representation on a single spin is projective

example: time-reversal symmetry for $S=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$

$$\hat{S}_j^\alpha \to -\hat{S}_j^\alpha$$

state

THEOREM 1: Consider a quantum spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and a short-ranged Hamiltonian that is invariant under translation and $\mathbb{Z}_2 \times \mathbb{Z}_2$ transformation. Then it can never be the case that the corresponding ground state is unique and accompanied by a nonzero gap.

it is only essential that the state is pure, translation invariant and satisfies the split property

any translation invariant pure state with area law entanglement is excluded Matsui 2013

general theorem

Yuji Tachikawa, private communication

THEOREM 2: In quantum spin chains, there can be no translation invariant pure states with area law entanglement and on-site symmetry whose representation on a single spin is projective

COROLLARY: In a translation invariant spin chain with $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ and time-reversal or $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, any "scar" state must be degenerate and break symmetry

Summary

- ☑ LSM-type no-go theorem is proved for quantum spin chains with translation and on-site symmetry whose representation is projective

- ☑ the proof is based on the inconsistency between the projective symmetry and the transformation property of the Cuntz algebra

- ☑ it is surprising (at least, to me) that such a mathematically abstract object as the von Neumann algebra is useful in proving physically natural theorems

  (cf. Ogata's fully rigorous index theorem for SPT)

background and related topics can be found in my book in preparation (see the workshop Slack or ask me)

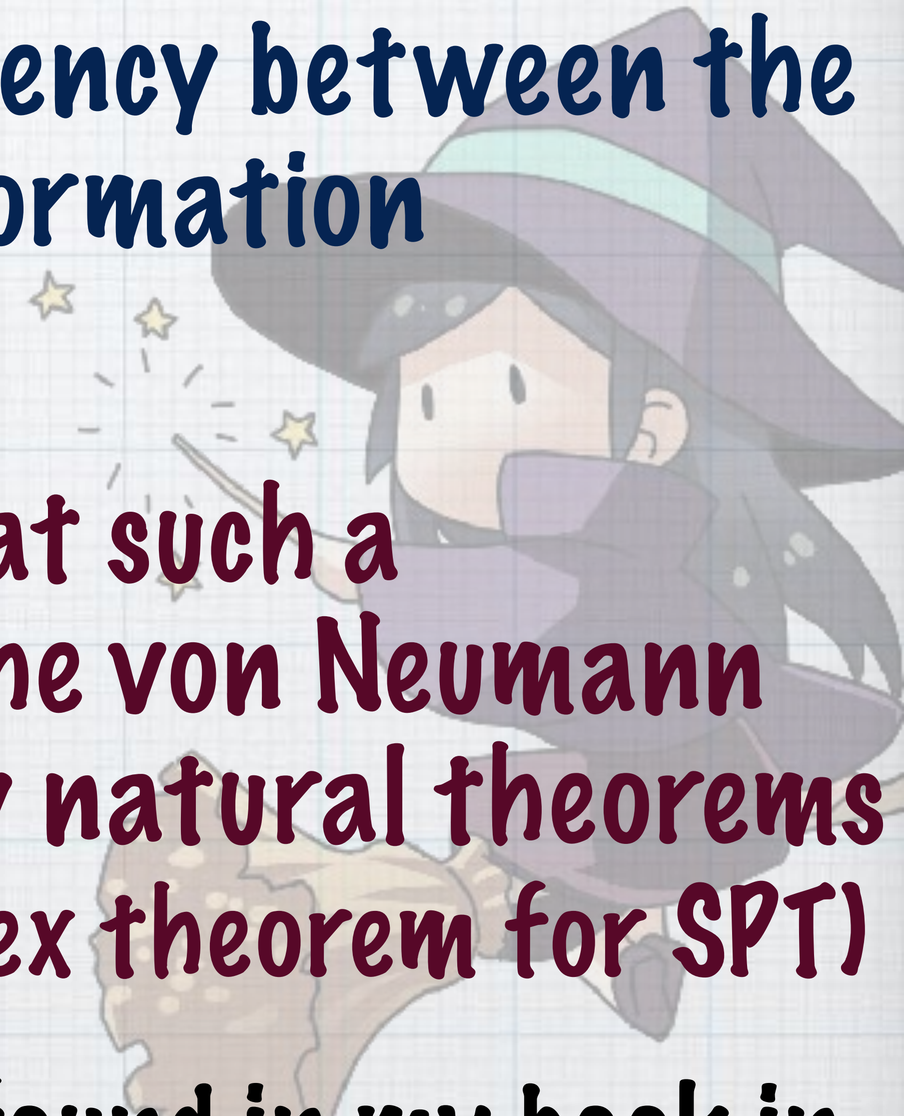

illustration by Chisato Naruse